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# *k*th Power of a Partial Sum

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## *k*th Power of a Partial Sum

Recently the following result appeared [1, Theorem 2.1].

**Theorem 1.** *For any finite sequence of positive numbers* $\left(a_j\right)_{j=1}^n$ *whose partial sums are* $\left(S_j\right)_{j=1}^n$ *we have* $\sum_{j=1}^{n}\left(a_j^2 + 2a_j S_{j-1}\right) = S_n^2$.

Here we prove the following extension of the previous result.

**Theorem 2.** *For any finite sequence of positive numbers* $\left(a_j\right)_{j=1}^n$ *whose partial sums are* $\left(S_j\right)_{j=1}^n$ *and for all integers* $k \geq 1$ *we have*

$$\sum_{j=1}^{n}\sum_{m=1}^{k}\binom{k}{m} S_{j-1}^{k-m} a_j^m = S_n^k.$$

*Proof.* The proof follows by telescoping. Since $S_j = S_{j-1} + a_j$, we have

$$S_j^k - S_{j-1}^k = \left(S_{j-1} + a_j\right)^k - S_{j-1}^k = \sum_{m=1}^{k}\binom{k}{m} S_{j-1}^{k-m} a_j^m.$$

Since $S_0 = 0$, we have $\sum_{j=1}^{n}\sum_{m=1}^{k}\binom{k}{m} S_{j-1}^{k-m} a_j^m = \sum_{j=1}^{n}\left(S_j^k - S_{j-1}^k\right) = S_n^k$. ■

**Corollary.** *For* $k = 3$ *we have* $\sum_{j=1}^{n}\left(a_j^3 + 3a_j^2 S_{j-1} + 3a_j S_{j-1}^2\right) = S_n^3$.

**Example.** Let $a_j = F_{2j-1}$. It is well known that $S_n = \sum_{j=1}^{n} F_{2j-1} = F_{2n}$. It follows that $a_j^3 + 3a_j^2 S_{j-1} + 3a_j S_{j-1}^2 = F_{2j-1}^3 + 3F_{2j-1}F_{2j-2}F_{2j}$, which implies

$$\sum_{j=1}^{n}\left(F_{2j-1}^3 + 3F_{2j-1}F_{2j-2}F_{2j}\right) = F_{2n}^3.$$

Many other identities may be found and proved using Theorem 2.


REFERENCE

[1] Treeby, D. (2016). Further physical derivations of Fibonacci summations. *Fibonacci Quart.* 54(4): 327–334.



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