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RESEARCH ARTICLE

A generalisation of the Rayleigh distribution with applications in wireless fading channels

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ABSTRACT

The signal received in a mobile radio environment exhibits rapid signal level fluctuations which are generally Rayleigh-distributed. These result from interference by multiple scattered radio paths between the base station and the mobile receptor. Fading-shadowing effects in wireless channels are usually modelled by means of the Rayleigh–Lognormal distribution (RL), which has a complicated integral form. The K-distribution (K) is similar to RL but it has a simpler form and its probability density function admits a closed form; however, due to the Bessel function, parameter estimates are not direct. Another possible approach is that of the Rayleigh-inverse Gaussian distribution (RIG). In this paper, an alternative is presented, a generalisation of the Rayleigh distribution which is simpler than the RL, K and RIG distributions, and thus more suitable for the analysis and design of contemporary wireless communication systems. Closed-form expressions for the bit error rate (BER) for differential phase-shift keying (DPSK) and minimum shift keying (MSK) modulations with the proposed distribution are obtained. Theoretical results based on statistically well-founded distance measurements validate the new distribution for the cases analysed. Copyright © 2011 John Wiley & Sons, Ltd.

KEYWORDS
wireless fading; Rayleigh distribution; K-distribution

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1. INTRODUCTION

Fading-shadowing effects in wireless channels are usually modelled using the Rayleigh–Lognormal distribution (RL) [1,2], which has a complicated integral form. The K-distribution [3], which results from a combination of the Rayleigh distribution (for the fading) and the gamma distribution (for the shadowing), is similar to the RL distribution but it has a simpler form and its probability density function (pdf) admits a closed form but, due to the Bessel function, the estimates of the parameters are not direct. An alternative, based on the Lognormal distribution and other than the RL distribution, is the Rayleigh-inverse Gaussian distribution (RIG) [4] with the same restriction as the above. New fading models and their corresponding pdf have been proposed recently (see Ref. [5–7]) and have been validated from theoretical and experimental results. For these models, second order statistics, mainly for level crossing rate (LCR) and average fade duration (AFD) have been obtained in a closed form and the physical models are also well-established [8–11]. Currently, the RL distribution is extensively applied to fading modelling, especially for the long-term signal variation, whereas the short-term signal variation can be described by the Rayleigh distribution (and others). Therefore, the goal of finding a new analytic distribution to better approximate the RL distribution, and which includes the Rayleigh distribution as a particular case, fully justifies the new distribution that is discussed in this paper. Such a distribution can be applied to model both long and short-term signal variations in a wireless fading channel.

The new distribution has two main advantages: it has a simple mathematical expression and it subsumes the Rayleigh distribution. Parameter estimation can be accomplished either by matching the first and the second order moments or by maximum likelihood estimation. In both cases, the parameters are calculated after solving a system of equations. Due to the simple mathematical form of the new distribution, closed-form expressions for the bit error rate (BER) for DPSK and MSK modulations can be easily computed and are provided in the paper.

The new distribution can be simulated in terms of phasors, which makes it suitable for the physical modelling of the fading. The fundamental statistical parameters of the new distribution, such as the median, the variance and higher order moments, as well as their estimation by maximum likelihood procedures, are also examined in this paper. Theoretical results based on statistically well founded measurements validate the new distribution for the cases analysed. Its application to the practical modelling of fading-shadowing effects in wireless channels is also discussed.

In order to make the paper self-contained, we start by recalling the expressions for the PDFs of the Rayleigh-Lognormal distribution, the K-distribution and the Rayleigh-inverse Gaussian distribution:

$$f_X(x) = \int_0^\infty \frac{x}{y}\exp\left(\frac{-x^2}{2y}\right)\frac{1}{y\sqrt{2\pi\sigma^2}}\exp\left[\frac{-(\log y - \mu)^2}{2\sigma^2}\right]\mathrm{d}y$$

$$f_X(x) = \frac{2}{c\Gamma(d+1)}\left(\frac{x}{2c}\right)^{d+1}K_d\left(\frac{x}{c}\right)$$

$$f_X(x) = 2\sqrt{\frac{\phi}{2\pi}}x\mathrm{e}^{\phi/\lambda}\left[\frac{(x^2+\phi)\lambda^2}{\phi}\right]^{-3/4} \times K_{-3/2}\left(\sqrt{\frac{x^2\phi+\phi^2}{\lambda^2}}\right)$$

respectively. Here $x \geq 0, y \geq 0, c > 0, d > -1, \phi > 0, \lambda > 0$, $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$ and $K_\nu(z)$ denotes the modified Bessel function of the second kind of order ν and argument z.

The organisation of this paper is as follows. First, the distribution and relevant statistical parameters are derived. Section 3 compares the new distribution to others that are widely applied to describe the statistics of mobile radio signals, and discusses specific parameters related to wireless communication channels. Section 4 is focussed on generating a random variate for the proposed density distribution, and Section 5 introduces the generalised Rayleigh phasor and presents some simulation plots. Finally, some conclusions are drawn in Section 6.

2. THE GENERALISED RAYLEIGH DISTRIBUTION

Rayleigh fading is a reasonable model when there are many objects in the environment that scatter the radio signal before it arrives at the receiver. The central limit theorem holds that if there is sufficient scattering, the channel impulse response will be well modelled as a Gaussian process irrespective of the distribution of the individual components. If there is no dominant component to the scattering, then such a process will have zero mean and phase, uniformly distributed between 0 and 2π radians. The envelope of the channel response will therefore be Rayleigh distributed, with the pdf

$$g(r;\alpha) = \frac{r}{\alpha}\exp\left\{-\frac{r^2}{2\alpha}\right\}, \quad r \geq 0, \alpha > 0, \tag{1}$$

where $\mathcal{R}(R^2) = 2a\mathrm{IE}(R^2) = 2\alpha$ is the expected value of R^2. In this case we write $R \sim R(\alpha)$.

Consider now the random variable $R|\mathcal{Z} \sim R(\alpha/\mathcal{Z})$, where $\alpha > 0$ and $\mathcal{Z}$ is a random variable taking values in the set $\{1, 2, \ldots\}$. If the random variable $\mathcal{Z}$ is allowed to follow a positive geometric distribution with the probability mass function (pmf)

$$\begin{aligned} p(z) &= \Pr(\mathcal{Z} = z) \\ &= \left(\frac{a}{a+1}\right)^{z-1}\frac{1}{a+1}, \quad a > 0, z = 1, 2, \cdots, \end{aligned} \tag{2}$$

the pdf of R, $f(r)$, can be obtained from the conditional Rayleigh pdf by using the pmf of $\mathcal{Z}$, and is

$$f(r;\alpha,a) = \frac{r(1+a)}{\alpha}\frac{\exp\{r^2/(2\alpha)\}}{\left[(1+a)\exp\{r^2/(2\alpha)\}-a\right]^2}, \quad r \geq 0, \alpha > 0, a > 0 \tag{3}$$

from which (1) is obtained from (3) when a tends to 0. Henceforth, this distribution is represented as GR (α, a).

A simple interpretation of the new model is given in the following conjecture. Suppose that $\{R_i\}_{i=1}^{Z}$ are independent and identically distributed random variables with density $R(\alpha)$ and that $\mathcal{Z}$ is random and independent of the $\{R_i\}_i$, and follows the pmf (2). It is then simple to show that the marginal pdf of $R = \min\{R_1, R_2, \cdots, R_Z\}$ is given by (3). For example, consider the variation of the amplitude of the diffuse component, which occurs due to the presence of an unknown number of scattering points $\mathcal{Z}$ of the same kind. The R_i can then represent the amplitude of different signals after reflection and scattering. The new model can then be used to estimate the minimum value of the received signal.

By computing the derivative of (3) and equating to zero, we obtain, after some simple algebra, the following equation:

$$h(r) = (\alpha - r^2)(1+a)\exp\left\{\frac{r^2}{2\alpha}\right\} - a(\alpha + r^2) = 0 \tag{4}$$

Now, because $h'(r) < 0$ for $r \in (0, \infty)$, we conclude that the new distribution is unimodal with a modal value satisfying Equation (4).

The following result shows that the pdf (3) can be represented as an infinite mixture of the classical Rayleigh pdf.

Proposition 1 *The GR(α,a) distribution can be rewritten as an infinite mixture of R(α/(1 + i)) in the form*

$$f(r;\alpha,a) = \frac{1}{1+a}\sum_{i=0}^{\infty}\left(\frac{a}{1+a}\right)^i g\left(r;\frac{\alpha}{1+i}\right) \tag{5}$$

Proof By expressing (3) as

$$f(r;\alpha,a)=\frac{r}{\alpha(1+a)}\frac{\exp\{-r^2/(2\alpha)\}}{[1-(a/(1+a))\exp\{-r^2/(2\alpha)\}]^2},$$
$$r\geq 0, \alpha>0, a>0 \qquad (6)$$

and using the series representation

$$(1-z)^{-k}=\sum_{i=0}^{\infty}\frac{\Gamma(k+i)}{\Gamma(i+1)\Gamma(k)}z^i, \quad |z|<1, k>0 \qquad (7)$$

we obtain the result after simple algebra.

The survivor and hazard functions of the random variable $R\sim \mathrm{GR}(\alpha,a)$ are given by

$$\bar{F}(r;\alpha,a)=\Pr(R>r)=\frac{1}{(a+1)\exp\{r^2/(2\alpha\}-a}$$
$$h(r;\alpha,a)=\frac{f(r)}{\bar{F}(r)}=\frac{r(1+a)}{\alpha}\frac{\exp\{(r^2/(2\alpha)\}}{(a+1)\exp\{r^2/(2\alpha\}-a} \qquad (8)$$

for $r\geq 0$, $\alpha>0$, $a>0$, respectively.

From (8) it is simple to derive the quantile $\gamma(r_\gamma)$, which is given by

$$r_\gamma=\sqrt{2\alpha\log\left[\frac{1+a\gamma}{\gamma(a+1)}\right]}$$

In particular, the median is $r_{0.5}=\sqrt{2\alpha\log[(2+a)/(1+a)]}$. This value is always lower than the median of the Rayleigh distribution, given by $\sqrt{2\alpha\log 2}$.

By conditioning, and starting from the s-th moment around the origin of the Rayleigh distribution, it is a simple exercise to show that the first and second moments around the origin of distribution (3) are given by,

$$\mathbb{E}(R;\alpha,a)=\frac{1}{a}\sqrt{\frac{\alpha\pi}{2}}Li_{1/2}\left(\frac{a}{1+a}\right)$$
$$\mathbb{E}(R^2;\alpha,a)=\frac{2\alpha}{a}\log(1+a) \qquad (9)$$

where $Li_n(z)=\sum_{k=1}^{\infty}z^k/k^n$ represents Euler's polylogarithm function (see Ref. [12]) which is readily available in standard software such as Mathematica [13].

Because $\log(1+a)/a<1$ the expected value of R^2 under (3) is smaller than that under (1).The variance is

$$\mathrm{Var}(R;\alpha,a)=\frac{\alpha}{a}\left\{\log(1+a)^2-\frac{\pi}{2a}\left(Li_{1/2}\left(\frac{a}{1+a}\right)\right)^2\right\} \qquad (10)$$

Expressions (9) and (10) can be used to estimate the parameters of the model by the method of moments. Moreover, for a sample $(r_1,\ldots,r_n)$ taken from model (3) and taking into account that the log-likelihood function is $\ell=\sum_{i=1}^{n}\log f(r_i)$ we can straightforwardly obtain the maximum likelihood estimates for distribution (3) by solving the system of equations

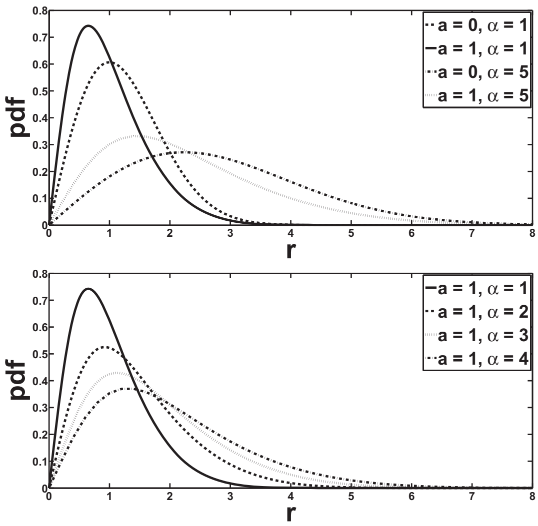

Figure 1. Illustration of the shape of the GR distribution for a set of different α,a parameter values (top) and the GR distribution plotted for $a=1$ and α as a parameter (bottom).

$$\frac{\partial\ell}{\partial\alpha}=n(2\alpha+s^2+\bar{r}^2)+2(1+a)\sum_{i=1}^{n}\frac{r_i^2\exp\{r_i^2/(2\alpha)\}}{(1+a)\exp\{r_i^2/(2\alpha)\}-a}=0$$
$$\frac{\partial\ell}{\partial a}=n-2(1+a)\sum_{i=1}^{n}\frac{\exp\{r_i^2/(2\alpha)\}-1}{(1+a)\exp\{r_i^2/(2\alpha)\}-a}=0$$

where $\bar{r}$ and s^2 are the sample mean and the variance, respectively.

The shape of the GR(α,a) distribution is shown in Figure 1, where the dependence of the shape on the parameters α and a can be appreciated.

3. COMPARISON WITH THE RL, K AND RIG DISTRIBUTIONS

In this section, some examples are given to show how the new generalised Rayleigh distribution works. It is well known that the distance or relative information between two probability distributions can be studied by using the Kullback–Leibler divergence measure [14], which is defined as follows. Let f and g be probability densities on IR^n such that f is absolutely continuous with respect to g (that is, $g(x)=0$ implies $f(x)=0$), then the relative information or Kullback–Leibler divergence, $D_{\mathrm{KL}}(f||g)$ of f with

Table I. Parameter estimates for $c = 2$.

d	-0.20	0.30	0.90
μ	1.216	1.965	2.460
σ	1.090	0.867	0.721
ϕ	1.279	3.034	5.570
λ	6.400	10.400	15.200
a	13.787	5.142	2.715
α	32.757	29.462	31.450

respect to g is

$$D_{\mathrm{KL}}(f||g) = \int_0^{\infty} f(x)\log\left\{\frac{f(x)}{g(x)}\right\}\mathrm{d}x \qquad (11)$$

with the convention that $0/0 = 1$. When f is not absolutely continuous with respect to g we define $D_{\mathrm{KL}}(f||g) = \infty$. One of the disadvantages of (11) is that the Kullback–Leibler divergence is not symmetric and therefore it is not a genuine distance metric. To overcome this problem, we use the Jensen–Shannon divergence [15] given by

$$D_{\mathrm{JSD}}(f||g) = \frac{1}{2}(D_{\mathrm{KL}}(f||m) + D_{\mathrm{KL}}(g||m))$$

where $m = (f + g)/2$. The integrated squared error (ISE) [16] is given by

$$D_{\mathrm{ISE}}(f||g) = \int_0^{\infty} (f(x) - g(x))^2\mathrm{d}x$$

To compare the RL(μ,σ), K(c,d) and RIG(ϕ,λ) distributions and the new model proposed in this paper, we used the values of $c = 2$ and $d = -0.2$, 0.3 and 0.9, as considered by Abdi and Kaveh [3]. The equivalent parameters of the above distributions were estimated by the method of moments. The set of parameters related to the distributions is shown in Table I. The pdf's of the above distributions are shown in Figure 2. Table II includes the Jensen–Shannon divergence for the GR, K and RIG distributions for the case $c = 2$. It can be seen that, except for the case $d = 0.90$, the GR distribution provides the lowest distance with respect to the RL distribution. Similar results are obtained for other combination of parameters corresponding to typical values of signal envelope fading (-40 to 15 dB).

In order to compare the fit of the distributions, the pdf of GR(α,a) and the pdf of the RL, for various parameter values, are illustrated in Figure 3. The parameters for the GR(α,a) distribution are estimated by the method of moments. The figure also illustrates the excellent fit between the RL distribution and the GR(α,a) distribution.

Note that the new distribution is more suitable than the K distribution and the RIG distribution, which both involve the Bessel function and therefore make analysis more complicated. The existence of a simpler form enables us to estimate specific channel parameters, such as BER performance (see next section) and diversity effects, as well as the outage probability for co-channel interferences. The conclusion that can be drawn from these results is that the proposed GR(α,a) distribution in general provides better results than do other, commonly used distributions and therefore it can be used to efficiently characterise wireless channel fading.

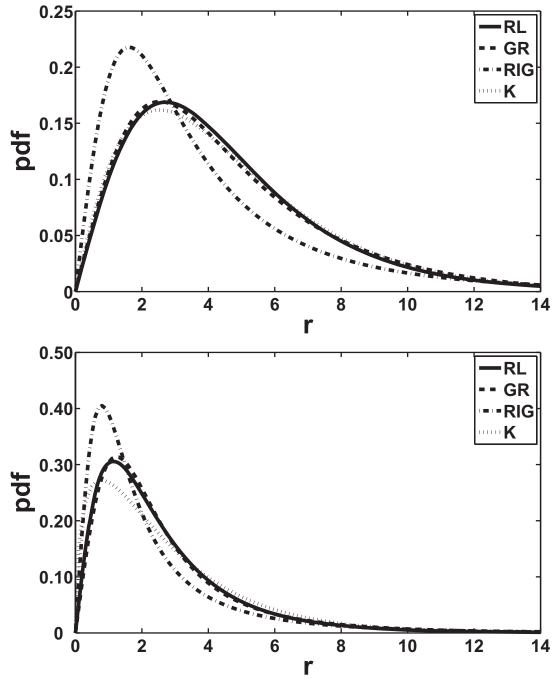

Figure 2. Illustration of the shape of the distributions (top: $c = 2$, $d = 0.9$, bottom: $c = 2$, $d = -0.2$).

Table II. Numerical values of JSD and ISE measures for K, RIG and GR distributions.

d	-0.20	0.30	0.90
JSD			
K	5.52×10^{-3}	1.36×10^{-3}	$4.67 \times 10^{-}$
RIG	1.32×10^{-2}	1.67×10^{-2}	1.92×10^{-2}
GR	5.75×10^{-4}	5.57×10^{-4}	$6.33 \times 10^{-}$
ISE			
K	6.20×10^{-3}	1.07×10^{-3}	2.83×10^{-4}
RIG	1.94×10^{-2}	1.73×10^{-2}	1.54×10^{-2}
GR	4.32×10^{-4}	1.85×10^{-4}	1.97×10^{-4}

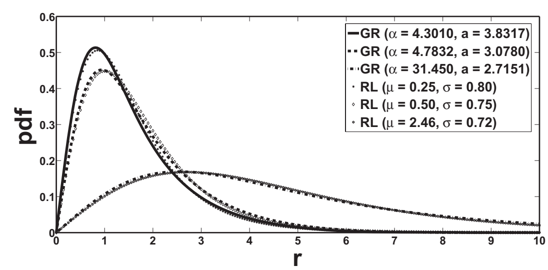

Figure 3. Shapes of probability density functions of GR(α,a) and RL for various parameter values.

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3.1. Some measures of interest

In this section, expressions for the Rényi entropy, the average BER for the DPSK and MSK signals transmitted over the GR(α,a) fading channel are derived.

It is well known that the Rényi entropy of the GR(α,a) is defined as $I_R(\gamma) = (1/(1-\gamma))\log\int_0^\infty f^\gamma(r;\alpha,a)\mathrm{d}r$, for $\gamma > 0$ and $\gamma \neq 1$. By using (6) and (7) we obtain

$$\int_0^\infty f^\gamma(r;\alpha,a)\mathrm{d}r$$

$$= \int_0^\infty \left[\frac{r}{\alpha(1+a)}\right]^\gamma \mathrm{e}^{-\frac{r^2\gamma}{2\alpha}}\left[1-\frac{a}{1+a}\mathrm{e}^{-\frac{r^2\gamma}{2\alpha}}\right]^{-2\gamma}\mathrm{d}r$$

$$= \frac{1}{[\alpha(1+a)]^\gamma}\sum_{i=0}^\infty \frac{\Gamma(2\gamma+i)}{i!\Gamma(2\gamma)}\left(\frac{a}{1+a}\right)^i \int_0^\infty r^\gamma \mathrm{e}^{-\frac{r^2(i+\gamma)}{2\alpha}}\mathrm{d}r$$

The last term can be fitted from the Nakagami distribution [10] and, it follows after some simple algebra that the Rényi entropy is given by

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\left(\frac{\alpha}{2}\right)^{\frac{1-\gamma}{2}}\frac{\Gamma((\gamma+1)/2)}{(1+a)^\gamma}\sum_{i=0}^\infty \frac{\Gamma(2\gamma+i)}{i!\Gamma(2\gamma)}\left(\frac{a}{1+a}\right)^i\left(\frac{1}{i+\gamma}\right)^{\frac{\gamma+1}{2}}\right\}$$

while the Shannon entropy is

$$h(g(r;\alpha,a)) = \frac{1}{1+a}\left\{\log[\alpha(1+a)] + \sum_{i=0}^\infty\left(\frac{a}{1+a}\right)^i \times\left[\frac{\gamma+\log[(1+i)/(2\alpha)]}{2} + \frac{1}{1+i} + 2(1+i)\left(\frac{a\,{}_2F_1(1,2+i,3+i,(a/(1+a)))}{(1+a)(2+3i+i^2)} - \frac{\log(1+a)}{1+i}\right)\right]\right\}$$

where ${}_2F_1(m,b,c,s) = \sum_{k=0}^\infty (m)_k(b)_k/(c)_k s^k/k!$ is the hypergeometric function and $(l)_k$ is Pochhammer's symbol. See Johnson *et al.* [17] for details.

Some measures of special interest, such as the amount of fading and average BER of DPSK and MSK for the generalised Rayleigh distribution proposed here and which are useful in wireless fading channels, can be obtained under closed form, as shown below.

For a single-input–single-output (SISO) system, the amount of fading, $\mathrm{AF} = \mathrm{Var}(R^2)/[E(R^2)]^2$, for the generalised Rayleigh distribution discussed here is given by

$$\mathrm{AF}_{\mathrm{GR}} = 2a\frac{Li_2(a/(1+a))}{\log^2(1+a)} - 1$$

which varies between 1 and ∞ for $a > 0$.

Now, using the BER expression for DPSK and MSK for the Rayleigh distribution in Ref. [18] we can obtain, by conditioning, the corresponding average BER of DPSK and MSK for the generalised Rayleigh distribution. These are given by

$$\bar{P}_{b,\mathrm{DPSK}} = \frac{1}{2} - \frac{\alpha\gamma}{(1+a)(1+2\alpha\gamma)} \times {}_2F_1\left(1, 1+2\alpha\gamma; 2+2\alpha\gamma; \frac{a}{1+a}\right)$$

$$\bar{P}_{b,\mathrm{MSK}} = \frac{1}{2} - \frac{1}{1+a}\sqrt{\frac{\gamma\alpha}{2}}\,\Phi\left(\frac{a}{1+a}, \frac{1}{2}, 1+2\alpha\gamma\right)$$

respectively. Here $\gamma = E_B/N_0$, where E_b is the transmitted energy per bit, N_0 is the noise power spectral density and $\Phi(z,s,a) = \sum_{k=0}^\infty z^k/(k+a)^s$ is the Lerch trascendent function, which also allows the following integral representation, $\Phi(z,s,a) = 1/\Gamma(s)\int_0^\infty t^{s-1}\mathrm{e}^{-at}/(1-z\mathrm{e}^{-t})\mathrm{d}t$.

In Figure 4, the average BERs are plotted for DPSK and MSK for the RL, K, RIG and and GR(α,a) distributions for the three sets of parameter values given in Table I. For the GR(α,a) distribution, the above expressions are applied. For the K distribution, the expressions from Ref. [18] are used. The BERs for the RL and RIG distributions were computed numerically for DPSK and MSK. Setting A is for $d = -0.20$, setting B is for $d = 0.30$ and setting C is for $d = 0.90$. For all cases $c = 2$, as shown in the same table.

The utility of the GR(α,a) distribution for BER prediction in multipath fading-shadow fading channels can be seen from these plots. Note the excellent fit between the proposed GR(α,a) distribution and the common RL distribution for all settings for the DPSK and MSK modulation schemes. It is clear for all the cases illustrated that the GR(α,a) distribution provides a better approximation to the RL than do the other fading distributions which are normally used. Note, too, that for the RL distribution, there is no closed-form expression for the average BER, which must be calculated by numerical integration methods (usually, the Gauss–Hermite method). For the RL distribution, an exact but complicated formula for estimating the BER in the DPSK case is reported in Ref. [19].

Comparison of the analytic expresions for BER estimation for the GR(α,a) distribution and the K distribution reveals a similar level of mathematical complexity. Therefore, the proposed distribution may be efficiently applied to capture fading shadowing aspects of wireless channels.

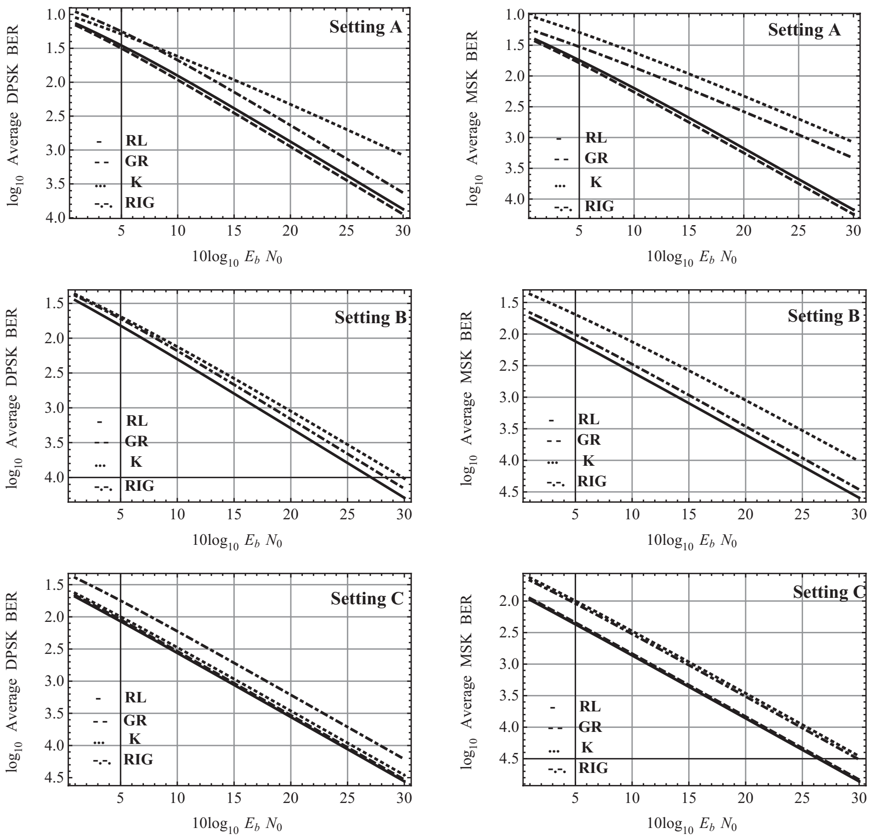

Figure 4. Average BERs of DPSK and MSK for the RL, GR, K and RIG distributions assuming the values for the parameters given in different settings as in Table I.

4. GENERATING A RANDOM VARIATE FOR THE PROPOSED DENSITY DISTRIBUTION

To test the capabilities of the proposed density distribution, it is necessary to generate, at low computational cost, a random variate according to the density distribution. To generate a set of data distributed with the GR(α,a) density, the inverse transform method [20] was applied. This method was also used in Ref. [21] but for the generation of bivariate Rayleigh and Nakagami-m fading envelopes. Although it is known that this method is not necessarily the most efficient for generating random variates, it was selected here on the basis of its straightforward implementation. The inverse transform method is quite simple and can be summarised as follows: let X be a random variable with cumulative probability distribution (CDF) $F_x(x)$. As $F_x(x)$ is a nondecreasing function, the inverse function $F_x^{-1}(y)$ may be defined for any value of y between 0 and 1 as: $F_x^{-1}(y)$ is the smallest x satisfying $F_x(x) \geq y$, that is,

$$F_x^{-1}(y) = \inf\{x : F_x(x) \geq y\}, \quad 0 \leq y \leq 1$$

To apply the inverse transform method, $F_x(x)$ must be available in a form for which the corresponding inverse transform can be found analytically, which fortunately is the case for the GR(α,a) distribution. When this is not possible, a numerical technique, for instance the well-established bisection approach, must be applied, as occurs in Ref. [21], which makes this powerful method less efficient in terms of computational time and the quality of the data obtained.

Hence, if $r \in U(0, 1)$ is uniformly distributed over the interval (0.1), from expression (7), each random variate y distributed with GR(α,a) pdf is explicitly given by,

$$U = \int_{-\infty}^{y} f_X(t)\mathrm{d}t = 1 - \frac{1}{(a+1)\exp(y^2/(2\alpha)) - a}$$

that is,

$$y = \sqrt{2\alpha \log\left(\frac{1}{a+1}\left(a - \frac{1}{r-1}\right)\right)}$$

This methodology works well, as can be seen in Figure 5, where a good fit between the analytic and simulated distributions is observed. The analytic median is provided by $r_{0.5} = \sqrt{2\alpha\log[(2+a)/(1+a)]}$, which, for $\alpha = 0.978$ and $a = 0.5$ gives a value of 1.0 (0 dB) which is very close to the simulated median (1.0019), and provides a relative error

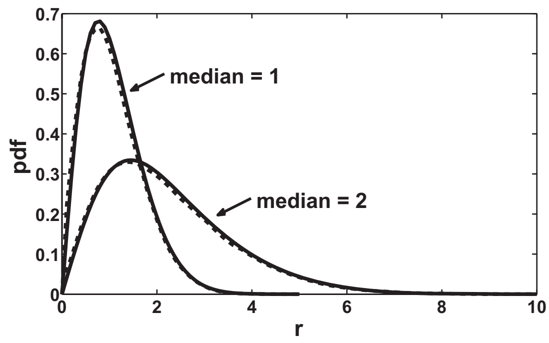

Figure 5. Analytic and simulated density distributions (solid line and dotted line, respectively) for the GR(α,a) distribution with median = 1 (parameters: $\alpha = 0.978$ and $a = 0.5$), and with median = 2 (parameters: $\alpha = 4.932$ and $a = 1.0$).

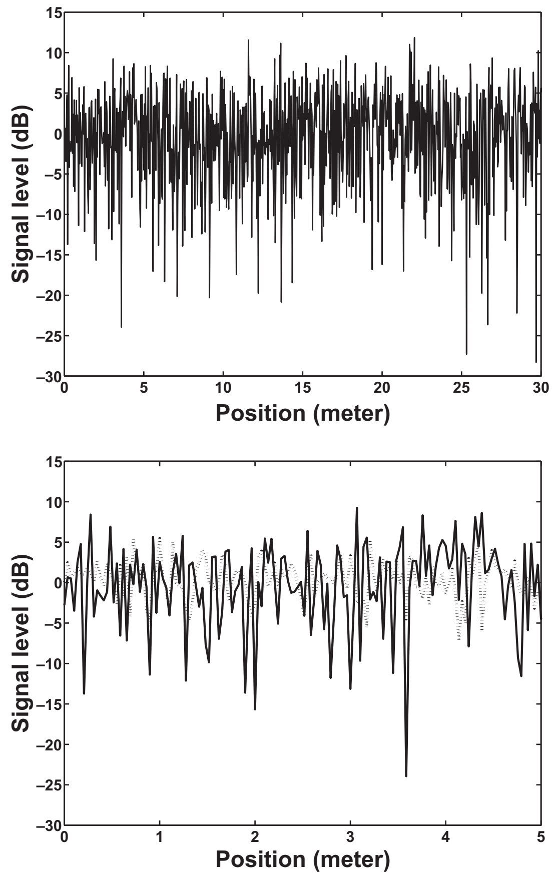

Figure 6. Simulated samples of GR(α,a) data set spaced 0.1 wavelength apart for the 0 dB median value ($\alpha = 0.978$ and $a = 0.5$) (top). Horizontal zoom for the same image with a simulated RL sample data set ($\mu = 0$ dB and $\sigma = 3$ dB) superimposed with dashed line (bottom).

of around 0.2%. For the variance (see expression 10), the results are: 0.3854 for the analytic variance and 0.3848 for the simulated variance, which represents a relative error of around 0.15%. The mean squared error is around 1.11. For the pdf with a median value of 2 (and a corresponding analytic variance of 1.8021), a simulated median of 1.9990 is obtained (a relative error of around 0.05%) and a simulated variance of 1.8118 (a relative error of around 0.53%). The mean squared error for this latter case is around 2.53. A C++ routine was written to implement the above random variate generator. A Monte Carlo simulation to generate $N = 100\,000$ random variates takes less than a second on a Pentium IV, 3 GHz-single core-computer, which makes this approach a practical one for fading channel analysis. For both distributions, a total of 10 000 samples were used and a transmitter frequency of 870 MHz was assumed.

The fading simulation for the GR(α,a) distribution is shown in Figure 6, where a RL distribution is also plotted for the sake of comparison. The RL distribution is simulated as explained in Ref. [22]. As shown in the figures, the fading level is concentrated at the 0 dB level. This, for the case of the Rician fading distribution, results from the dominant line-of-sight (LOS) signal, unlike in the Rayleigh fading distribution. For the GR(α,a) distribution, this must be analysed. The peak-to-peak fading level spans those reported in the relevant references for well-established fading distributions. Note that the deep fading level (−25 dB) is also taken into consideration in the simulated data. Fading level values lower than −25 dB can be easily generated from the parameters α and a.

5. THE GENERALISED RAYLEIGH PHASOR

In this section, we show that the generalised Rayleigh distribution can be obtained in an exact form as a sum of mutually independent Gaussian stochastic processes, as is required in order to account for the simulation of the fading channel, that is, to simulate the signal envelope.

It is known that Rayleigh fading envelopes can be generated from zero-mean complex Gaussian random variables. Other fading distributions (see for instance Ref. [23] for the Nakagami-m case) are obtained in a similar manner after some mathematical considerations. Hence, it is necessary to prove that the phase and the amplitude of a given propagating signal are distributed according to a uniform pdf in the $[0,2\pi]$ interval and to the generalised Rayleigh distribution, respectively.

Following Ref. [24], consider the sum

$$S = \mathrm{R}e^{i\theta} = \sum_{j=1}^{n} A_j e^{i\Phi_j} = (X, Y) = (R\cos\theta, R\sin\theta)$$

where the terms X (the in-phase phasor) and the Y (quadrature phasor) are independent uniformily distributed phasors (UDP) and the A_j are all distributed identically. When n is large and A_j is not correlated with the Φ_j, both X and Y will be distributed normally with mean 0 and variance $(1/2)n = \sum_{j=1}^{n} E(A_j^2)$. Let us now assume $(1/2)n = \sum_{j=1}^{n} E(A_j^2) = \sigma^2/Z$, $Z = \{1, 2, \cdots\}$. Then, the joint distribution of X and Y is

$$p(x, y) = \frac{Z}{2\pi\sigma^2}\exp\left\{-\frac{(x^2+y^2)Z}{2\sigma^2}\right\} \quad (12)$$

Let now that $\mathcal{Z} = \{1, 2, \cdots\}$ is a discrete random variable following the pmf (2). Then, by expressing (12) in polar coordinates we have

$$p(r, \theta|z) = \frac{r\mathcal{Z}}{\alpha\pi}\exp\{-r^2\mathcal{Z}/\alpha\}, \;\; 0 \leq \theta \leq 2\pi, \;\; r \geq 0.$$

It is straightforward to obtain the following marginal distributions:

$$p(\theta|z) \equiv p(\theta) = \int_0^\infty p(r, \theta|z)\mathrm{d}r = \frac{1}{2\pi},$$

$$p(r|z) = \int_0^{2\pi} p(r, \theta|z)\mathrm{d}\theta = \frac{2rz}{\alpha}\exp\{-r^2 z/\alpha\}.$$

Hence, the nonconditional distributions (independent of z) are

$$p(\theta) = \sum_{z=1}^\infty p(\theta|z)\pi(z) = \frac{1}{2\pi}$$

$$p(r) = \sum_{z=1}^\infty p(r|z)\pi(z)$$

$$= \frac{r}{\alpha(1+a)}\frac{\exp\{-r^2/(2\alpha)\}}{1-(a/(1+a))\exp\{-r^2/(2\alpha)\}}$$

It is clear that the phase distribution is uniform, i.e. $p(\theta) = (1/(2\pi)), 0 \leq \theta \leq 2\pi$, and the amplitude distribution is, with $\alpha/2 \equiv \alpha$, as in (3). In consequence, a generalised Rayleigh distribution as in (3) is always a UDP phasor. Otherwise, the generalised Rayleigh distribution can be obtained as a sum of phasors directly from expression (5).

Therefore, once a set of Rayleigh distributions $g(r;\alpha)$ with parameter $\alpha' = \alpha/(1+i)$ is generated from the zero-mean complex Gaussian random variables, the generalised Rayleigh distribution, GR(α,a) can be obtained from the infinite mixture of the Rayleigh distributions. The pdf of the generalised Rayleigh distribution, GR(α,a) obtained from expression (5) is illustrated in Figure 7 (top) in order to show that all that is required is a value of $i = 10$, and a Rayleigh distribution of $g(r;\alpha')$ to accurately fit the desired pdf. In this case, each $g(r;\alpha')$ pdf is evaluated from its analytic expression. In the same figure (bottom), a similar result is obtained but in this case, each $g(r;\alpha')$ is obtained through simulation using the phasors described above.

To comply with the fading channel characteristics for the physical simulation of the channel, the random process (random data set) must be correlated in time but uncorrelated between processes. Therefore, to generate the GR(α,a) data set, we first generated a Rayleigh data set using a random number generator to obtain the random phasor sum in the complex plane, as follows:

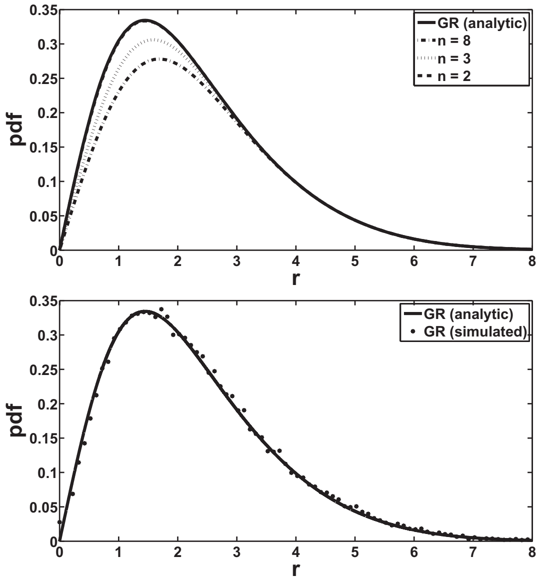

Figure 7. PDF of the Generalised Rayleigh ($\alpha = 4.932606$, $a = 1$), from the sum of analytic Rayleigh distributions (top). Comparison of the generalised Rayleigh ($\alpha = 4.932606$, $a = 1$) magnitude pdf and the simulated data set using expression (5) (bottom).

$$X = \mathrm{Re}(R_{\mathrm{Rayleigh}}) = r\cos\theta = \sum_{j=1}^n A_j\cos\phi_j = \sum_{j=1}^n X_j$$

$$Y = \mathrm{Im}(R_{\mathrm{Rayleigh}}) = r\sin\theta = \sum_{j=1}^n A_j\sin\phi_j = \sum_{j=1}^n Y_j$$

where A_j accounts for the jth random amplitude and ϕ represents the jth random phase. The Rayleigh data set can be obtained, after fitting the desired amplitude as,

$$R_{\mathrm{Rayleigh}} = r\mathrm{e}^{j\theta} = \sum_{j=1}^n A_j \mathrm{e}^{j\theta_j}$$

where r is a random variable for the amplitude of the Rayleigh distribution and θ is the phase of the distribution with a uniform distribution pdf, $p(\theta) = (1/(2\pi)), 0 \leq \theta \leq 2\pi$. In the simulation represented in Figure 7, 15 scattered random phasors and 15 000 samples were used.

The physical model is completed by reformulating the phasors as are the fading models widely applied in previous studies (see Ref. [23]) as follows,

$$x_i(t) = \sum_{j=1}^N A_{ij}\cos(w_{ij}t - \phi_{ij})$$

$$y_i(t) = \sum_{j=1}^N A_{ij}\sin(w_{ij}t - \phi_{ij})$$

In both expressions, A_{ij} accounts for the amplitude, and the ensemble average is $\left\langle \sum_{j=1}^N A_{ij}^2) \right\rangle = 1$ and the phase

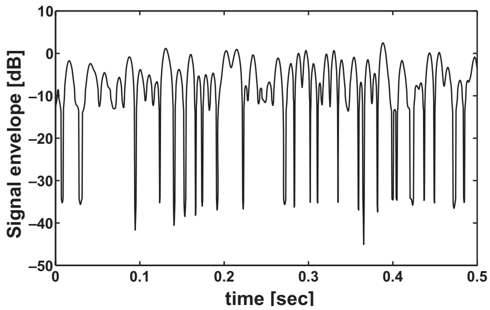

Figure 8. Fading signal for the generalised Rayleigh ($\alpha = 0.4450$, $a = 2$); frequency = 900 MHz, speed = 40 km/h.

θ is now replaced by the term $\theta = w_{ij}t - \phi_{ij}$. The phase ϕ_{ij} is the random phase uniformly distributed in $[-\pi,\pi]$, and $w_{ij} = \beta v \cos(\psi_{ij})$ is the Doppler shift, where v represents the vehicle speed, $\beta = 2\pi/\lambda$ the wave number, λ the wavelength and βv the maximum Doppler shift (in radians per second), respectively. The angle of arrival of the signal is ψ_{ij}, which is also distributed in $[-\pi,\pi]$, while N is the number of sinusoidal waves; if this value is large enough then both $x_i(t)$ and $y_i(t)$ can be considered Gaussian processes. Consequently, the Rayleigh signal envelope is given by $r^2 = x_i^2 + y_i^2$, and $i = 1,2,\ldots$. Then, using (5) the amplitude of the envelope is fitted to the GR(α,a) amplitude.

A simulated fading signal for the GR(α,a) distribution is shown in Figure 8, which corresponds to a signal with a mean amplitude value of −5.8516 dB. Six Rayleigh processes were simulated to obtain the GR(α,a) envelope, and the simulated envelope pdf of the data set fits well with the analytical one (not shown here). The envelope in this case extends to very deep fading levels of around −50 dB which, although rather infrequent have been reported in rapid fading in HF long-distance propagation, see Ref. [23].

Although in this paper, we have not discussed the physical model related to the new distribution, it seems clear from expression (5) that there must exist a straightforward relation between the α parameter and the carrier-to-multipath power appearing in the Rayleigh distribution. However, the relation for the parameter a does not seem so evident, but we believe that it may be related to the power of the scattered waves.

A simulator block implementing the GR(α,a) signal has been developed (coded in C++) and included in a more general mobile radio channel software simulator. The simulator performs the Gaussian processes to account for the Rayleigh simulation to obtain the signal envelope distributed with GR(α,a) pdf. Some typical routines are also included, such as an n-pole Tchebicheff filter block and a simple RF combiner (for the equal gain and maximum-ratio cases).

6. CONCLUSIONS

A new distribution, GR, which can be used to replace the RL distribution, is presented. The proposed distribution also includes the Rayleigh distribution as a particular case. The expressions for the parameter estimation of the new distribution are discussed. The new distribution can be used to replace the K, the RIG or other equivalent RL fading distributions and it is considerably simpler from a mathematical point of view. It is important to note that the new GR distribution can be applied to model both long and short-term signal variations, and hence there is no need to include any other distribution to account for the fading in current wireless channels. Two methods to obtain the simulated envelope are discussed, one based strictly on the pdf of the distribution and the other on a physical model built from the Rayleigh physical model. The latter is exact and hence all the attributes of the GR are retained. A closed-form expression for the BER for DPSK and MSK modulations for the proposed distribution are also derived. An area for future work is the development of analytic models to enable us to understand the physical model related to the new fading distribution, in order to give a physical meaning to the new parameters and to provide exact expressions for the second-order statistics quantities, such as LCR (average LCR) and AFD.

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