On the reception and spread of metaphysical explanations of imaginary numbers in Spain

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Abstract

The introduction in Spain of the ideas on complex quantities is documented through the work of José María Rey Heredia, a Professor of Logic and self-taught mathematician who wrote during his last life years the book *Teoría Transcendental de las Cantidades Imaginarias*. Rey was a follower of Kant and Krause, and tried to inscribe the development of the theory into the intellectual framework of Transcendental Logic. His attempt is a lengthy and sometimes erroneous comment, on the *Mémoire* on imaginary quantities published by the Abbé Buée in the Philosophical Transactions the year 1806. Rey had a number of followers who introduced his ideas in teaching through the second half of the 19th Century as an aide to the introduction of various geometrical concepts.

Resumen

La introducción en España de las ideas filosóficas sobre los números complejos se encuentran documentadas en la obra de José María Rey Heredia, Profesor de Lógica y matemático, quien escribió una especie de testamento matemático filosófico durante los últimos años de su vida, la *Teoria Transcendental de las Cantidades Imaginarias*. Rey fue seguidor de Kant y de Krause e intentó situar los números complejos en el marco de la Lógica Trascendental kantiana. El texto de Rey es un largo y desigual comentario, a la luz del esquema de las categorías, de la Memoria publicada por el Abate Buée en las *Philosophical Transactions* el año 1806. Rey tuvo varios seguidores que introdujeron sus ideas en la enseñanza media durante la segunda mitad del XIX como apoyo al estudio de la Geometria.

AMS Classifications: 01A55, 01A90

Keywords: Imaginary quantities, complex numbers, Kant, Spain, 19th Century

1. Preliminaries

The introduction of rigour in Mathematics is a topic that can be addressed from at least two different viewpoints. A first one, mostly encountered in contemporary Mathematics, stems from the explosive developments which took place during the 19th Century, and is represented by the fact that practitioners are trained to absorb certain techniques amounting to a carefully prepared mixture of computing capabilities, cross references, and the ability to derive results from well authorised sources. As a rule this is considered as *the* standard and to a certain extent it has been able to ban most intuitive reasoning patterns in large mathematical areas. The second viewpoint is the attempt to base mathematical derivations on sound principles regarding the philosophical status of the ideas involved. History offers a dearth of nomenclature such as *negative*, *impossible*, *surd*, and *imaginary* numbers, to quote just a few examples showing that some operations performed within the usual –for their times- number domains could yield results without existence in them. All the above names suggest some sort of strangeness that should be managed in order to "justify" computations that would otherwise remain in the realm of purely formal ones.

Both standpoints have quite different natures and areas of application. While the first one is emphasised when a theory is axiomatically developed and has a very syntactic flavour, the second one is reserved to more foundational matters and tries to preserve unity of Mathematics by establishing conceptual connections between succesive enlargements and developments. Here a case study of the second class is presented, namely an episode related to the introduction of complex numbers in Spain during the 19th Century. It will be shown that rigour in the second sense can be accompanied by a lack of it in the first sense. Nevertheless, quite revealing pieces of Philosophy and Mathematics will appear along this paper.

The philosophical status of negative and imaginary numbers was a recurrent concern for some enlightened philosophers and mathematicians during the second half of the 18th Century, even though their use can be tracked in mathematical practice back to the earlier days of Algebra in the works of Girolamo Cardano (1501-1576) and Raffaelle Bombelli (1526-1578) when dealing with the classical problem of dividing a given number into two numbers whose product is known. It must be noted that the interest of these considerations was enhanced due to the emergence of Algebra as a sort of Universal Arithmetic deeply related with metaphysical considerations on the validity of the items under manipulation. A very interesting observation on these precursors is that they considered complex roots of an equation not as solutions thereof *strictu sensu*, but in the sense of complying with the usual symmetric functions of the roots, an idea still used by 19th Century theoreticians, see *e.g.* [Gilbert 1831]: This is a very early appearance of the idea of a weak solution, much in the same sense that non-differentiable functions can be considered solutions of differential equations within the framework of Distribution Theory. Moreover, by 1800 Science was coming of age for the introduction of vector magnitudes, where the mere idea of adding them amounts to the simultaneous consideration of several properties embodied in the components.

The reception of all these advances took place in Spain quite late, well into the 19th Century, and is found in textbooks used for several decades in secondary teaching, and even at the university level, until the first years of the 20th Century.

The history of Spanish Mathematics during the 19th Century, though possibly not as exciting as that in other European countries -there is no reference to Spain in [Klein 1926]-, is intimately related with the turbulent political and cultural atmosphere that prevailed through the Century and was responsible for a variety of educational and research policies in Spain. A detailed account is still missing, though an interesting attempt has been recently undertaken in [Suárez 2006].

2. Rey Heredia: a biographical sketch

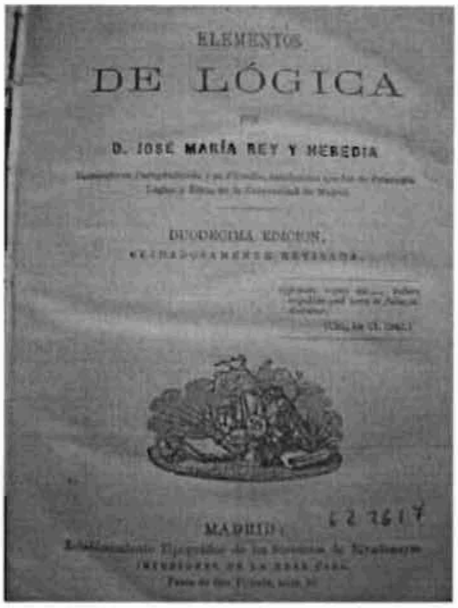
The history of the study of complex numbers in Spain from a philosophical viewpoint is that of the endeavour of the Andalusian philosopher and self-taught mathematician José María Rey Heredia, who lived between 1818 and 1861 –a contemporary of Bernhard Riemann (1826-1866)-, and is contained in his posthumous book *Teoría Transcendental de las Cantidades Imaginarias* (*A Transcendental Theory of Imaginary Quantities*, for short *Teoría* in what follows) published the year 1865 after the author's death, at the expenses of the Spanish Education Ministry [Rey 1865]. The book was presented by Rey's estate to a national contest for texts with ideas applicable in improving quality of the Spanish Secondary Education, following the wake of the 1857 Act known as *Ley Moyano* after the Minister Claudio Moyano (1809-1890) who promoted it. Most biographical data on the author are found on the long foreword to *Teoría* by the Academician Pedro Felipe Monlau (1808-1871), a politician and physician who co-authored with him an earlier two-volume book entitled *Curso de Psicología y Lógica* published during 1849 in Madrid.

Rey was born in Córdoba and at the age of fifteen he entered the Seminar, where he read Philosophy and Theology for nearly eleven years in order to become a priest. This was a rather common practice at the time for lower class families when they happened to have a son with good intellectual aptitude, for children were educated and taken care of for several years with little or no cost to the parents. He did not receive the orders, but during his last years at the Seminar he collaborated in teaching younger students matters such as Philosophy and French. For sure the ecclesiastical build up of Rey influenced his texts, where Latin quotations are unforgivingly presented without translation. On leaving the Seminar he was offered a chair at a local school for teaching Logic, but he declined it and went instead to Ciudad Real to occupy the chair of Logic at an official secondary school. After obtaining in 1846 the civil degree of *Bachiller en Filosofía*, he was appointed *Catedrático* -the Spanish title for full professor- of Logic at the *Instituto del Noviciado* in Madrid, a high school which is still open as such under the name of *Instituto Cardenal Cisneros*, where he wrote the Logic volume of the *Curso* and met in 1850 the mathematician and Professor Acisclo Vallín Bustillo (1825-1895), who introduced him to the world of complex numbers. He went on obtaining different degrees: *Bachiller en Jurisprudencia* (1852), *Licenciado en Jurisprudencia* (1854), and *Licenciado en Filosofía y Letras* (1857). In his private life he was a respectable, not very healthy man who died from tuberculosis, the same illness from which his young wife had passed away in 1854, and at his early death a street in Córdoba received his name. Now it is a busy and long street and very few locals, if any, do know that it wears the name of a mathematician. A portrait of Rey can be seen in Figure 1.

Figure 1: The Spanish philosopher and mathematician José María Rey Heredia (1818-1861), and his signature.

3. Rey's cultural and philosophical background

Rey's *Elementos de Lógica* (*Lógica* in what follows, see Figures 2 and 3) is a text published in 1853 when the two-volume textbook written with Monlau evolved into two separate books, and is the first source to know about the author's cultural and philosophical background and his opinions. *Lógica* had many succesive editions until the end of the 19^{th} Century. For this study a copy of the 12^{th} edition [Rey 1849] has been employed, published by the classical printers Sucesores de Rivadeneyra in Madrid the year 1883, see Figure 2. The text is a treatise on classical Logic encompassing Grammar and Dialectics for use at secondary institutions, and it is aimed to mastering the art of enchaining plausible clauses and maintaining, and hopefully winning, dialectic challenges. The exposition shows a remarkable unity and is written in very good Spanish with the usual 19^{th} Century flourished style. It is divided into four parts or books and a long introductory chapter entitled *Prenociones* where in a schematic though clear way the main concepts and ideas are presented. Figure 3 shows the definition of Logic in the first line of this introduction.



ELEMENTOS

DE LÓGICA

POR

D. JOSÉ MARÍA REY Y HEREDIA

DUODÉCIMA EDICIÓN,

ESMERADAMENTE REVISADA.

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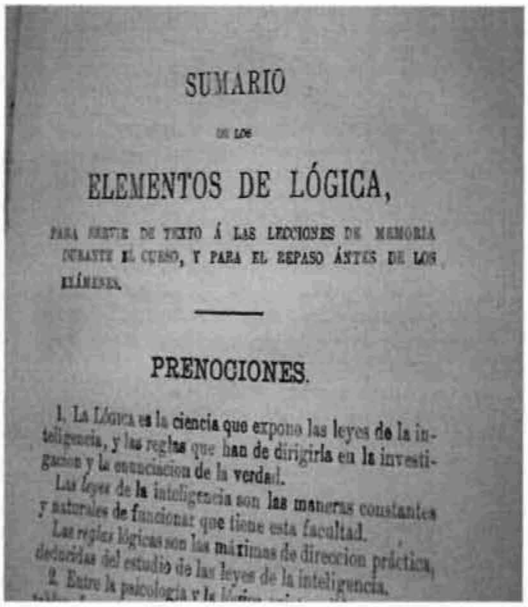

SUMARIO

DE LOS

ELEMENTOS DE LÓGICA,

PARA SERVIR DE TEXTO Á LAS LECCIONES DE MEMORIA DURANTE EL CURSO, Y PARA EL REPASO ÁNTES DE LOS EXÁMENES.

PRENOCIONES.

1. La Lógica es la ciencia que expone las leyes de la inteligencia, y las reglas que han de dirigirla en la investigacion y la enunciacion de la verdad.

Las leyes de la inteligencia son las maneras constantes y naturales de funcionar que tiene esta facultad.

Las reglas lógicas son las máximas de direccion práctica, deducidas del estudio de las leyes de la inteligencia.

2. Entre la psicología y la

Figures 2 and 3: *Lógica*, and the definition of Logic, as expressed in *Lógica*.

From the study of *Lógica* it is difficult to ascribe Rey to any particular philosophical current. Indeed, by the time he wrote the book, in his early thirties, he was aware of various classical problems and of the work of the more known authors, but he seems not to have made up his mind in any definite sense. Moreover, and as the 12^{th} edition shows, Rey did not include anything related with the new trends in Mathematical Logic developed by his contemporary George Boole (1815-1864), whose *An investigation into the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities* was published in 1854. Nevertheless, in p. 256-257 he states, when dealing with the *sorites*

$$(p_1 \text{ } is \text{ } p_2) \wedge \ldots \wedge (p_i \text{ } is \text{ } p_{i+1}) \wedge \ldots \wedge (p_{n-1} \text{ } is \text{ } p_n) \Rightarrow (p_1 \text{ } is \text{ } p_n)$$

or inference through enchained propositions, that :

> *Puede compararse el sorites á una serie de ecuaciones en que concluimos la igualdad de dos extremos por ser iguales á varios términos medios...*[1]

[1] *Sorites* can be compared with a sequence of equations, where we conclude equality of the extremes through the equality of the intermediate terms...

Some studies [Calero 1994] point out at Rey as a late *idéologue*, a follower of the philosophical, linguistic and pedagogical school developed around Antoine Destutt-Tracy (1754-1836) in the final years of the 18^{th} Century on the steps of John Locke (1632-1704), the Abbé Condillac (1715-1780) and others [Picavet 1891]. The group was a most influencing one in the design of educative policies in pre-napoleonic France and had a strong influence in some other countries. For instance, Destutt's book *Élements d'Idéologie* was the source of the Spanish text by the mathematician Juan Justo García (1752-1830) *Elementos de Lógica Verdadera*, practically a translation of Destutt [Cuesta 1980]. published in Madrid the year 1821. García is credited to be among the first to introduce Calculus in Spanish Higher Education [Cuesta 1985]. The influence of this school can be noticed even in texts such as [Colburn 1831] in North America:

> *Success in reasoning depends very much upon the language which is applied to the subject, and also upon the choice of the words that are to be used* (p. 241).

For this study it is quite relevant to quote Rey's *définition idéologique* of a sign (or symbol) in page 114 of *Lógica*:

> *[un signo es una] cosa cualquiera considerada como medio que nos conduce al conocimiento de otra*[2],

an idea used later in *Teoría* when dealing with the various interpretations of the imaginary unit $i = \sqrt{-1}$. Maybe the best choice would be to assess Rey as an eclectic, as shown by the authors cited in *Lógica*, like Pierre Royer-Collard (1763-1845) and Victor Cousin (1792-1867) or Felicité Lamennais (1782-1854), by some dubitative expressions on the still prevalent Port Royal Logic –he supports it on page 124 but is against it in page 118, and by his familiarity with some philosophers of the Scottish School of Common Sense, like Thomas Reid (1710-1796) [Reid 1764].

On reading *Lógica*, the name of Emmanuel Kant (1724-1804) is found only once in the following phrase (p. 232):

> *El paso de las proposiciones a sus contrarias las llama Kant "raciocinio del entendimiento"*[3],

[2] *[A sign is] any thing considered as a means towards the knowledge of another thing.*
[3] *According to Kant, the passage from a proposition to its contrary is a "judgement of our understanding"*

and no mention is made of Transcendental Logic or any other kantian ideas. Thus, the enthusiastic application -even to the point of including an Appendix containing a translation of the first part of the Transcendental Logic extracted from the *Critique of Pure Reason*- of Kant's methods and concepts made by Rey in *Teoría* is a novelty, showing that he must have been among the first to have read Kant, possibly in the early partial translations made after French versions[4] or even directly in French. See [Albares 1996] for an account of the reception of Kant in Spain during the first half of the 19th Century. Therefore, it is uncertain how deeply Rey knew about Kant, though he was later considered and classified as a kantian [Méndez Bejarano 1927]. Indeed he does not doubt when it comes to apply the Theory of Categories to the treatment of imaginary quantities.

Another striking fact about Rey shown in *Teoría*, apart from his quoting Reid, is his knowledge of two British authors counting among the developers of Algebra in a modern sense: George Peacock (1791-1858) and John Warren (1796-1852). The main impression is that during the last decade of his life Rey was able to update his philosophical and mathematical background, quite possibly through preparation to the succesive exams he took in order to obtain his academic degrees.

4. Rey's *Teoría*

Teoría is a beautifully printed book (Figure 4) with XX+330 pages produced the year 1865 by the Imprenta Nacional at Madrid. Rey started working on it during 1855 and devoted the last five years of his life to writing the book, whose Introduction was the part he wrote last. *Teoría* is a piece of very cultivated Spanish and is illustrated with a large amount of line drawings and lithographies supervised by Vallín, who also checked formulas and computations, with little success in various cases, as shown by a detailed study of the book. To Rey, the task of the philosopher / mathematician is stated in the motto attributed to Pierre Simon de Laplace (1749-1824) through F. Coyteux –the author of a book published in Paris by Moreau in 1845: *Exposé d'un systéme philosophique suivi d'une théorie des sentiments ou perceptions*- he quotes at the very beginning of the book:

[4] The first substantial direct translation of the *Critique of Pure Reason* from German into Spanish, although not a complete one, was the work of the Spanish-Cuban politician and philosopher José del Perojo Figueras (1852-1908) and appeared in 1883. There exist several editions of this translation published by Losada in Buenos Aires, the last one dating from 1979. Perojo was elected

Figure 4. Cover page of *Teoria*

Il faut refaire les Mathématiques,les placer sur un nouveau piédestal : Il serait à désirer que ce fût l'oeuvre d'un homme nouveau, qui fût étranger aux mouvements et aux progrès des sciences et n'en connût que les premiers éléments.

This programme, in a certain sense a forerunner of the logicist schools around 1900, seems to be at the basics of all subsequent studies on mathematical foundations. Of course Rey insists on it several times along *Teoría*, and especially in the final part of the Introduction where he wrote his most deep belief on this idea and on the adequacy of the Transcendental Logic to deal with complex numbers.

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twice to the Congress as a representative of Las Palmas, where a statue of him as well as a street with his name can be found.

Rey's *Teoría* cannot be considered as an isolated attempt. There exist more books and surveys of his time trying to make clear the role of imaginary quantities and other sorts of mathematical objects from a foundational viewpoint. In addition to the earlier *Treatise on the geometrical interpretation of the square roots of negative quantities* of 1828 by Warren, there is the long report [Matzka 1851], quoted in [Hoüel 1874], and the interesting Dutch dissertation [de Haan 1863]. In the present state of knowledge, it is impossible to assess whether Rey knew about these two authors, Matzka or de Haan, and most possibly the answer is in the negative, because Rey's sources are usually French or British writers, although no mention, not even an indirect one, is found of William Rowan Hamilton (1805-1865), to quote but one important absence.

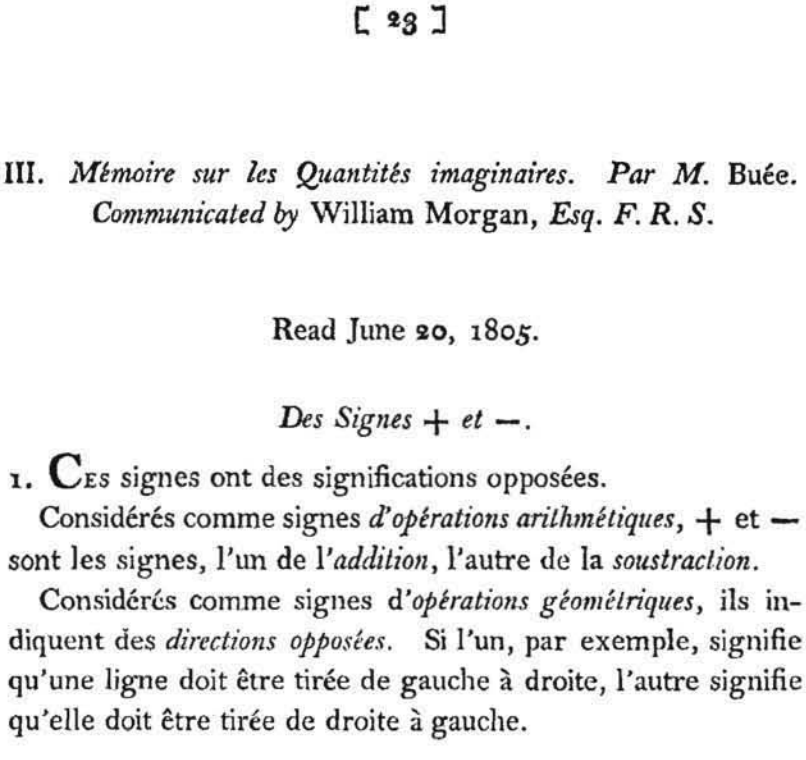
5. The French Connection: Rey and Buée, and Pascal

In his very interesting Introduction to *Teoría* Rey attributes to a certain *Mr. Buée, emigrado francés*[5] the glory of discovering or pointing out, that imaginary quantities can be used to represent perpendicularity.

Adrien-Quentin Buée (1748-1826) was a French priest belonging to the large group of *émigrés*, mainly aristocrats and clergymen who fled from France into England from 1792 onwards as a result of the French Revolution. After that, by 1814 most of them were back in France. Among them there was a number who were intellectuals welcome by the British learned people, and Buée is a clear example. He went to Bath, where he tried to publish in 1799 a leaflet titled *Recherches Mathématiques sur la texture intime des corps*, though it seems that it never came into print. Later on he wrote an article for the Journal of Natural Philosophy, Chemistry and the Arts (1804): *Outlines of Mineralogical Systems of Romé de l'Isle and the Abbé Haüy, with Observations* –by the way, Haüy was another *émigré* and *haüyne*, a bright blue mineral appearing very often in volcanic lava, is named after him- and in 1806 he published in the Philosophical Transactions of the Royal Society the memoir on imaginary numbers [Buée 1806] (Figure 5) that was to inspire Rey's research. It is remarkable that it was published in French; only very few papers are written in languages other than English in this Journal: putting apart the Latin which coexisted with English until well into the

[5] Mr. Buée, a French *émigré*.

18th Century, no more than ten articles were published in French in the whole history of the Transactions, and from 1810 onwards only English is used. Buée appears again in Paris taking care of the 1821 new edition of a *Dictionnaire des termes de la Révolution* he had anonimously printed in 1792 before escaping to England. Information on Buée can be found in [Flament 1982].



[23]

III. *Mémoire sur les Quantités imaginaires. Par M.* Buée. *Communicated by* William Morgan, *Esq. F. R. S.*

Read June 20, 1805.

Des Signes + *et* −.

1. CES signes ont des significations opposées.

Considérés comme signes *d'opérations arithmétiques*, + et − sont les signes, l'un de l'*addition*, l'autre de la *soustraction*.

Considérés comme signes d'*opérations géométriques*, ils indiquent des *directions opposées*. Si l'un, par exemple, signifie qu'une ligne doit être tirée de gauche à droite, l'autre signifie qu'elle doit être tirée de droite à gauche.

Figure 5: First lines of Buée's memory on imaginary numbers.

Buée's views on imaginary numbers conform one of the last episodes in the acceptation of negative and imaginary numbers as valid mathematical entities on metaphysical grounds, a topic that had been dealt with along the previous Century by authors like Abraham de Moivre (1667-1754), a follower of John Wallis (1616-1702) [Smith 1959], Jean Argand (1768-1822) or Caspar Wessel[6] (1745-1818)

[6] In Wessel's book (translation by Zeuthen) the following construction is found: « *Désignons par* +1 *la unité rectiligne positive, par* +ε *une autre unité perpendiculaire à la première et ayant la même origine : alors l'angle de direction de* +1 *sera égal à 0°, celui de -1 à 180°, celui de* +ε *à 90°, et celui de* -ε *to -90° ou à 270° ; et selon la règle que l'angle de direction du produit est égal à la somme de ceux des facteurs, on aura* : (+1)(+1) = +1; (+1)(-1)=-1; (-1)(-1)=+1; (+1)(+ε)=+ε; (+1)(-ε)=-ε; (-1)(+ε)=-ε; (-1)(-ε)=+ε; (+ε)(+ε)=-1; (+ε)(-ε)=+1;(- ε)(+ε)=-1. *Il en résulte que* ε *est égal à* $\sqrt{-1}$ *et que la déviation du produit est déterminée de telle sorte qu'on ne tombe en contradiction avec aucune des règles d'opération ordinaires* ».

[Smith 1953, Wessel 1797] on the side of supporters, while one of the strongest opponents was William Frend (1757-1841) who published his *Principles of Algebra* in 1796. The ideas distilled by Buée and Robert Woodhouse (1773-1827) [Woodhouse 1801 and 1802] were to inspire more than one book, *e.g.* that of Warren, another source for Rey, and parts of Peacock's *Treatise on Algebra* of 1830. Buée is credited as well in Hoüel's *Théorie Élementaire des Quantités Imaginaires*, published in Paris as late as 1874 [Hoüel 1874].

It is surprising the neopithagoric flavour employed by Rey, insisting on it along *Teoría*, emphasising on the single phrase of Blaise Pascal (1623-1662) taken from *Pensées* (Série XXV, nº 592)[Pascal 1977, Vol II, p 138] and cited five times in the book:

Los números imitan al espacio, aunque son de naturaleza tan diferente

which he uses as the ultimate authoritative argument in various chapters of *Teoría*. The original French reads *Les nombres imitent l'espace, qui sont de nature si différente.* A point of interest could be Rey's translation, where the French *qui* is translated into the Spanish *aunque* (though, although). Possibly it is a matter of overemphasising on the different nature of numbers and space. Rey is aware that numbers are the mathematical representation of time, as opposed to space, and he is shocked by the application of numbers to the description of space. Maybe the preparation of *Lógica* with its survey of the Port-Royal Logic induced Rey to a study of Pascal, where he found condensed in a single phrase the Cartesian idea of translating Geometry into equations which he was to follow and develop in *Teoría*. Later on, he must have found it most adequate as a vivid representation of the application of kantian categories to the study of complex numbers.

6. A survey of *Teoría*

Teoría is essentially a large and erudite commentary of Buée's *Mémoire* of 1806, which is closely followed in the introductory parts of the book. From the mathematical and historical perspective the points of higher interest are found in the Introduction, in Book I (Chapters I and II), Book II (Chapter II), and in Book IV. The extant chapters are devoted to an exposition of the usual algebraic algorithms and geometrical interpretations of complex numbers and the most interesting features are the line figures and wonderful lithographies in white on a black background, as well as the intricacies of some complicated explanations.

6.1. The Introduction

The Introduction is a long programmatic text extending along twenty-two pages of very fluid and elaborated Spanish. Divided into thirteen sections devoted to several aspects on the general ideas of Mathematics, the notion of imaginary quantities, their history, the necessity of Metaphysics in Mathematics, some authoritative quotations and phrases with an oratorial flavour, it is the most brilliant part of *Teoría* and is worth reading and enjoying its phrasing.

The first section deals with the exactness of Mathematics and asserts that in some future no obscurities or mysteries will be left in this discipline due to the application of the Transcendental Philosophy:

> *Y esta esperanza se funda en el hecho, cada vez más patente, de que los puntos más obscuros, por menos explorados o más difíciles para la ciencia, son precisamente aquellos que la ponen en contacto con la filosofía del espíritu humano. De esta filosofía transcendental y crítica es de donde únicamente puede venir la luz: hágase un esfuerzo, remuévase el obstáculo, y ella se derramará a torrentes, inundando con claridad igual todo el cuerpo de la ciencia* (p. 2)[7].

Next come the presentation of one of those obscure points, which Rey names *imaginarismo*. In his own words: *el imaginarismo es el "scandalum mathematicum"*. And he concludes:

> *Es necesaria una teoría transcendental del imaginarismo, que salve todas las contradicciones y dé a la ciencia matemática aquel esplendor e integridad a que tiene derecho con mejores títulos que ninguna otra ciencia*[8].

A historical survey of the theoretical development of imaginary quantities follows in sections III and IV. According to it, purely mathematical reasons are not enough basis for the understanding of imaginaries: Some think of them as the expression of the impossibility of solving certain equations, Condillac is an idéologue who thinks that they are senseless signs [Condillac 1769], some others think of them as logical

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[7] And this hope is based on the very patent fact that obscure points [in Mathematics], possibly because they are less explored or more difficult, are those establishing contact with the philosophy of the human soul. Light will be thrown only from this transcendental and critical philosophy: Make en effort, remove the obstacle, and it will torrentially spread, flooding with clarity all the body of science.

[8] There is a need of a transcendental theory of imaginarism to overcome all contradictions, and to endow mathematical science with the splendour and integrity it deserves, for it has better right to them than any other science.

symbols, Hoëne Wronski (1778-1853) is described as trying to submerge imaginary quantities in the realm of infinity, and finally Buée is credited to have discovered the true interpretation of imaginary quantities as a representation of perpendicularity. Other authors who favour this viewpoint are cited, like Joseph Gergonne (1771-1859), Warren and Peacock, M. F. Vallès, who appears twice as the author of an *Essai sur la philosophie du Calcul* (he was also the author of *Des formes imaginaires en Algèbre: Leur interpretation en abstrait et en concret* (Gauthier-Villars, Paris, 1869 and 1876)) and C. V. Mourey, who published in Paris *La vraie théorie des quantités négatives et des quantités prétendues imaginaires* the year 1818. According to Rey he introduced the word *versor* as a name for the argument of a complex number. Two more authors, opposed to this advance, are also considered: Charles Renouvier (1815-1903) [Renouvier 1950], and Augustin Cournot (1801-1877).

Sections V and VI are devoted to a reflection on the nature of Algebra and its ability to cope with the representation of ideas more complex than Arithmetic. Rey decidedly asks and answers this question in a single paragraph:

> *¿Puede haber afección o relación geométrica que no sea representable por el Álgebra? Yo digo resueltamente que no: y presento como demostración el contenido material de este libro...*[9],

and goes on to assert in p. 6 that according to Pierre Simon de Laplace (1749-1824) "*the Metaphysics of these sciences has not yet been made*", and that algebraic notation is endowed with a better analytical ability than purely arithmetical manipulations:

> *Como lengua, alcanza el Álgebra esa superioridad a la que no tiene derecho como ciencia limitada a números*[10].

Finally, and here the *Idéologie* is rather apparent, he states that Algebra is more than a universal language for relationships and properties of numbers, and its universality is not merely a philological one: It is objective, and Algebra deals with matters that cannot be studied only by way of Arithmetic or Geometry, either isolated or jointly considered.

[9] Are there any geometrical relationships or affections not describable through Algebra? Definitely, I say no: And the proof is the material content of this book...[to Rey, *affection* means modification of the meaning of a arithmetical entity by some additional information, *e.g.* the minus sign in front of a positive number changes the sense of its measure on the line]

[10] As a language, Algebra attains a superiority to which it has no right as a science limited to numbers. This is taken straight away from p. 25 of Buée's *Mémoire* .

In order to introduce the kantian scheme, two more sections, VII and VIII, follow where doing mathematics is postulated as an innate quality of human mind:

> *...este conocimiento es formado por la virtualidad propia de nuestro espíritu, sin deber a la experiencia más que la ocasionalidad de su excitación; verdad contra la cual tanto se rebela el espíritu sensualista y empírico que heredaron de la Filosofía del siglo anterior los matemáticos, más que ningún otro linaje de pensadores.* [11]

According to Rey our minds are more mathematical than we know, because our mental structures rely more on the logical truth of judgements and propositions, in opposition to empirical science, which proceeds slowly through experience. This has so reduced an extension that the universality and necessity of mathematical propositions overflows it. Transcendental Philosophy, when trying to explain this special nature of mathematical knowledge, is forced to recognise the purely subjective and formal nature of the grounds of geometrical and arithmetical intuitions: *space* and *time*. These two concepts do not derive from experience, they are prior to it and are transcendental conditions thereof.

The intuition of space, once stripped of any phenomenic or sensible element, is that of a simultaneous, infinite, and homogeneous capacity where nothing is predetermined. On the other hand, the pure intuition of time is that of an infinite sequence of essentially succesive and transitory moments. Rey notes that

> *En esta serie están determinadas las cosas como numerables: la numeración es imposible sin la síntesis sucesiva de la unidad consigo misma...* [12]

The idea of synthesis of the unit with itself is clearly a translation of addition, and is found at the basis of axiomatic descriptions of natural numbers, like the well known Peano formulation. Rey uses this conception along *Teoría*, in a picturesque language where *e.g.* the fact that $1 \times 1 = 1$ is described like "the barrenness of unity under production" and other similar expressions.

A succint historical description of Algebra is found in section IX, where after considering Descartes as the discoverer of Scientific Algebra and Euclid as the first instance where Arithmetic and Geometry are sister disciplines, he

[11] This knowledge [the mathematical one] is built by the innate powers of our spirit, it owes to experience only the occasion to excite them; and against this truth rebels the sensualist and empiricist spirit inherited by mathematicians –more than by any other scientific lineage- from last Century's Philosophy.

concludes with the above quoted phrase of Pascal, which appears here for the first time in the book (p. 16).

Section X starts by asking the question of how could the problem of imaginary numbers be settled in his Pascalian programme. Here is the solution:

> *La teoría geométrica de la posición, o analysis sitûs, que era el gran desiderátum de Leibnitz[...]llegará a convertirse en una teoría algébrica ordinaria, desde el momento en que se nos den signos para todas las posiciones de las rectas en el plano indefinido o en el espacio, con tal que estos signos no sean arbitrarios, sino engendrados por el mismo cálculo como resultados algorítmicos necesarios de las teorías de los números.*[13]

Therefore, a theory of signs must be built, and the basics thereof must take into account the concept of quality in order to represent every "affection" or direction: The road is now free for the appearance of Transcendental Logic and the kantian categories. This is the aim of the last two sections, XII and XIII, where a rather hard vindication of Metaphysics as the root of Mathematics is presented. The following excerpt makes it clear (p. 20):

> *¡Horror a la Metafísica los matemáticos, cuando ellos sin quererlo, y sin saberlo, son los primeros metafísicos! ¡Cuando las ideas de espacio, tiempo, movimiento, nada, é infinito aparecen a cada momento como enredadas entre sus teorías, ó siendo la trama de sus cálculos!*[14]

And he ends: The best mathematicians praised by History were deep metaphysicists as well, and it is certain that Gottfried Leibnitz (1646-1716), René Descartes (1596-1650), Isaac Newton (1642-1727), Pascal or Leonhard Euler (1707-1783) would not have been so higher mathematicians if their philosophical insight had not been as deep as it was. Moreover: Who ignores that Kant was a mathematician before becoming a criticist? And the same can be said of Reid, Georg Hegel (1770-1831) and Karl Krause (1781-1832). And in p. 22:

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[12] In this sequence things are enumerably determined: enumeration is impossible without the *succesive synthesis* of unit with itself... Synthesis is a very frequent word in *Teoría*, where it is used with the general meaning of "sum, combination,..."

[13] Geometrical position theory or *analysis sitûs*, a *desideratum* of Leibnitz [...] will become an ordinary algebraic theory from the moment we are given signs for every possible direction in the plane or in the space, taking care that these signs are not arbitrarily chosen, but derived algorithmically from computations carried on with numbers.

[14] How come that mathematicians abhorr Metaphysics, when they are –without wanting or knowing it- the very first metaphysicists! Moreover, when ideas like space, time, force, movement, *nothing*, and infinity appear like braided in their calculations!

La crítica kantiana es un admirable trabajo matemático[15]

Finally, a defence of the kantian critique based on its mathematical value is expounded and the Introduction closes with a wish for a long and fruitful alliance between Philosophy and Mathematics.

6. 2. Some aspects in the body of *Teoría*

The contents in Book I are very close to those of Buée's *Mémoire*, and Rey's goal is to state the definitions and results as stemming in a natural way from the intellectual framework of categories.

According to Rey and his inspirer Buée, imaginary quantities must be understood through an application of the quality concept. Here, quality is an addition to the quantity itself, considered as a modifying attribute or adjective, in such a way that the arithmetical property of number known as quantity is completed with a geometrical viewpoint as well. The simplest case is that of negative numbers, where the minus sign indicates reversal of the sense used to represent the original numbers as segments emanating from a definite point in a line. Observe that an important difference between the nomenclature used by Buée or Rey and contemporary use is that *line* was used to mean a finite segment, and *direction* meant a choice in the sense used to define the segment once a fixed end was given. The status of negative quantities was thoroughly studied by the end of the 18th Century, *e.g* in the long time forgotten memory of Wessel published in Copenhagen, and in 1802 the philosopher and later freemason Krause, quoted by Rey in pp. 22 and 32 of *Teoría*, read an *Habilitationsschrift* in Jena dealing with this topic with the title *De philosophiae et matheseos notione et earum intima coniunctione*. An interesting account on the romantic character of the subject can be read in [Dhombres 2003]. Krause, who was to become an influential philosopher in some intellectual Spanish and Latin American circles during the 19th Century, wrote a second essay in 1804 on the relationship between Philosophy and Mathematics under the title *Grundlagen eines philosophischen Systems der Mathematik*. The sound logical foundations were established by Carl Friedrich Gauss (1770-1855) [Gauss 1831], and the field structure of the complex numbers, by Hamilton around the same time.

[15] Kant's critique is an admirable mathematical piece.

When a mathematical object such as a number, obtained as a result of manipulations, appears endowed with both a quantitative meaning and a qualitative one, the main concern is the accomodation of computation rules to this new situation in such a way that the old rules are conserved as particular cases in this new, more general, framework. This property, already observed by Wessel (see footnote 6), was first stated by Peacock, and is known as "the conservation of formal laws". It deserved a long commentary by Hermann Hankel (1839-1873) published in 1867, the *Prinzip der Permanenz der formalen Gesetzten*[16]. The rule for signs in multiplication exemplifies a particular case[17] and the rules of Arithmetic for complex numbers are another one. Rey´s attempt was to base these rules in the kantian categorical framework, to whose object he chose two categories and their associated judgements:

Mathematical Categories	**Subcategories**	**Judgements**
Quantity	Unity	Universal
	Plurality	Particular
	Totality	Singular
Quality	Reality	Affirmative
	Negation	Negative
	Limitation	**Limitative**

Rey's definitive tool is provided by limitative judgements, where he finds the necessary distinction between "not (to be *B*)" and "to be (not *B*)" he exploits as the origin of the qualitative interpretation of the sign $\sqrt{-1}$. All possible directions in the plane[18] are the representation of the various qualities, so the choice of a given

[16] *„Wenn zwei in allgemeinen Zeichen der arithmetica universalis ausgedrückte Formen einander gleich sind, so sollen sie einander auch gleich bleiben, wenn die Zeichen aufhören, einfache Grössen zu bezeichnen und daher auch die Operationen einen irgend welchen anderen Inhalt bekommen".*

[17] Rey cites Jean LeRond D'Alembert (1717-1783) and Lazare Carnot (1753-1823) in p. 33 of *Teoría* in relationship with the problematic interpretation of this rule.

[18] A natural question is why restrict the theory to the plane. Extending it to say three dimensional space could have led Rey to quaternions...

axis will represent a particular quality B, where the origin O separates the ideas of "positively having it" or "negatively having it". Thus, if $R(A,B)$ means that a point A has been chosen in the positive half of the B axis, $R(A,\neg B)$ will mean that A has been chosen in the negative half. For any direction other than B, the actual situation is $\neg R(A,B)$, so necessarily the relationship

$$R(A,\neg B) \neq \neg R(A,B)$$

will hold. Nevertheless, degrees in being or not being B do actually exist[19], and they are represented by all directions defined by the origin and the points in a circle passing through O and whose centre C is such that $R(C,B)$. Therefore, the only direction whose representative line has no common point with the circle other than the origin will be the perpendicular to the axis through O, and it will represent complete indifference with respect to the quality B. This leads Rey to the provisional and arbitrary adoption of $\sqrt{-1}$ as a sign of quality, on equal foot with + or –, denoting *perpendicularity as the representation of total indifference* towards the quality B.

The above idea, though an appealing one, is of little use in everyday mathematical practice. Thus Rey embarks in a justification of this interpretation by combining it with the classical problem of extracting the square root of a negative number. In his own words (*Teoría*, p. 51):

> *En el anterior capítulo he considerado el símbolo* $\sqrt{-1}$ *como un signo de limitación ó de neutralidad perfecta, entre la afección positiva y la negativa, ó como la expresión más propia de un grado máximo de indiferencia respecto de aquellas direcciones fundamentales. Sin embargo, por su forma radical revela el signo* $\sqrt{-1}$ *un origen algorítmico potencial, y simboliza la totalidad de una teoría inmensa, de la cual no son sino determinaciones particulares los tres momentos típicos representados por los signos* $+, -, \pm\sqrt{-1}$ *correspondientes a los tres conceptos, realidad, negación y*

[19] In a certain sense, Fuzzy Logic was been used many years before it came into being.

limitación. En la teoría de la potencialidad, ó sea de la graduación, está el germen de la teoría cualitativa[20].

This excursion into the symbolic quality of $\sqrt{-1}$ and its possible connection with algebraic, nonmetaphysical questions, is the core of Rey's argumentation and reveals a philological, semiotic and *idéologique* origin in his research.

Now to the construction of the square root of -1. Rey argues that the square root of -1 cannot be either -1 or $+1$, for *según la doctrina comun y verdadera de los algebristas*[21], or rule of signs, the square of any of them will be $+1$. Therefore –in the same way that Wessel and Buée had indicated sixty years before- the root must lie in some direction different from the one defined by -1 and $+1$, and he goes on in his rhetoric style:

> *...la posición indirecta, que corresponde a la raíz $\sqrt{-1}$, debe constituir un grado tal de evolución cualitativa respecto de la unidad positiva, que repetido este grado en igual razón y manera progresiva, constituya a su vez la segunda potencia -1...*[22]

Note in this text the inadequate usage of $\sqrt{-1}$, in an instance where the expression "the square root of -1", rather than the symbol, should have been used. This error shows that Rey had some difficulties in distinguishing concepts when he tried to put them in mathematical formulation. In any case, the rationale in this paragraph is that the positive unity $+1$ suffers a qualitative evolution that succesively takes it into other directions before arriving at the negative part of the axis. Of course Rey does not mention it explicitly, but it is plain that he is thinking of rotating the positive axis around the origin without leaving the plane, an idea due to Carl Friedrich Gauss (1777-1855) [Gauss 1831]. According to this picture, the square root of -1 will be obtained after some particular evolution, and upon iteration of this same evolution on the square root, the square -1 will be obtained

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[20] In the previous chapter I considered the symbol $\sqrt{-1}$ as a sign of limitation or perfect neutrality between the positive and the negative affections, or as the most adequate expression of the maximum degree of indifference towards these fundamental directions. Nevertheless, the radical form of the symbol reveals a potential (with the form of a power) algorithmic origin, and is the symbol of a immense theory, from which the three typical moments $+, -, \pm\sqrt{-1}$ are particular determinations corresponding to the concepts of *reality*, *negation*, and *limitation*. The germ of the qualitative theory is in the theory of potentiality, *i.e.* of graduation (raising to powers).

[21] ...according to the common and true doctrine of algebrists...

in a supposedly natural way. But the only direction having this property is that at right angles, and he concludes:

Luego la perpendicular representa la raíz segunda de -1, *y el símbolo radical* $\sqrt{-1}$ *es a su vez genuina expresión de la perpendicularidad.*[23]

Next, the above deduction is performed in mathematical notation, with a pretension of generality: Let a be a quantity representing the square root of $-a$, and let p be the *coefficient or particular evolution* that acts on a for it to become the square root of $-a$, *i.e.* $p(a) = pa = SQRT(-a)$. Then,

...es claro que, repitiendo esta modificación sobre la raíz, quedará constituida la segunda potencia $-a$, *y se tendrá...*[24]

In formulas, this is written as:

$$p(p(a)) = p(pa) = p^2(a) = p^2 a = -a$$

and on dividing the last equation through a the desired result will appear:

$$p^2 = -1\text{, or equivalently, } p = \pm\sqrt{-1}$$

A critical study of this argument is worth, because there are several flaws or obscure points in it:

- No mention is made to the point that a must be a positive quantity. Otherwise the argument, which amounts to two consecutive rotations of equal angle, will fail.
- More important is the fact that the above argument will not work for $a \neq 1$. In order to comply with the usual rules of extracting roots, a modification on the magnitude of a must happen simultaneously to the rotation. Suppose *e.g.* that $a > 1$. Then, in the first application, the action of p is split into a rotation and a shrinking $a \to +\sqrt{a}$. Therefore in the second application p

[22] ...the indirect (oblique) position corresponding to the square root must be obtained by a *degree of qualitative evolution of the positive unity* such that, on repeating it, the square number -1 will be obtained...

[23] Therefore the perpendicular represents the square root of -1, and the radical symbol $\sqrt{-1}$ is in turn a genuine expression of perpendicularity.

should perform a stretching $+\sqrt{a} \to a$ and a rotation. In other words, it would be a different p!

- The equation $p^2 = -1$ is not an equation involving numbers: it is a purely formal one, and its correct meaning is that the iteration of some operator acting on a positive quantity yields the opposite thereof, so the algebraic origin of $\sqrt{-1}$ remains hidden in the dark.

The original argument in Buée's *Mémoire* (p. 28 and 29), on which this explanation is inspired, is much simpler and less elaborate. Stripped of rhetoric, the essential of both approaches amounts to applying the numerical idea of proportional means by extending it to operator-like entities whose models are rotations of the plane around a fixed point.

Algebraic manipulation of complex numbers is studied in Books II, III, and IV of *Teoría*. The names used by Rey are: *Synthesis* for addition, *production* for multiplication, and *graduation* for raising to powers. Apart from this nomenclature, Rey presents and explains the componentwise addition and the multiplicative rule as *syncategorematic* operations, thus indicating that they consider jointly both quantitative and qualitative aspects of complex numbers. Already in 1831 Cauchy had studied complex numbers as ordered pairs [Cauchy 1831] [Pérez-Fernández 1999]. Apart from this philosophical touch, all topics in Books II and III are commonplace and are classically considered. Book IV, dealing with *graduation* or raising to a power, is much more interesting.

6. 3. Book IV and Rey's mathematical abilities

The reader can observe in Book IV the real mathematical background of Rey and to which extent he had understood it. The Book starts with some general ideas on graduation and how it works for positive quantities, or "quantitative graduation" and for negative ones, or "qualitative graduation". Indeed the difference is that qualitative graduation displays an oscillating behaviour. On extending graduation to imaginary quantities (p. 169 ff.) the periodic behaviour of powers of $\sqrt{-1}$ is established after a lengthy preamble where the validity of the operation according

[24] ...it is clear that by repeating this modification upon the root, $-a$, the square, will be constituted...

to Rey's original plan is acknowledged. A few pages are devoted to *involution* or root extraction, as the operation inverse to graduation.

In p. 179, in order to find the binomial form of $\left[\sqrt{-1}\right]^{\frac{1}{m}}$ Rey does not hesitate to use *an infinite number of times* the binomial or Newton series expansion for

$$\left[\sqrt{-1}\right]^{\frac{1}{m}} = \left[1-(1-\sqrt{-1})\right]^{\frac{1}{m}},$$

in a first instance for the global expression, and then substituting the corresponding expansion for every appearance of the expression $(1-\sqrt{-1})^{p}$. The reader is forced to conclude that to Rey the authority argument represented by invoking the name of Newton dominates every other consideration on convergence, actual infinities or any other mathematical ideas. This is not a case of Metaphysics: he is simply using well established rules, and this is reason enough to accept them without further thought. This was all right in Buée's times, but it should be reminded that Augustin Cauchy (1789-1857) had already attempted to introduce rigour in these matters forty years before.

Cyclotomic polynomials are also extensively considered, and a trigonometric theory of imaginaries is presented (p. 233 ff.), a topic addressed by all authors studying complex quantities. It is found in Buée, and had been common fare in academic sessions, as shown *e.g.* in [Svanberg 1812]. Graphical constructions also deserve a lot of attention, and then Chapter VII on infinite graduation of imaginary quantities starts.

Under the rather mysterious title *Inevolubilidad esencial de la unidad bajo el concepto de cantidad*[25] Rey explains that the unit 1 remains the same regardless of the exponent –even an infinite one- when graduation $(1)^m$ is performed. Now Rey asserts that if unity is perturbed by the addition of an infinitesimal quantity, infinite graduation will fruitfully evolve, so he proceeds on to produce the marvellous formula [Euler 1748, Chapter VII] :

$$\left(1+\frac{\mu}{\infty}\right)^{\infty} = 1+\frac{\infty}{1}\times\frac{\mu}{\infty}+\frac{\infty^2}{1\cdot 2}\times\frac{\mu^2}{\infty^2}+\frac{\infty^3}{1\cdot 2\cdot 3}\times\frac{\mu^3}{\infty^3}+\ldots$$

where working *à la* Euler infinite quantities cancel, and upon setting $\mu = 1$ the number e is obtained. Then the fundamental property

$$e^{p} = \left(1 + \frac{p}{\infty}\right)^{\infty}$$

is established and a remark on the practical computation of logarithms is presented.

In p. 203 a new version of the mysterious heading appears: *Inevolubilidad de la unidad positiva bajo el concepto de calidad. Número e cualitativo*[26]. It is natural that Rey should try to eliminate such an undesirable non evolutionary behaviour by adapting the above reasoning for the case $\mu = \sqrt{-1}$. In the same mood he defines the *qualitative* number :

$$\mathsf{E} = \left(1 + \frac{\sqrt{-1}}{\infty}\right)^{\infty} = 1 + \frac{\sqrt{-1}}{1} - \frac{1}{1 \cdot 2} - \frac{\sqrt{-1}}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$$

And when it comes to find the binomial expression for E, he wishfully writes:

> *En esta serie, cuya estructura es ya conocida [...] es posible reconocer la convergencia de los términos reales cuya suma es α, y de los coeficientes de los imaginarios cuya suma es β, a un punto común en que ambas sumas se hacen iguales; de suerte que en el límite en que esto se verifica es un punto de tal naturaleza, que respecto de él se tiene $\alpha = \beta$ y la expresión sumatoria de la serie infinita es* $\left(1 + \frac{\sqrt{-1}}{\infty}\right)^{\infty} = \alpha + \alpha\sqrt{-1}$ [27].

Of course this is not true. Rey's phrasing "it is possible to recognise..." would have better been applied to recognise in E the exponential

$$e^{\sqrt{-1}} = 0.540302\ldots + 0.841471\ldots\sqrt{-1},$$

[25] The essential lack of evolution of unity under the concept of quantity.
[26] The lack of evolution of unity under the concept of quality. Qualitative version of e.
[27] In this expansion it is possible to recognise the convergence of the real terms and of the coefficients of the imaginary ones to a common point where both sums become equal...

the unit-like complex number $1_{\theta=1}$ with unit module and argument one radian. This spoils the beautiful symmetry of the theory to be developed from Rey's expected value $\mathbf{E}=\sqrt{\sqrt{-1}}$.

The final paragraphs of the book are equally disappointing. The ideas of Buée and Argand suggest immediately the possibility of extending to space the theory of perpendicularity: Is it possible to represent a line perpendicular to a given plane by means of some particular complex number $a+b\sqrt{-1}$? Many years before, in 1813, François Servois (1768-1847) had tried to answer the question in the negative, establishing a precedent of the theory of quaternions later developed by Hamilton. According to Rey, the double exponential $\left(\sqrt{-1}\right)^{\sqrt{-1}}$ does the job, and he justifies it in this cryptic paragraph (p. 284):

> *Que tal debe ser la evolución, se infiere del efecto natural de los exponentes reales sobre una base imaginaria. Si el exponente positivo le imprime una moción progresiva, el exponente cero la coloca sobre el radio, y el negativo la hace retrogradar por bajo de esta posición; el imaginario la elevará desde este origen de los arcos por un arco que puede llamarse imaginario como perpendicular a los arcos positivo y negativo.*[28]

Therefore the six quantities,

$$\left\{+1,-1,+\sqrt{-1},-\sqrt{-1},+\left(\sqrt{-1}\right)^{\sqrt{-1}},-\left(\sqrt{-1}\right)^{\sqrt{-1}}\right\}$$

are postulated as a system from which Space Geometry could be derived. Needless to say, Rey never attempts to prove that $\left(\sqrt{-1}\right)^{\sqrt{-1}}$ can be expressed in the form $a+b\sqrt{-1}$: without further check, he simply thinks of it as a new species of complex or imaginary quantity. And so did some of his followers (see below), even if they knew that $\left(\sqrt{-1}\right)^{\sqrt{-1}}$ is a real number.

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[28] That this must be the evolution can be inferred from the natural effect of real exponents upon an imaginary basis. If a positive exponent impresses a progressive motion, a null one will leave it on the radius, and a negative one will impress a retrograde movement under this position; an imaginary one will elevate it from this origin of all arcs following an arc that can be called imaginary because it is perpendicular to the positive and the imaginary arcs.

In p. 291, just before the translation of Kant and the final glossary, the author presents a synthesis of his work in three mottos and their attribution to various intellectual realms:

1. The symbol $\sqrt{-1}$ is a sign of perpendicularity (Buée)[29].
2. Numbers imitate space, though their natures are so different (Pascal).
3. The tableau of categories indicates every moment of any projected speculative, even providing its ordering and regime (Kant)[30].

And then, to where they belong:

- The first one is a pure mathematical thought.
- The second belongs to Mathematical Philosophy.
- The third, to Transcendental Philosophy.

7. Some epigons of Rey

The theories of Rey enjoyed a certain popularity in the following years, and several mathematicians incorporated some of his ideas in their texts, as stated in the survey [García de Galdeano 1899] and in the words of Julio Rey Pastor (1888-1962), one of the founders of contemporary Spanish Mathematics [Rey Pastor 1915]. Here some of those followers are presented, even at the risk of forgetting more than one:

7.1. Rochano

José Antonio Rochano Alemany, who was a Professor in Granada, appears as a very devoted follower of Rey, with whom he had been apparently acquainted. In 1870 he published his *Elementos de Algebra* [Rochano 1870], a text which is a mixture of a close copy of *Teoría* and more common algebraic topics, like GCDs and other usual concepts. Rochano cites Rey explicitly several times along the text,

[29] Ainsi $\sqrt{-1}$ est le signe de la perpendicularité, dont... [Buée 1806], p. 28.

[30] Rey offers a compressed version of the original sentence of Section 3, Paragraph 11 in *Des transzendentales Leitfadens der Entdeckung aller reinen Verstandbegriffe*: *„Denn daß diese Tafel im theoretischen Teile der Philosophie ungemein dienlich, ja unentbehrlich sei, den Plan zum Ganzen einer Wissenschaft, sofern sie auf Begriffen a priori beruht, vollständig zu entwerfen, und sie mathematisch nach bestimmten Prinzipien abzuteilen, erhellt schon von selbst daraus, daß gedachte Tafel alle Elementarbegriffe des Verstandes vollständig, ja selbst die Form eines Systems derselben im menschlichen Verstande enthält, folglich auf alle Momente einer vorhabenden spekulativen Wissenschaft, ja sogar ihre Ordnung, Anweisung gibt, wie ich denn auch davon anderwärts eine Probe gegeben habe".*

and emphasises in the most polemic aspects presented in *Teoría*, such as spherical imaginaries, but goes no further than Rey. During this study no other mathematical activity of Rochano has been found. Next, a sample of his writing (p. 91) is shown, the introduction to a comment on the fact that $\sqrt{-a} \times \sqrt{-b}$ is a real number:

> *Consecuencia importante. Luego el producto de dos imaginarios es REAL. ¿Cómo se ha obrado este milagro? ¿Con que dos cantidades simbólicas, quiméricas, imposibles, visionarias, combinadas por la vía de la producción, dan una cantidad que ya no es quimérica y visionaria?...*[31]

7.2. Domínguez Hervella

The Navy officer Modesto Domínguez Hervella, published some manuscript lecture notes around 1870 under the title *Elementos de Geometría analítica*, which were later put in print with the same title in 1879 [Domínguez 1879]. A copy of this last one has been used. The book starts with a quotation from the Introduction of *Teoría*, and many illustrations are clearly inspired in the original ones of Rey's text. Hervella is much more technical than Rey or Rochano, but he still sticks to the use of $\left(\sqrt{-1}\right)^{\sqrt{-1}}$ for the study of space Geometry, even when he is aware of the non-imaginary nature of this quantity. In a report of the *Real Academia de Ciencias* included in the book[32], an anonymous reviewer reproaches the author for not adopting Hamilton's system instead of the old fashioned, complicated, and erroneous double imaginary.

7.3. Navarro

Luciano Navarro Izquierdo, a Professor at Salamanca, published in 1874 a *Tratado de Geometría elemental y Trigonometría rectilínea y esférica*, a book that had two more editions in 1883 and 1887. For this study a copy of the second edition has been used [Navarro 1883]. This author uses freely ideas taken from Rey, and although he does not explicitly cite him, he employs complex numbers in routine trigonometric computations, as in deriving the sine and cosine of the double angle. Moreover, he was the author of a *Tratado de Aritmética y Algebra*

[31] An important consequence: How did this miracle happen? How come that the combination through production of two symbolic, chimerical, impossible, visionary quantities can yield a quantity that is no more chimerical and visionary?...

[32] A positive report opened the door to automatically selling three hundred copies for public libraries all over Spain.

appeared in 1875, and a report in 1886 on the procedure of Bernardino de Sena for the approximation of square roots. Here is a sample of Navarro's style in *Geometría* (p. 23), where he explains the concept of direction considered as a quality:

> *Puesto que la cualidad de la cantidad no es otra cosa que su modo de ser u obrar con relación a la tomada por unidad, es evidente que toda recta, cuya dirección sea la misma que la de la unidad, es de la misma cualidad que ésta, siendo de cualidad distinta si su dirección es diferente: la cualidad de una recta no es otra cosa, pues, que la dirección de la misma respecto de la unidad.*[33]

Then he proceeds with the same formal calculation yielding $\sqrt{-1}$ as the mark of perpendicularity.

7.4. Lubelza and Pérez de Muñoz

Next come the mining engineer Joaquín Lubelza, who was a Professor at the *Escuela de Minas* in Madrid. He was the author of a nearly modern Calculus text published in 1905 [Lubelza 1905] with the title *Cálculo Infinitesimal*, a text he had previously distributed in lecture notes form around 1901. Quite interestingly, the Introduction deals with quaternions, and Rey is cited as having discovered the syncategorematic character of vector sums (instead of vector, the word *rector* is used throughout). Lubelza wrote a booklet on Mechanics in 1901, as well as an article in 1902 on centrifugal forces.

To end with Rey's epigons, another mining engineer, Ramón Pérez de Muñoz published in 1914 his *Elementos de Cálculo Infinitesimal*, [Pérez de Muñoz 1914] where the Introduction on quaternions was changed into a survey of Cantor's set theory, the name vector was adopted and quaternions were relegated to a small Appendix. Significantly enough, no mention to Rey is made in this book.

[33] Given that the quality of a quantity is nothing but the way it behaves in relation to the one taken as unity, it is obvious that every segment whose direction is the same as that of unity, has the same quality as this one...: the quality of a segment is its direction relative to unity [This is a complicated and obscure paragraph, much in the style of Rey].

8. Conclusions

It is known (see *e.g.* [Pacheco *et al.* 2005]) that during the first half of the 19th Century the study, teaching and dissemination of Mathematics in Spain was by no means an organised activity, and only a handful of isolated characters worked in this field of knowledge, mostly in military schools or in the secondary teaching system. Advances occurring in Europe arrived late and often through second-hand reports, and it is no surprise that Spanish authors stuck to certain attitudes preferring either to follow 18th Century Mathematics in its most practical aspects, or to study some problems half way between Mathematics and Philosophy which at first sight did not require much mathematical ability.

In this paper one such case has been considered: The reception of metaphysical explanations of complex numbers and how they influenced Spanish Mathematics for over fifty years. The foundational crisis of the second half of the 19th Century had little impact in Spain [Pacheco 1991], very possibly due to the poor technological development of the country, for it must be reminded that a great deal of problems in the foundations of Mathematics had deep roots in the flourishing of several classes of machines [Pacheco and Fernández 2005]. One of the outcomes of the crisis was the exigence of rigour in both senses addressed in the Introduction in order to consistently study the mathematical models underlying the functioning of the new technologies. The work of Rey tried to be rigorous in the metaphysical and in the logical side, but failed in the application of "mathematical" rigour, as shown in Nº 6. Thus, it is no wonder that Rey's book could enjoy a lasting popularity well above his mathematical value, as shown in [García de Galdeano 1899] and in [Rey Pastor 1915]. This seems to be a case where the distinction pointed out between history and heritage in [Grattan-Guinness 2004] is perfectly applicable. Nevertheless, the merit of Rey must be recognised when facing a difficult and hard problem with the naïve appreciation that it could be solved from scratch through the straightforward application of purely philosophical techniques. His courage deserves a place in the history of Spanish Mathematics of the 19th Century, even if the final result was not as brilliant as expected and many a problem remained unsolved.

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