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Solutions of a functional integral [1] equation in $\mathbf{BC}(\mathbb{R}_+)$

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Abstract

The aim of this paper is to prove an existence theorem for a functional-integral equation in the space of continuous and bounded functions on $\mathbb{R}_+$. The main tool used in our considerations is the technique associated with measures of noncompactness.

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1 Introduction

Integral equations create a very important and significant part of mathematical analysis and their applications to real world problems ([1, 2, 6, 7, 8], among others). The theory of integral equations is now well developed with the help of several tools of functional analysis, topology and fixed-point theory. In this paper we are going to investigate a functional-integral equation and we will show that such equation is solvable in the space of continuous and bounded functions on $\mathbb{R}_+$. The main tool used in our study is associated with the technique of measures of noncompactness which has been successfully applied in the solvability of some integral equations [4, 5, 6].

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2 Notation and auxiliary facts

Assume E is a real Banach space with norm $\|\cdot\|$ and zero element 0. Denote by $B(x,r)$ the closed ball centered at x and with radius r and by B_r the ball $B(0,r)$. If X is a nonempty subset of E we denote by $\overline{X}$, $ConvX$ the closure and the closed convex closure of X, respectively. The symbols λX and $X+Y$ denote the usual algebraic operations on sets. Finally, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Throughout this paper, we will also accept the following definition of the concept of regular measure of noncompactness [3].

Definition 1. . A function $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+ = [0,\infty)$ is said to be a *measure of noncompactness* in the space E if it satisfies the following conditions:

1. The family $ker\mu = \{X \in \mathfrak{M}_E \ : \ \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$.
4. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
5. If $\{X_n\}_n$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

For further facts concerning measures of noncompactness and their properties we refer to [3].

Now let us assume that Ω is a nonempty subset of a Banach space E and $T : \Omega \to \Omega$ is a continuous operator transforming bounded subsets of Ω to bounded ones. We say that T satisfies the Darbo condition with constant $k \geq 0$ with respect to a measure of noncompactness μ if

$$\mu(TX) \leq k\mu(X)$$

for each $X \in \mathfrak{M}_E$ such that $X \subset \Omega$.

If $k < 1$ then T is called a contraction with respect to μ.

In what follows we will need the following fixed point theorem which is a version of the classical fixed point theorem for lipschitzian mappings in the context of measures of noncompactness [3].

Theorem 1. *Let Ω be a nonempty, bounded, closed and convex subset of E, μ a measure of noncompactness in E and $T : \Omega \longrightarrow \Omega$ a contraction with respect to μ. Then T has at least one fixed point in Ω.*

In the sequel, we will work in the space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions defined bounded and continuous on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+, \mathbb{R})$ is equipped with the standard norm $\|x\| = \sup\{|x(t)| : t \geq 0\}$.

In our considerations we will use a measure of noncompactness defined in [3]. This measure is defined by

$$\mu(X) = w_0(X) + \lim_{t\to\infty} \sup diamX(t)$$

where

$$diamX(t) = sup\{|x(t) - y(t)| : x, y \in X\},$$

$$w(x, \varepsilon) = sup\ \{|\ x(t) - x(s)\ | :\ t, s \in \mathbb{R}_+,\ |\ t - s\ | \leq \varepsilon\},$$

$$w(X, \varepsilon) = sup\ \{w(x, \varepsilon)\ :\ x \in X\}$$

$$w_0(X) = \lim_{\varepsilon\to 0} w(X, \varepsilon).$$

3 Existence Theorem

In this section we will study the solvability of the following functional-integral equation

$$x(t) = f\left(t, \int_0^t v(t, s, x(s))ds, x(t)\right) \tag{1}$$

for $t \in \mathbb{R}_+$.

In what follows we formulate the assumptions under which equation (1) will be studied. Namely, we assume the following assumptions.

(i) The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition with constant $k < 1$ with respect to each variable, i.e.

$|f(t, y, x_1) - f(t, y, x_2)| \leq k|x_1 - x_2|$ for $t \in \mathbb{R}_+$, and $y, x_1, x_2 \in \mathbb{R}$.

$|f(t_1, y, x) - f(t_2, y, x)| \leq k|t_1 - t_2|$ for $t_1, t_2 \in \mathbb{R}_+$, and $x, y \in \mathbb{R}$.

$|f(t, y_1, x) - f(t, y_2, x)| \leq k|y_1 - y_2|$ for $t \in \mathbb{R}_+$, and $x, y_1, y_2 \in \mathbb{R}$.

(ii) There exists a constant $m \geq 0$ such that

$|f(t, y, 0)| \leq m$ for $t \in \mathbb{R}_+$, and $y \in \mathbb{R}$.

(iii) The function $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

- $\lim_{t\to\infty} a(t) = 0$,
- b is a bounded function and $b \in L^1(\mathbb{R}_+)$,
- $|v(t, s, x)| \leq a(t) \cdot b(s)$ for $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$

(iv) There exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi \in L^1(\mathbb{R}_+)$ such that

$$|v(t_1, s, x) - v(t_2, s, x)| \leq \varphi(s)|t_2 - t_1|,$$

for $t_1, t_2, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

Before we formulate our main result we can obtain the following remarks.

Remark 1. *Note that assumption (iii) implies that the function a is bounded and by $\|a\|$ we denote the supremum of the function a on $\mathbb{R}_+$.*

Remark 2. *By assumption (iii) we infer that*

$$\left|\int_0^t v(t, s, x(s))ds\right| \leq \int_0^t |v(t, s, x(s))|ds \leq a(t) \int_0^t b(s)ds$$

and as $\lim_{t\to\infty} a(t) = 0$ and $b \in L^1(\mathbb{R}_+)$ we deduce that

$$0 \leq \lim_{t\to\infty} a(t) \int_0^t b(s)ds \leq \lim_{t\to\infty} a(t)\|b\|_1 = 0.$$

Hence, $\lim_{t\to\infty} a(t) \int_0^t b(s)ds = 0$.

Now we will prove the following lemma which be needed further on.

Lemma 1. *Suppose that $x \in BC(\mathbb{R}_+, \mathbb{R})$ and $\varepsilon > 0$. Then*

$$w(x, \varepsilon) = \sup_{L>0} w^L(x, \varepsilon)$$

where $w^L(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, L], |t - s| \leq \varepsilon\}$.

Proof. Obviously

$$w(x, \varepsilon) \geq \sup_{L>0} w^L(x, \varepsilon).$$

Suppose that

$$w(x, \varepsilon) > \sup_{L>0} w^L(x, \varepsilon),$$

this means that there exist $t_1, t_2 \in \mathbb{R}_+$ with $|t_1 - t_2| \leq \varepsilon$ such that

$$\sup_{L>0} w^L(x,\varepsilon) < |x(t_1) - x(t_2)|.$$

Taking $L_0 = \max\{t_1, t_2\}$ we get

$$\sup_{L>0} w^L(x,\varepsilon) < |x(t_1) - x(t_2)| \leq w^{L_0}(x,\varepsilon)$$

which is a contradiction. Thus the proof is complete. □

Now we present our existence result

Theorem 2. *Under assumptions (i) − (iv), equation (1) has at least one solution $x = x(t)$ which belong to the space $BC(\mathbb{R}_+, \mathbb{R})$.*

Proof. Let us define the operator F on the space $BC(\mathbb{R}_+)$ by

$$(Fx)(t) = f\left(t, \int_0^t v(t,s,x(s))ds, x(t)\right).$$

Now we will prove that if $x \in BC(\mathbb{R}_+, \mathbb{R})$ then Fx is a continuous function on $\mathbb{R}_+$.

In order to do this firstly we show that the function G defined by

$$(Gx)(t) = \int_0^t v(t,s,x(s))ds$$

is a continuous function on $\mathbb{R}_+$.

Let us fix $x \in BC(\mathbb{R}_+)$, $t_0 \in \mathbb{R}_+$, $t_0 \neq 0$ and $\varepsilon > 0$. Let $t \in \mathbb{R}_+$ be such that $|t - t_0| < \delta$ where $\delta < \frac{\varepsilon}{\|\varphi\|_1 + M}$, being $M = \sup\{|v(t_0, s, x)| : s \in [0, t_0], x \in [-\|x\|, \|x\|]\}$ (this supremum exists in virtue of the continuity of v (assumption (iii)). Without loss of generality we can assume $t_0 < t$. Then taking into account our assumptions we have the following estimate:

$$\begin{aligned}
&|(Gx)(t) - (Gx)(t_0)| = \left|\int_0^t v(t,s,x(s))ds - \int_0^{t_0} v(t_0,s,x(s))ds\right| \leq \\
&\leq \left|\int_0^t v(t,s,x(s))ds - \int_0^t v(t_0,s,x(s))ds\right| + \\
&+ \left|\int_0^t v(t_0,s,x(s))ds - \int_0^{t_0} v(t_0,s,x(s))ds\right| \leq \\
&\leq \int_0^t |v(t,s,x(s)) - v(t_0,s,x(s))|ds + \int_{t_0}^t |v(t_0,s,x(s))|ds \leq \\
&\leq \int_0^t (t-t_0)\varphi(s)ds + M\int_{t_0}^t ds = (t-t_0)\|\varphi\|_1 + M(t-t_0) = \\
&= (\|\varphi\|_1 + M)(t-t_0) < (\|\varphi\|_1 + M)\delta < \varepsilon
\end{aligned}$$

The case $t_0 = 0$ is analogous. Hence Gx is continuous if $x \in BC(\mathbb{R}_+, \mathbb{R})$.
Moreover, taking into account the composition of the following continuous functions

$$\mathbb{R}_+ \xrightarrow{\Phi} \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$t \mapsto (t, (Gx)(t), x(t)) \mapsto f(t, (Gx)(t), x(t)) = (Fx)(t)$$

we derive that the function Fx is continuous if $x \in BC(\mathbb{R}_+, \mathbb{R})$.

In the sequel we show that Fx is a bounded function for $x \in BC(\mathbb{R}_+, \mathbb{R})$.

In fact, for $t \in \mathbb{R}_+$ and taking into account our hypotheses we can get

$$\begin{aligned} |(Fx)(t)| = \; & \left| f(t, \int_0^t v(t,s,x(s))ds, x(t)) \right| \leq \\ & \leq \left| f(t, \int_0^t v(t,s,x(s))ds, x(t)) - f(t, \int_0^t v(t,s,x(s))ds, 0) \right| + \\ & + \left| f(t, \int_0^t v(t,s,x(s))ds, 0) \right| \leq k|x(t)| + m \leq k\|x\| + m \end{aligned}$$

Consequently, Fx is bounded on $\mathbb{R}_+$ for $x \in BC(\mathbb{R}_+, \mathbb{R})$.
Moreover, from the above estimate we obtain

$$\|Fx\| \leq k\|x\| + m$$

Since $k < 1$ (assumption (i)) we deduce that the operator F transforms the ball B_r into itself for $r = \frac{m}{1-k}$.

Now we prove that F is continuous on $BC(\mathbb{R}_+, \mathbb{R})$.

In order to do this, let us fix $x \in BC(\mathbb{R}_+, \mathbb{R})$ and $\varepsilon > 0$. Taking into account Remark 2,

$$\lim_{t \to \infty} a(t) \int_0^t b(s)ds = 0$$

so, for our $\varepsilon > 0$ we can find $\tau > 0$ such that if $t > \tau$ then

$$a(t) \int_0^t b(s)ds < \frac{\varepsilon}{4k}. \tag{2}$$

Moreover, by the uniform continuity of the function $v(t, s, x)$ on the set $[0, \tau] \times [0, \tau] \times [-\frac{\varepsilon}{2k} - \|x\|, \frac{\varepsilon}{2k} + \|x\|]$ there exists $\delta_1 > 0$ such that if $|t - t'| \leq \delta_1$, $|s - s'| \leq \delta_1$ and $|x - x'| \leq \delta_1$ with $t, t', s, s' \in [0, \tau]$ and $x', x'' \in [-\frac{\varepsilon}{2k} - \|x\|, \frac{\varepsilon}{2k} + \|x\|]$, then

$$|v(t, s, x'') - v(t', s', x')| < \frac{\varepsilon}{2k\tau} \tag{3}$$

Let us consider $\delta < min\{\frac{\varepsilon}{2k}, \delta_1\}$ and let $y \in BC(\mathbb{R}_+, \mathbb{R})$ such that $\|x - y\| \leq \delta$. Then for $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| = \\
&\left| f(t, \int_0^t v(t,s,x(s))ds, x(t)) - f(t, \int_0^t v(t,s,y(s))ds, y(t)) \right| \leq \\
&\leq \left| f(t, \int_0^t v(t,s,x(s))ds, x(t)) - f(t, \int_0^t v(t,s,x(s))ds, y(t)) \right| + \\
&+ \left| f(t, \int_0^t v(t,s,x(s))ds, y(t)) - f(t, \int_0^t v(t,s,y(s))ds, y(t)) \right| \leq \\
&\leq k|x(t) - y(t)| + k \int_0^t |v(t,s,x(s)) - v(t,s,y(s))| ds \leq \\
&\leq k\delta + k \int_0^t |v(t,s,x(s)) - v(t,s,y(s))| ds.
\end{aligned}$$

Now we can consider two cases:

1. If $t > \tau$ then by (2) we obtain

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq k\delta + k \left(\int_0^t |v(t,s,x(s))|ds + \int_0^t |v(t,s,y(s))|ds \right) \leq \\
&\leq k\delta + k \cdot 2a(t) \int_0^t b(s)ds \leq \\
&\leq k\delta + k \cdot 2 \cdot \frac{\varepsilon}{4k} \leq k \cdot \frac{\varepsilon}{2k} + k \cdot \frac{2\varepsilon}{4k} = \varepsilon.
\end{aligned}$$

2. If $t \leq \tau$ as $\|x - y\| \leq \delta < \delta_1$ by (3):

$$|v(t,s,x(s)) - v(t,s,y(s))| < \frac{\varepsilon}{2k\tau},$$

so we have that

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq k\delta + k \cdot \frac{\varepsilon}{2k\tau} \int_0^t ds \leq k\delta + k \cdot \frac{\varepsilon}{2k\tau} \cdot \tau \leq \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The above established facts say us that the operator F is continuous on $BC(\mathbb{R}_+, \mathbb{R})$. (Note that the same reasoning proves us that F is uniformly continuous on B_r taking the compact set $[0, \tau] \times [0, \tau] \times [-\frac{\varepsilon}{2k} - r, \frac{\varepsilon}{2k} + r]$).

In what follows we prove that the operator F satisfies the Darbo condition with respect to the measure of noncompactness introduced in section 2.

Firstly, we study the term related to the oscillation.

Let us take a nonempty subset X of the ball B_r and $x \in X$. Then for a fixed $L > 0$, $\varepsilon > 0$ and for $t_1, t_2 \in [0, L]$ such that $t_1 < t_2$ and $t_2 - t_1 < \varepsilon$ we get

$$\begin{aligned}
&|(Fx)(t_2) - (Fx)(t_1)| = \\
&= \left| f(t_2, \int_0^{t_2} v(t_2, s, x(s))ds, x(t_2)) - f(t_1, \int_0^{t_1} v(t_1, s, x(s))ds, x(t_1)) \right| \leq \\
&\leq \left| f(t_2, \int_0^{t_2} v(t_2, s, x(s))ds, x(t_2)) - f(t_2, \int_0^{t_2} v(t_2, s, x(s)), ds, x(t_1)) \right| + \\
&+ \left| f(t_2, \int_0^{t_2} v(t_2, s, x(s))ds, x(t_1)) - f(t_2, \int_0^{t_1} v(t_1, s, x(s))ds, x(t_1)) \right| + \\
&+ \left| f(t_2, \int_0^{t_1} v(t_1, s, x(s))ds, x(t_1)) - f(t_1, \int_0^{t_1} v(t_1, s, x(s))ds, x(t_1) \right| \leq \\
&\leq k|x(t_2) - x(t_1)| + k \left| \int_0^{t_2} v(t_2, s, x(s))ds - \int_0^{t_1} v(t_1, s, x(s))ds \right| + \\
&+k(t_2 - t_1) \leq kw^L(x, \varepsilon) + k \int_0^{t_1} |v(t_2, s, x(s)) - v(t_1, s, x(s))|ds + \\
&+k \int_{t_1}^{t_2} |v(t_2, s, x(s))|ds + k(t_2 - t_1) \leq \\
&\leq kw^L(x, \varepsilon) + k(t_2 - t_1) \int_0^{t_1} \varphi(s)ds + ka(t_2) \int_{t_1}^{t_2} b(s)ds + k(t_2 - t_1) \leq \\
&\leq kw^L(x, \varepsilon) + k(t_2 - t_1)\|\varphi\|_1 + k\|a\| \cdot \|b\|(t_2 - t_1) + k(t_2 - t_1) \leq \\
&\leq kw^L(x, \varepsilon) + \varepsilon[k\|\varphi\|_1 + k\|a\| \cdot \|b\| + k]
\end{aligned}$$

Thus we have,

$$w^L(Fx, \varepsilon) \leq kw^L(x, \varepsilon) + \varepsilon[k\|\varphi\|_1 + k\|a\| \cdot \|b\| + k]$$

and applying supremum in L

$$\sup_L w^L(Fx, \varepsilon) \leq k \cdot \sup_L w^L(x, \varepsilon) + \varepsilon[k\|\varphi\|_1 + k\|a\| \cdot \|b\| + k].$$

By Lemma 1

$$w(Fx, \varepsilon) \leq k \cdot w(x, \varepsilon) + \varepsilon \cdot [k\|\varphi\|_1 + k\|a\| \cdot \|b\| + k].$$

Consequently,

$$\sup_{x \in X} w(Fx, \varepsilon) \leq k \cdot \sup_{x \in X} w(x, \varepsilon) + \varepsilon[k\|\varphi\|_1 + k\|a\| \cdot \|b\| + k]$$

and applying limit when $\varepsilon \to 0$

$$w_0(FX) \leq k \cdot w_0(X) \tag{4}$$

In the sequel, we study the term related to the diameter which appears in the expression of the measure μ.

Let us take a nonempty subset X of B_r, $x, y \in X$ and $t \in \mathbb{R}_+$. Then

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&= \left| f(t, \int_0^t v(t,s,x(s))ds, x(t)) - f(t, \int_0^t v(t,s,y(s))ds, y(t)) \right| \leq \\
&\leq \left| f(t, \int_0^t v(t,s,x(s))ds, x(t)) - f(t, \int_0^t v(t,s,x(s))ds, y(t)) \right| + \\
&+ \left| f(t, \int_0^t v(t,s,x(s))ds, y(t)) - f(t, \int_0^t v(t,s,y(s))ds, y(t)) \right| \leq \\
&\leq k|x(t) - y(t)| + k \left| \int_0^t v(t,s,x(s)) - v(t,s,y(s))ds \right| \leq \\
&\leq k|x(t) - y(t)| + k \int_0^t |v(t,s,x(s)) - v(t,s,y(s))|ds \leq \\
&\leq k|x(t) - y(t)| + k \int_0^t |v(t,s,x(s))|ds + k \int_0^t |v(t,s,y(s))|ds \leq \\
&\leq k|x(t) - y(t)| + k \cdot a(t) \cdot \int_0^t b(s)ds + k \cdot a(t) \cdot \int_0^t b(s)ds \leq \\
&\leq k|x(t) - y(t)| + 2 \cdot k \cdot a(t) \cdot \int_0^t b(s)ds
\end{aligned}$$

Applying supremum in x and y we obtain

$$\sup_{x,y \in X} |(Fx)(t) - (Fy)(t)| \leq k \sup_{x,y \in X} |x(t) - y(t)| + 2 \cdot k \cdot a(t) \int_0^t b(s)ds$$

or, equivalently

$$diam(FX)(t) \leq k \cdot diam X(t) + 2 \cdot k \cdot a(t) \int_0^t b(s)ds.$$

Applying upper limit when $t \to \infty$

$$\limsup_{t \to \infty} diam(FX)(t) \leq k \limsup_{t \to \infty} diam(X)(t). \tag{5}$$

Now, linking (4) and (5) we obtain

$$\mu(FX) \leq k \cdot \mu(X).$$

Finally, applying Theorem 1 we complete the proof.

□

In what follows we present some examples where existence can be established by using theorem 2.

Example 1. Consider the integral equation

$$x(t) = \frac{1}{2}\cos t + \frac{1}{2}\cos\left(\int_0^t \frac{1}{t+1}e^{-s}\cos x(s)ds\right) + \frac{1}{2}\cos x(t). \tag{6}$$

Observe that in this case the function f is given by

$$f(t,y,x) = \frac{1}{2}(\cos t + \cos y + \cos x).$$

It is easy to prove that such function f satisfies assumption (i) with constant $k = \frac{1}{2}$. Moreover, $f(t,y,0) = \frac{1}{2}(\cos t + \cos y + 1)$ and this gives us $|f(t,y,0)| \leq \frac{3}{2}$. Thus, f satisfies assumption (ii) with $m = \frac{3}{2}$. The function v is given by the expression

$$v(t,s,x) = \frac{1}{t+1}e^{-s}\cos x$$

where $a(t) = \frac{1}{t+1}$ and $b(s) = e^{-s}$. Consequently, v satisfies assumption (iii). Moreover,

$$\begin{aligned} |v(t_2,s,x) - v(t_1,s,x)| & \leq e^{-s}\left|\tfrac{1}{t_2+1} - \tfrac{1}{t_1+1}\right| \leq \tfrac{e^{-s}|t_2-t_1|}{(t_2+1)\cdot(t_1+1)} \leq \\ & \leq e^{-s}|t_2 - t_1| \end{aligned}$$

and we can take $\varphi(s) = e^{-s}$.

Theorem 2 says us that our equation (6) has a solution in $BC(\mathbb{R}_+, \mathbb{R})$ which belongs to the set B_3.

Example 2. Let us consider the function f defined by

$$f(t,y,x) = \frac{1}{2}\left(e^{-t} + \frac{1}{|y|+1} + e^{-x}\right).$$

Obviously, the function f verifies assumptions (i) and (ii) with $k = \frac{1}{2}$ and $m = \frac{3}{2}$. Consider as v the function given in example 1 and the integral equation

$$x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}\frac{1}{\left|\int_0^t \frac{1}{t+1}e^{-s}\cos x(s)ds\right| + 1} + e^{-x(t)}.$$

Theorem 2 guarantees that this equation has a solution in $BC(\mathbb{R}_+, \mathbb{R})$ which belongs to the set B_3.

4 Some Remarks

Now we will study the solvability of the functional-integral equation

$$x(t) = f\left(t, \int_0^t v(t,s,x(s))ds, x(t)\right) \cdot g\left(t, \int_0^t u(t,s,x(s))ds, x(t))\right) \qquad (7)$$

in $BC(\mathbb{R}_+, \mathbb{R})$.

Previously, we need the following lemma.

Lemma 2. *Assume that Ω is nonempty, bounded, convex and closed subset of $BC(\mathbb{R}_+, \mathbb{R})$ and the operators P and T transform continuously the set Ω into $BC(\mathbb{R}_+, \mathbb{R})$ in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, assume that the operator $S = P{\cdot}T$ transforms Ω into itself. If P and T satisfy on Ω the Darbo condition (with respect to the measure of noncompactness μ introduced in section 2) with constants k_1 and k_2, respectively, then the operator S satisfies on Ω the Darbo condition with constant $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1$, where*

$$\|P(\Omega)\| = \sup\{\|Px\| : x \in \Omega\}$$

$$\|T(\Omega)\| = \sup\{\|Tx\| : x \in \Omega\}.$$

Proof. Let us take a nonempty subset X on Ω and $x \in X$. Then for a fixed $L > 0$, $\varepsilon > 0$ and for $t_1, t_2 \in [0, L]$ such that $t_1 < t_2$ and $t_2 - t_1 < \varepsilon$ we can obtain

$$\begin{aligned}
&|(Sx)(t_2) - (Sx)(t_1)| = |(PTx)(t_2) - (PTx)(t_1)| = \\
&= |(Px)(t_2) \cdot (Tx)(t_2) - (Px)(t_1) \cdot (Tx)(t_1)| \le \\
&\le |(Px)(t_2) \cdot (Tx)(t_2) - (Px)(t_2) \cdot (Tx)(t_1)| + \\
&+ |(Px)(t_2) \cdot (Tx)(t_1) - (Px)(t_1) \cdot (Tx)(t_1)| \le \\
&\le |(Px)(t_2)| \cdot |(Tx)(t_2) - (Tx)(t_1)| + \\
&+ |(Tx)(t_1)| \cdot |(Px)(t_2) - (Px)(t_1)| \le \\
&\le \|P\Omega\| \cdot w^L(Tx, \varepsilon) + \|T\Omega\| \cdot w^L(Px, \varepsilon).
\end{aligned}$$

By applying supremum in L, taking into account lemma 1 and taking limit when $\varepsilon \to 0$ we get

$$w_0(SX) \le \|P\Omega\| \cdot w_0(TX) + \|T\Omega\| w_0(PX). \qquad (8)$$

On the other hand, we take $x, y \in X$ and $t \in \mathbb{R}_+$, then

$$\begin{aligned}
&|(Sx)(t) - (Sy)(t)| \le |(Px)(t)| \cdot |(Tx)(t) - (Ty)(t)| + \\
&+ |(Ty)(t)| \cdot |(Px)(t) - (Py)(t)| \le \\
&\le \|P\Omega\| \cdot diam(TX)(t) + \|T\Omega\| \cdot diam(PX)(t)
\end{aligned}$$

and, consequently,

$$\limsup_{t\to\infty} diam(SX)(t) \leq \|P\Omega\| \limsup_{t\to\infty}(TX)(t) + \|T\Omega\| \cdot \limsup_{t\to\infty}(PX)(t) \quad (9)$$

Now, linking (8) and (9), and taking into account our hypotheses on T and P we obtain

$$\begin{aligned}\mu(SX) &\leq \|P\Omega\| \cdot \mu(TX) + \|T\Omega\| \cdot \mu(PX) \leq \\ &\leq \|P\Omega\| \cdot k_1 \cdot \mu(X) + \|T\Omega\| \cdot k_2 \cdot \mu(X) = \\ &= (\|P\Omega\| \cdot k_1 + \|T\Omega\| \cdot k_2) \cdot \mu(X)\end{aligned}$$

and this fact completes the proof. □

In the sequel, we formulate the assumptions under which equation (7) will be studied.

(i) The functions $f, g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition with the same constant k with respect to each variable.

(ii) There exists a constant $m \geq 0$ such that

$$|f(t, y, 0)| \leq m; \qquad |g(t, y, 0)| \leq m$$

for $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$.

(iii) The functions $v, u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist continuous functions $a_i, b_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ such that

$$\lim_{t\to\infty} a_i(t) = 0 \qquad (i = 1, 2)$$

$$b_i \in L^1(\mathbb{R}_+) \quad \text{and bounded} \qquad (i = 1, 2)$$

$$|v(t, s, x)| \leq a_1(t) \cdot b_1(s), \quad |u(t, s, x)| \leq a_2(t) \cdot b_2(s) \text{ for } t, s \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}$$

(iv) There exist functions $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $(i = 1, 2)$ with $\varphi_i \in L^1(\mathbb{R}_+)$ such that

$$|v(t_1, s, x) - v(t_2, s, x)| \leq \varphi_1(s)|t_2 - t_1|$$

$$|u(t_1, s, x) - u(t_2, s, x)| \leq \varphi_2(s)|t_2 - t_1|$$

for $t_1, t_2 \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(v) $4km < 1$.

Then we obtain the following result.

Theorem 3. *Under assumptions* (i)–(v) *equation (7) has at least one solution in* $BC(\mathbb{R}_+, \mathbb{R})$.

Proof. Let us consider the operators F and G defined on $BC(\mathbb{R}_+,\mathbb{R})$ by

$$(Fx)(t) = f\left(t, \int_0^t v(t,s,x(s))ds, x(t)\right)$$

$$(Gx)(t) = g\left(t, \int_0^t u(t,s,x(s))ds, x(t)\right)$$

In a similar way that in theorem 1 we can prove that the operators F and G transform $BC(\mathbb{R}_+,\mathbb{R})$ into itself. Hence, the operator $Tx = Fx \cdot Gx$ also transforms $BC(\mathbb{R}_+,\mathbb{R})$ into itself.

Now, let us fix $x \in BC(\mathbb{R}_+,\mathbb{R})$. Then, taking into account our assumptions for $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
|(Tx)(t)| &= |(Fx)(t)\cdot(Gx)(t)| = \\
&= \left|f\left(t, \textstyle\int_0^t v(t,s,x(s))ds, x(t)\right)\right| \cdot \left|g\left(t, \textstyle\int_0^t u(t,s,x(s))ds, x(t)\right)\right| \leq \\
&\leq \left\{\left|f\left(t, \textstyle\int_0^t v(t,s,x(s))ds, x(t)\right) - f\left(t, \textstyle\int_0^t v(t,s,x(s))ds, 0\right)\right| + \right.\\
&+ \left.\left|f\left(t, \textstyle\int_0^t v(t,s,x(s))ds, 0\right)\right|\right\} \cdot \left\{\left|g\left(t, \textstyle\int_0^t u(t,s,x(s))ds, x(t)\right) - \right.\right.\\
&- \left.\left. g\left(t, \textstyle\int_0^t u(t,s,x(s))ds, 0\right)\right| + \left|g\left(t, \textstyle\int_0^t u(t,s,x(s))ds, 0\right)\right|\right\} \leq \\
&\leq (k|x(t)| + m)\cdot(k|x(t)| + m) \leq (k\|x\| + m)^2. \qquad (10)
\end{aligned}$$

Thus, $\|Tx\| \leq (k \cdot \|x\| + m)^2$.

From estimate (10), we infer that the operator T transforms the ball B_r into itself for $r = r_1$ where

$$r_1 = \frac{1 - 2km - \sqrt{1-4km}}{2k^2}.$$

Note that from estimate (10) we have

$$\|FB_r\| \leq kr + m \qquad \text{and} \qquad \|GB_r\| \leq kr + m. \qquad (11)$$

In order to prove that the operator T is continuous on B_r we use the same reasoning that in theorem 2 for the operator F and G independently.

As in the proof of theorem 2, we show that the operators F and G satisfy the Darbo condition with constant k.

Finally, taking into account lemma 2, the estimates (11) and

$$\begin{aligned}
2k(kr+m) &= 2k\left(k \cdot \tfrac{1-2km-\sqrt{1-4km}}{2k^2} + m\right) = \\
&= 1 - 2km - \sqrt{1-4km} + 2km = 1 - \sqrt{1-4km} < 1
\end{aligned}$$

Theorem 1 implies that the operator T has a fixed point in the all B_r. This finishes the proof.

□

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