Algebra Colloquium **1?** : ? (200?) ??–??

Algebra Colloquium

Local Cohomology with Supports in the Non-free Locus*

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Received 9 May 2007
Revised 17 July 2007
Communicated by Zhongming Tang

Abstract. The groups of local cohomology with supports in the non-free locus of a module are used in order to obtain three classifications and one characterization of four classes of modules.

2000 Mathematics Subject Classification: primary 13C05; secondary 13E05, 13E15

Keywords: ideal module, local cohomology, non-free locus, projective dimension, reflexive module

1 Introduction and Preliminaries

Let R be a Noetherian ring and let M be a finitely generated R-module. As is well known (see [5, (11.1.1)]), the set of points $\mathfrak{p} \in \operatorname{Spec} R$ such that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module is an open subset in the Zariski topology. Hence, its complement C is a closed subset, called the non-free locus of M, whose corresponding radical ideal (cf. [1, II, §4.3, Proposition 11(iii)]) is denoted by $\mathfrak{a} = \mathfrak{a}(M) = \mathfrak{I}(C)$ throughout the paper.

Proposition 1.1. *Let R be a Noetherian ring and let $F_1 \xrightarrow{\varphi} F_0 \to M \to 0$ be a finite free presentation of the R-module M. If $\operatorname{rank} \varphi = r$ and $\operatorname{rank} F_0 = n$, then*

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*Supported by Ministerio de Ciencia y Tecnología of Spain, under grants BFM2001-2718 and MTM2005-00173.

the ideal $\mathfrak{a}(M)$ *coincides with the radical of* $(n-r)$*-th Fitting invariant of* M*; i.e.,* $\mathfrak{a}(M) = \operatorname{rad} I_r(\varphi)$.

Proof. As $\mathfrak{a}$ is a radical ideal, we only need to prove that $V(\mathfrak{a}) = V(I_r(\varphi))$, or conversely, that $\mathfrak{a} \not\subseteq \mathfrak{p}$ if and only if $I_r(\varphi) \not\subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p}$ does not contain $\mathfrak{a}$, then $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module and hence $I_r(\varphi)_\mathfrak{p} = R_\mathfrak{p}$ (for example, see [2, Proposition 1.4.9]); therefore $I_r(\varphi) \not\subseteq \mathfrak{p}$. The converse also follows from the previous reference. □

Below, we use the ideal $\mathfrak{a}$ in order to obtain classifications of two classes of modules and a characterization of k-th syzygies. More precisely, in Section 2, a reflexive finitely generated module M over a Noetherian local domain whose dual module is of projective dimension one is shown to be completely determined by $H^2_\mathfrak{a}(M)$. Similarly, in Section 3, we obtain a classification of the ideal modules (in the sense of [7, Proposition 5.1]) over a regular local ring by means of $H^1_\mathfrak{a}(M)$. As a consequence of such a result (see Corollary 3.2), from a decomposition $H^1_\mathfrak{a}(M) = \bigoplus_{i=1}^r R/\mathfrak{a}_i$ for certain ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$, we deduce that M is stably equivalent to $\bigoplus_{i=1}^r \mathfrak{a}_i$. By applying these results and from the existence of a dualizing functor, in Section 4, we also obtain a classification of the torsion-free finitely generated and non-free modules of projective dimension one over a regular local ring.

Finally, in Section 5, we obtain a characterization of k-th syzygies within the class of reflexive finitely generated R-modules over a regular local ring by the vanishing of the groups $H^i_\mathfrak{a}(M)$ for $i = 0, \ldots, k-1$. To a certain extent, this result can be considered as a generalization of the characterization of the vector bundles on the punctured spectrum, which are k-th syzygies (see [4, Lemma 6.5]).

2 Reflexive Modules with Dual of Projective Dimension One

Lemma 2.1. *If* R *is a Cohen–Macaulay ring, then* $H^1_\mathfrak{q}(R) = 0$ *for every ideal* $\mathfrak{q}$ *with* $\operatorname{height} \mathfrak{q} \geq 2$.

Proof. By virtue of the hypothesis, we know that $\operatorname{depth} R_\mathfrak{p} = \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R$ (see [2, Theorem 2.1.3]). Therefore, $\operatorname{depth}_\mathfrak{q} R = \min_{\mathfrak{p} \in V(\mathfrak{q})} \operatorname{depth} R_\mathfrak{p} = \min_{\mathfrak{p} \in V(\mathfrak{q})} \dim R_\mathfrak{p} \geq 2$, and we can conclude by simply applying the cohomological interpretation of depth. □

Theorem 2.2. *Let* $(R, \mathfrak{m})$ *be a regular local ring of dimension* $d \geq 3$. *Then every reflexive finitely generated* R*-module* M *with* $\operatorname{proj\,dim} M^\vee = 1$ *is completely determined by* $H^2_\mathfrak{a}(M)$.

Proof. Let

$$0 \to F_1 \xrightarrow{\varphi} F_0 \to M^\vee \to 0 \tag{1}$$

be a minimal free resolution of $M^\vee$. As M is reflexive, dualizing (1), we obtain the exact sequence

$$0 \to M \to F_0^\vee \xrightarrow{\varphi^\vee} F_1^\vee \to \operatorname{Ext}^1(M^\vee, R) \to 0, \tag{2}$$

which breaks into two short exact sequences

$$0 \to \operatorname{Im}\varphi^\vee \to F_1^\vee \to \operatorname{Ext}^1(M^\vee, R) \to 0, \tag{3}$$

$$0 \to M \to F_0^\vee \to \operatorname{Im}\varphi^\vee \to 0. \tag{4}$$

First, we prove that $\operatorname{Ext}^1(M^\vee, R)$ is isomorphic to $H^2_{\mathfrak{a}}(M)$. Owing to reflexivity, M and $M^\vee$ have the same non-free locus; hence support $\operatorname{Ext}^1(M^\vee, R) \subseteq V(\mathfrak{a})$, and accordingly, $H^0_{\mathfrak{a}}(\operatorname{Ext}^1(M^\vee, R)) = \operatorname{Ext}^1(M^\vee, R)$. Moreover, as $\operatorname{Im}\varphi^\vee$ and $F_1^\vee$ are torsion-free modules, both $H^0_{\mathfrak{a}}(\operatorname{Im}\varphi^\vee)$ and $H^0_{\mathfrak{a}}(F_1^\vee)$ vanish, and taking cohomology with supports in $V(\mathfrak{a})$ in (3), by virtue of Lemma 2.1, we obtain $\operatorname{Ext}^1(M^\vee, R) \cong H^1_{\mathfrak{a}}(\operatorname{Im}\varphi^\vee)$. Similarly, by taking cohomology with supports in $V(\mathfrak{a})$ in (4), we obtain an isomorphism $H^1_{\mathfrak{a}}(\operatorname{Im}\varphi^\vee) \cong H^2_{\mathfrak{a}}(M)$. Hence, $\operatorname{Ext}^1(M^\vee, R) \cong H^2_{\mathfrak{a}}(M)$, and consequently, M is a second syzygy module of $H^2_{\mathfrak{a}}(M)$. In order to conclude, we only need to prove that the minimality of the resolution (1) implies that of (2). In fact, if a linear form $\omega_0 \in F_0^\vee$ exists such that $\varphi^\vee(\omega_0) \notin \mathfrak{m}F_1^\vee$ (cf. [2, Proposition 1.3.1]), then by Nakayama's lemma, $\varphi^\vee(\omega_0)$ belongs to a basis of $F_1^\vee$ and hence there exists an element $x_1 \in F_1$ such that $\varphi^\vee(\omega_0)(x_1) = \omega_0(\varphi(x_1)) = 1$. Hence, $\operatorname{Im}\varphi \not\subset \mathfrak{m}F_0$ and (1) is not minimal. □

3 Ideal Modules

Theorem 3.1. *Let M be an ideal module over a regular Noetherian local ring $(R, \mathfrak{m})$ of dimension $d \geq 2$, and let $r(M)$ be the greatest rank of a free direct summand in M. Then M is completely determined by $r(M)$ and $H^1_{\mathfrak{a}}(M)$.*

Proof. The module M embeds into its bidual and the quotient $T = M^{\vee\vee}/M$ is a torsion module. First of all, we prove that $H^1_{\mathfrak{a}}(M) = T$. For every prime ideal $\mathfrak{p}$ of R with height $\mathfrak{p} \leq 1$, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module as R is regular and M is torsionless. Therefore, $M_{\mathfrak{p}} = M_{\mathfrak{p}}^{\vee\vee}$ and accordingly $\mathfrak{p} \notin V(\mathfrak{a})$; hence height $\mathfrak{a} \geq 2$. Since $M^{\vee\vee}$ is free, by virtue of Lemma 2.1, we conclude that $H^1_{\mathfrak{a}}(M^{\vee\vee}) = 0$. This fact proves our claim by simply taking cohomology with supports in $\mathfrak{a}$ in the sequence $0 \to M \to M^{\vee\vee} \to T \to 0$, and recalling that $H^0_{\mathfrak{a}}(M) = H^0_{\mathfrak{a}}(M^{\vee\vee}) = 0$ as M and $M^{\vee\vee}$ are torsionless, and $H^0_{\mathfrak{a}}(T) = T$ as the support of T is contained in $V(\mathfrak{a})$.

If M and M' are two isomorphic ideal modules, then T and $T' = M'^{\vee\vee}/M'$ are also isomorphic. Moreover, from the very definition of $r(M)$, it follows that there exist two submodules $F, \bar{M} \subseteq M$ such that F is free of rank $r(M)$ and $M = F \oplus \bar{M}$. Hence, we only need to prove that if M and M' are two ideal modules such that $r(M) = r(M')$ and $T \cong T'$, then $\bar{M} \cong \bar{M}'$. Let $\pi\colon \bar{M}^{\vee\vee} \to T = \bar{M}^{\vee\vee}/\bar{M}$ (resp., $\pi'\colon \bar{M}'^{\vee\vee} \to T' = \bar{M}'^{\vee\vee}/\bar{M}'$) be the quotient map. As $\bar{M}^{\vee\vee}$ and $\bar{M}'^{\vee\vee}$ are free R-modules, every isomorphism $\phi\colon T \to T'$ induces a homomorphism $\Phi\colon \bar{M}^{\vee\vee} \to \bar{M}'^{\vee\vee}$ making the following diagram commutative:

$$\begin{array}{ccc} \bar{M}^{\vee\vee} & \xrightarrow{\pi} & T \to 0 \\ \Phi\downarrow & & \downarrow\phi \\ \bar{M}'^{\vee\vee} & \xrightarrow{\pi'} & T' \to 0 \end{array}$$

We claim $\bar{M} \subset \mathfrak{m}\bar{M}^{\vee\vee}$ (resp., $\bar{M}' \subset \mathfrak{m}\bar{M}'^{\vee\vee}$), otherwise, every element $\bar{x}$ in $\bar{M}$ not belonging to $\mathfrak{m}\bar{M}^{\vee\vee}$ generates a submodule $R\bar{x}$, which is a direct summand in $\bar{M}$

by Nakayama's lemma, thus contradicting the definition of $r(M)$. Hence, rank $\bar{M}^{\vee\vee} = \dim_{R/\mathfrak{m}}(T/\mathfrak{m}T)$, and similarly for the rank of $\bar{M}'^{\vee\vee}$. Accordingly, Φ is an isomorphism that induces an isomorphism from $\bar{M} = \ker\pi$ onto $\bar{M}' = \ker\pi'$. □

Corollary 3.2. *With the same hypotheses as in Theorem 3.1, assume a decomposition* $H^1_{\mathfrak{a}}(M) = \bigoplus_{i=1}^r R/\mathfrak{a}_i$ *holds for certain ideals* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ *in* R. *Then there exists a free module* F *such that* $M \cong F \oplus \left(\bigoplus_{i=1}^r \mathfrak{a}_i\right)$.

Proof. Let $M = F \oplus \bar{M}$ be as in the proof of the previous theorem. As R is a unique factorization domain by virtue of our assumption, and each $\mathfrak{a}_i^\vee$ is a reflexive R-module of rank 1 (cf. [6, Corollary 1.2, Proposition 1.9]), we conclude $\mathfrak{a}_i^\vee \cong R$; hence $\left(\bigoplus_{i=1}^r \mathfrak{a}_i\right)^{\vee\vee} \cong R^r$. Moreover, from the first part of the proof of Theorem 3.1, we know that $T = \bigoplus_{i=1}^r R/\mathfrak{a}_i$, and proceeding as in the second part of that proof, we obtain $\bar{M} \cong \bigoplus_{i=1}^r \mathfrak{a}_i$. □

Example 3.3. Let s, t be two variables over the field k, and let $A = k[s^4, s^3t, st^3, t^4]$ be the k-algebra, which is a classical example of a non-Cohen–Macaulay ring. We also set $R = k[s^4, t^4]$ and $\mathfrak{m} = (s^4, t^4) \cdot R$. The inclusion of R into A converts A into an R-module generated by $1, s^3t, st^3, s^2t^2$, and we have $A = s^2t^2\mathfrak{m} \oplus F$, where F is the R-module generated by $1, s^3t, st^3$. As $\mathfrak{m}^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(\mathfrak{m}, R)) \cong R$, we readily conclude that A is an ideal R-module and, in this case, we have $r(A) = 3$ and $H^1_{\mathfrak{a}}(A) = k$.

4 Torsion-free Modules of Projective Dimension One

Let R be a Noetherian local domain and let $F \xrightarrow{\pi} M$ be a minimal epimorphism of a torsion-free finitely generated and non-free R-module M, where F is a free R-module, which is completely determined by M up to an isomorphism. Dualizing the short exact sequence

$$0 \to N \xrightarrow{\varphi} F \xrightarrow{\pi} M \to 0, \tag{5}$$

we obtain $0 \to M^\vee \xrightarrow{\pi^\vee} F^\vee \xrightarrow{\varphi^\vee} N^\vee$. The 'codual module' of M is defined to be the R-module $\mathrm{cd}\, M = \mathrm{Im}\,\varphi^\vee$, which is also a torsion-free finitely generated and non-free R-module, as follows from the following exact sequence by virtue of the assumptions on M:

$$0 \to M^\vee \xrightarrow{\pi^\vee} F^\vee \xrightarrow{\varphi^\vee} \mathrm{cd}\, M \to 0. \tag{6}$$

Proposition 4.1. *Let* R *be a Noetherian local domain and let* M *be a torsion-free finitely generated and non-free* R*-module. We have:*

(i) $\mathrm{cd}\,(\mathrm{cd}\, M) = M$.
(ii) $\mathfrak{a}(M) = \mathfrak{a}(\mathrm{cd}\, M)$.

Proof. (i) Dualizing (6), we obtain $0 \to (\mathrm{cd}\, M)^\vee \xrightarrow{\varphi^{\vee\vee}} F \xrightarrow{\pi^{\vee\vee}} M^{\vee\vee}$. From this sequence and (5), taking the natural inclusion $M \subseteq M^{\vee\vee}$ into account, we obtain an injection $N \subseteq (\mathrm{cd}\, M)^\vee$. As N and $(\mathrm{cd}\, M)^\vee$ have the same rank (equal to rank F − rank M), there exists $r \in R$ such that $r \cdot (\operatorname{cd} M)^\vee \subseteq N$, and r must be invertible because M is torsion-free.

(ii) If $(\operatorname{cd} M)_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module, then $(M^\vee)_\mathfrak{p}$ (and hence $M_\mathfrak{p}$) is also free, as follows from (6). Hence, $\mathfrak{a}(M) \subseteq \mathfrak{a}(\operatorname{cd} M)$, and we can conclude by virtue of (i). □

Remark 4.2. More formally, we can state that 'taking the codual module' is a dualizing functor in the sense of [3, §21.1].

Proposition 4.3. *Let R be a regular Noetherian local domain of dimension $d \geq 2$. Then every torsion-free finitely generated non-free R-module M of projective dimension one is completely determined by $r(\operatorname{cd} M)$ and $H^1_\mathfrak{a}(\operatorname{cd} M)$.*

Proof. By virtue of Theorem 3.1, we only need to prove that 'taking the codual module' is an equivalence between the category of torsion-free finitely generated non-free R-modules of projective dimension one and that of ideal torsion-free finitely generated non-free R-modules. Let $0 \to F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\pi} M \to 0$ be a minimal free resolution of M. From [7, Proposition 5.1(e)], we know that $\operatorname{cd} M = \operatorname{Im} \varphi^\vee$ is an ideal module. Conversely, if M is an ideal torsion-free finitely generated non-free R-module, then again from the reference above, it follows that $\operatorname{cd} M$ is of projective dimension one. □

5 Reflexive Modules over a Regular Ring Which Are k-th Syzygies

Theorem 5.1. *Let R be a regular Noetherian local ring. A reflexive finitely generated R-module M is a k-th syzygy if and only if $H^i_\mathfrak{a}(M) = 0$ for every $i = 0, \ldots, k-1$.*

Proof. Let M be a reflexive finitely generated R-module. Because of reflexivity (e.g., see [2, 1.4.19(c)]), we can assume $k \geq 2$. Let

$$0 \to F_j \xrightarrow{\varphi_{j-1}} F_{j-1} \xrightarrow{\varphi_{j-2}} \cdots \xrightarrow{\varphi_1} F_1 \xrightarrow{\varphi_0} F_0 \to M^\vee \to 0 \tag{7}$$

be a minimal free resolution of $M^\vee$. If M is a k-th syzygy, then by dualizing (7), since M is reflexive, we obtain an exact sequence (see the proof of [4, Lemma 5.1])

$$0 \to M \to F_0^\vee \xrightarrow{\varphi_0^\vee} F_1^\vee \xrightarrow{\varphi_1^\vee} \cdots \xrightarrow{\varphi_{k-3}^\vee} F_{k-2}^\vee \xrightarrow{\varphi_{k-2}^\vee} F_{k-1}^\vee \to N \to 0 \tag{8}$$

with $N = \operatorname{coker} \varphi_{k-2}^\vee$, which breaks into the following k short exact sequences:

$$\begin{aligned} S_0: &\quad 0 \to M \to F_0^\vee \to \operatorname{Im} \varphi_0^\vee \to 0, \\ S_i: &\quad 0 \to \operatorname{Im} \varphi_{i-1}^\vee \to F_i^\vee \to \operatorname{Im} \varphi_i^\vee \to 0 \quad (i = 1, \ldots, k-2), \\ S_{k-1}: &\quad 0 \to \operatorname{Im} \varphi_{k-2}^\vee \to F_{k-1}^\vee \to N \to 0. \end{aligned}$$

Let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime ideal with height $\mathfrak{p} \leq k$. As M is a k-th syzygy, it satisfies Serre's S_k condition (e.g., see [4, Theorem 3.8(b)]), and accordingly, we have depth $M \geq$ height $\mathfrak{p}$. From the Auslander–Buchsbaum formula [2, Theorem 1.3.3], we thus conclude that proj dim $M_\mathfrak{p} = 0$, or in other words, $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module. Therefore, $\mathfrak{p} \in V(\mathfrak{a})$ implies height $\mathfrak{p} \geq k+1$, so that height $\mathfrak{a} \geq k+1$ and hence $d = \dim R \geq k+1$. Recalling the cohomological interpretation of depth, from the sequence S_0 above, we obtain $H_{\mathfrak{a}}^{p-1}(\operatorname{Im}\varphi_0^\vee) \cong H_{\mathfrak{a}}^p(M)$ for $p = 1,\dots,d-1$, and similarly, from S_i for $i = 1,\dots,k-2$, we have $H_{\mathfrak{a}}^{p-1}(\operatorname{Im}\varphi_i^\vee) \cong H_{\mathfrak{a}}^p(\operatorname{Im}\varphi_{i-1}^\vee)$ for $p = 1,\dots,d-1$. In particular, for every $p = 1,\dots,k-1$, we have

$$\begin{aligned} H_{\mathfrak{a}}^p(M) &\cong H_{\mathfrak{a}}^{p-1}(\operatorname{Im}\varphi_0^\vee) \cong H_{\mathfrak{a}}^{p-2}(\operatorname{Im}\varphi_1^\vee) \cong \cdots \\ &\cong H_{\mathfrak{a}}^{p-j}(\operatorname{Im}\varphi_{j-1}^\vee) \cong \cdots \cong H_{\mathfrak{a}}^0(\operatorname{Im}\varphi_{p-1}^\vee). \end{aligned}$$

Moreover, we have $H_{\mathfrak{a}}^0(M) = H_{\mathfrak{a}}^0(\operatorname{Im}\varphi_0^\vee) = \cdots = H_{\mathfrak{a}}^0(\varphi_{k-2}^\vee) = 0$ as all these modules are torsion-free. Hence, $H_{\mathfrak{a}}^p(M) = 0$ for $p = 0,\dots,k-1$, and we can conclude the first part of the proof.

Conversely, assume $H_{\mathfrak{a}}^i(M) = 0$ for $i = 0,\dots,k-1$. Again by virtue of [4, Theorem 3.8], in order to prove that M is a k-th syzygy, we only need to state that M satisfies Serre's S_k condition. Given $\mathfrak{p} \in \operatorname{Spec} R$, we are led to distinguish two cases: If $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module, then $\operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}}$ and the S_k condition certainly holds. If $M_{\mathfrak{p}}$ is not free as an $R_{\mathfrak{p}}$-module, then $\mathfrak{p} \in V(\mathfrak{a})$ and we have $\operatorname{depth} M_{\mathfrak{p}} \geq \operatorname{depth}_{\mathfrak{a}} M = \min_{\mathfrak{p}\in V(\mathfrak{a})} \operatorname{depth} M_{\mathfrak{p}} \geq k$, so that M satisfies the S_k condition again. The theorem is thus established. □

Corollary 5.2. *With the same assumptions as in Theorem 5.1, the module M satisfies the S_k condition if and only if* $\operatorname{depth}_{\mathfrak{a}} M \geq k$.

Remark 5.3. If M is a k-th syzygy strictly, i.e., if M is a k-th syzygy but not a $(k+1)$-th syzygy, then the torsion submodule T of the module N in (8) does not vanish, and localizing (8) at any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{height} \mathfrak{p} \leq k$, we obtain an exact sequence of free $R_{\mathfrak{p}}$-modules; hence $T_{\mathfrak{p}} = 0$, and accordingly, $\operatorname{support} T \subseteq V(\mathfrak{a})$. Taking cohomology with supports in $\mathfrak{a}$ in the sequence S_{k-1}, we have $H_{\mathfrak{a}}^k(M) \cong H_{\mathfrak{a}}^0(N) = T$.

Acknowledgement. We would like to thank Professor Peter Schenzel for his valuable comments and suggestions in preparing the manuscript.

References

[1] N. Bourbaki, *Elements of Mathematics, Commutative Algebra*, Hermann, Paris, 1972.
[2] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.
[3] D. Eisenbud, *Commutative Algebra. With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.
[4] Gr. Evans, Ph. Griffiths, *Syzygies*, Cambridge University Press, Cambridge, 1985.
[5] A. Grothendieck, J. Dieudonné, *EGA IV. Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES, 20, 24, 28, 32, IHES, Paris, 1964, 1965, 1966, 1967.
[6] R. Hartshorne, Stable reflexive sheaves, *Math. Ann.* **254** (1980) 121–176.
[7] A. Simis, B. Ulrich, W.V. Vasconcelos, Rees algebras of modules, *Proc. London Math. Soc.* **87** (2003) 610–646.