The American Mathematical Monthly

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/uamm20

Problems and Solutions

Daniel H. Ullman, Daniel J. Velleman, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

To cite this article: Daniel H. Ullman, Daniel J. Velleman, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou. (2020) Problems and Solutions, The American Mathematical Monthly, 127:10, 944-952, DOI: 10.1080/00029890.2020.1822718

To link to this article: https://doi.org/10.1080/00029890.2020.1822718

Published online: 11 Dec 2020.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman**, and **Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by April 30, 2021, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12216. *Proposed by Zachary Franco, Houston, TX.* A regular icosahedron with volume 1 is rotated about an axis connecting opposite vertices. What is the volume of the resulting solid?

12217. *Proposed by Giuseppe Fera, Vicenza, Italy.* Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$\frac{3}{2} < \frac{AI}{AG} + \frac{BI}{BG} + \frac{CI}{CG} \le 3.$$

12218. *Proposed by Richard Stong, Center for Communications Research, La Jolla, CA, and Stan Wagon, Macalester College, St. Paul, MN.* For which positive integers n does there exist an ordering of all permutations of $\{1, \dots, n\}$ so that their composition in that order is the identity?

12219. *Proposed by Brad Isaacson, New York City College of Technology, New York, NY.* Let k and m be positive integers with $k < m$. Let $c(m, k)$ be the number of permutations of $\{1, \dots, m\}$ consisting of k cycles. (The numbers $c(m, k)$ are known as unsigned Stirling numbers of the first kind.) Prove

$$\sum_{j=k}^{m} \frac{(-2)^j \binom{m}{j} c(j, k)}{(j-1)!} = 0$$

whenever m and k have opposite parity.

12220. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $a_n = \sum_{k=1}^{n} 1/k^2$ and $b_n = \sum_{k=1}^{n} 1/(2k-1)^2$. Prove

$$\lim_{n\to\infty} n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{3}{\pi^2}.$$

doi.org/10.1080/00029890.2020.1822718

For $t > 0$, the substitution $y = xt$ gives

$$\int_0^\infty \frac{\log(1+xt)}{x^{p+1}}\,dx = t^p \int_0^\infty \frac{\log(1+y)}{y^{p+1}}\,dy.$$

This formula remains valid for $t = 0$, since both sides then vanish. Therefore

$$I = \int_0^a f(s)^p \int_0^\infty \frac{\log(1+y)}{y^{p+1}}\,dy\,ds = J(p)\int_0^a f(s)^p\,ds,$$

where

$$J(p) = \int_0^\infty \frac{\log(1+y)}{y^{p+1}}\,dy.$$

To evaluate $J(p)$, first integrate by parts:

$$J(p) = \frac{1}{p}\left[-\frac{\log(1+y)}{y^p}\right]_{y=0}^\infty + \frac{1}{p}\int_0^\infty \frac{dy}{y^p(1+y)} = \frac{1}{p}\int_0^\infty \frac{dy}{y^p(1+y)}.$$

This calculation uses $p < 1$ and $\log(1+y) = O(y)$ as $y \to 0$.

The beta function B is well known to satisfy

$$B(u,v) = \int_0^\infty \frac{y^{u-1}}{(1+y)^{u+v}}\,dy$$

for $u, v > 0$. Letting $(u,v) = (1-p, p)$ yields

$$B(1-p,p) = \int_0^\infty \frac{dy}{y^p(1+y)}\,dy.$$

Expressing the beta function in terms of the gamma function gives

$$J(p) = \frac{B(1-p,p)}{p} = \frac{\Gamma(1-p)\Gamma(p)}{p\Gamma(1)} = \frac{\Gamma(p)\Gamma(1-p)}{p}.$$

The reflection formula for the gamma function implies $J(p) = \dfrac{\pi}{p\sin(p\pi)}$. Therefore

$$I = \frac{\pi}{p\sin(p\pi)}\int_0^a f(s)^p\,ds,$$

which is equivalent to the desired formula.

Also solved by K. F. Andersen (Canada), P. Bracken, H. Chen, G. H. Chung, K. Gatesman, N. Hodges, F. Holland (Ireland), O. Kouba (Syria), O. P. Lossers (Netherlands), M. A. Prasad (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, T. Wiandt, FAU Problem Solving Group, and the proposer.

The Limit of an Infinite Product

12110 [2019, 371]. *Proposed by Pedro Jesús Rodríguez de Rivera (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* Let $\alpha_k = (k + \sqrt{k^2+4})/2$. Evaluate

$$\lim_{k\to\infty} \prod_{n=1}^\infty \left(1 - \frac{k}{\alpha_k^n + \alpha_k}\right).$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. The limit is $1/2$. Since $\alpha_k - k = \alpha_k^{-1}$, the product equals

$$\prod_{n=1}^\infty \frac{\alpha_k^n + \alpha_k^{-1}}{\alpha_k^n + \alpha_k},$$

which in turn is

$$\prod_{n=1}^{\infty} \frac{1+\alpha_k^{-n-1}}{1+\alpha_k^{-n+1}}.$$

This product telescopes, so it equals $(1+\alpha_k^0)^{-1} \cdot (1+\alpha_k^{-1})^{-1}$. Since $\alpha_k^0 = 1$ and $\alpha_k^{-1} \to 0$, it follows that the limit is $1/2$.

Editorial comment. Some solvers noted that the product for $k = 1$ appears in *The Fibonacci Quarterly* as Elementary Problem B-1237 [2018, 366] by Hideyuki Ohtsuka.

Also solved by K. F. Andersen (Canada), F. R. Ataev (Uzbekistan), M. Bataille (France), P. Bracken, B. Bradie, N. Caro (Brazil), R. Chapman (UK), G. Fera (Italy), D. Fleischman, K. Gatesman, R. Guadalupe (Philippines), E. A. Herman, F. Holland (Ireland), M. Kaplan & M. Goldenberg, O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, X. Liu (China), R. Molinari, A. Natian, H. Ohtsuka (Japan), M. Omarjee (France), A. Pathak, M. A. Prasad (India), T. de Souza Leao, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Terr, T. Wiandt, J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

Harmony on the n-Cube

12111 [2019, 468]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* A line segment AB can be oriented in two ways, which we denote (AB) and (BA). A square $ABCD$ can be oriented in two ways, which we denote $(ABCD)$ (the same as $(BCDA)$, $(CDAB)$, and $(DABC)$) and $(DCBA)$ (the same as $(CBAD)$, $(BADC)$, and $(ADCB)$). We say that the orientation $(ABCD)$ of a square *agrees with* the orientations (AB), (BC), (CD), and (DA) of its sides. Suppose that each edge and 2-dimensional face of an n-dimensional cube is given an orientation.
(a) What is the largest possible number of 2-dimensional faces whose orientation agrees with the orientations of its four sides?
(b) What is the largest possible number of edges whose orientation agrees with the orientations of all 2-dimensional faces containing the edge?

Solution by GCHQ Problem Solving Group, Cheltenham, UK. Denote by N_F the answer to part (a) and by N_E the answer to part (b). We prove $N_F = 2^{n-2}\lfloor n^2/4 \rfloor$ and $N_E = 2^n$.

(a) Choose a vertex A of the n-cube and consider all n edges and all $\binom{n}{2}$ 2-dimensional faces containing A. For a face $ABCD$ to agree with the orientations of its four sides, the edges AB and AD must have opposite orientations (one toward A and one away from A). If exactly t of the edges containing A are oriented away from A, at most $t(n-t)$ faces containing A can agree with the orientations of their four sides. It follows that the number of faces containing A whose orientations agree with the orientations of the four sides is at most $\lfloor n^2/4 \rfloor$, the largest possible value of $t(n-t)$. Summing over all vertices, we have

$$4N_F \le 2^n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

since the left side counts each face four times.

The following construction attains this bound. Let $B_1, \dots, B_n$ be the vertices adjacent to A, and let $t = \lfloor n/2 \rfloor$. For $i \in \{1, \dots, t\}$ orient the edges as AB_i, and for $j \in \{t+1, \dots, n\}$ orient the edges as B_jA. Let C_{ij} be the remaining vertex in the face containing the sides AB_i and AB_j. Orient that face as $AB_iC_{ij}B_j$. (The remaining faces containing A may be oriented arbitrarily.)

Every edge PQ not containing A is parallel to exactly one edge AB_r, for some $r \in \{1, \dots, n\}$. Let d be the Hamming distance from the edge PQ to AB_r. Give PQ the orientation parallel to AB_r if d is even, antiparallel to AB_r if d is odd.