INTERNATIONAL UNION OF PURE AND APPLIED CHEMISTRY

MACROMOLECULAR DIVISION
COMMISSION ON MACROMOLECULAR NOMENCLATURE*

DEFINITION OF TERMS RELATING TO THE NON-ULTIMATE MECHANICAL PROPERTIES OF POLYMERS

(IUPAC Recommendations 1998)

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Definition of terms relating to the non-ultimate mechanical properties of polymers (IUPAC Recommendations 1998)

DEFINITION OF TERMS RELATING TO THE NON-ULTIMATE MECHANICAL PROPERTIES OF POLYMERS

SUMMARY

The document gives definitions of terms related to the non-ultimate mechanical behaviour of polymeric materials, in particular bulk polymers and concentrated solutions and their elastic and viscoelastic properties.

The terms which have been selected are those met in the conventional mechanical characterization of isotropic polymeric materials. They have additionally been limited to those which can be defined precisely and with mathematical rigour. They are arranged in sections dealing with basic definitions of stress and strain, deformations used experimentally, stresses observed experimentally, quantities relating stress and deformation, linear viscoelastic behaviour, and oscillatory deformations and stresses used experimentally for solids.

An index, an alphabetical list of terms and a glossary of symbols are included for ease of reference.

INTRODUCTION

This document gives definitions of terms related to the non-ultimate mechanical behaviour or mechanical behaviour prior to failure of polymeric materials, in particular bulk polymers and concentrated solutions and their elastic and viscoelastic properties.

The terms are arranged into sections dealing with basic definitions of stress and strain, deformations used experimentally, stresses observed experimentally, quantities relating stress and deformation, linear viscoelastic behaviour, and oscillatory deformations and stresses used experimentally for solids. The terms which have been selected are those met in the conventional mechanical characterization of polymeric materials.

To compile the definitions, a number of sources have been used. A number of the definitions were adapted from an International Standards Organization (ISO) manuscript on Plastics Vocabulary[1]. Where possible, the names for properties, their definitions and the symbols for linear viscoelastic properties were checked against past compilations of terminology[2-6]. Other documents consulted include ASTM publications [7-13]

The document does not deal with the properties of anisotropic materials. This is an extensive subject in its own right and the reader is referred to specialized texts [14,15] for information.

In the list of contents, main terms separated by / are alternative names, and terms in parentheses give those which are defined in the context of main terms, usually as notes to the definitions of main terms, with their names printed in bold type in the main text. Multicomponent quantities (vectors, tensors, matrices) are printed in bold type. Names printed in italics are defined elsewhere in the document and their definitions can be found by reference to the alphabetical list of terms.

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USED EXPERIMENTALLY FOR SOLIDS

1. BASIC DEFINITIONS

In this section, quantities are expressed with respect to rectangular Cartesian co-ordinate axes, Ox_1, Ox_2, Ox_3, except where otherwise stated. The components of a vector V are denoted V_1, V_2, and V_3 with respect to these axes.

1.1 traction
stress vector

Recommended symbol: t

A vector force per unit area on an infinitesimal element of area that has a given normal and is at a given point in a body.

Unit: Pa

Note

1. The components of t are written as t_1, t_2, t_3.

2. t is sometimes called true stress. The term *traction* (or *stress vector*) is preferred to avoid confusion with *stress tensor* (see 1.2 note 5)

1.2 stress tensor
stress

Recommended symbol: σ

The tensor with components σ_{ij} which are the components of the traction in the Ox_i direction on an element of area whose normal is in the Ox_j direction.

Unit: Pa

Notes

1. A unit vector area with normal n can be resolved into three smaller areas equal to n_1, n_2, and n_3 with normals in the directions of the respective co-ordinate axes. Accordingly, each component of the traction on the original area can be considered as the sum of components in the same direction on the smaller areas to give

$$t_i = \sum_{j=1}^{3} \sigma_{ij} n_j \quad , \quad i = 1, 2, 3.$$

2. In usual circumstances, in the absence of body couples, $\sigma_{ij} = \sigma_{ji}$.

3. For a homogeneous stress σ is the same at all points in a body.

4. For an inhomogeneous stress $\sigma_{ij} = \sigma_{ij}(x_1, x_2, x_3)$.

5. σ is a **true stress** because its components are forces per unit current area (cf. 3,4).

6. If $\sigma_{13} (= \sigma_{31}) = \sigma_{23} (= \sigma_{32}) = \sigma_{33} = 0$ then the stress is called a **plane stress**. Plane stresses are associated with the deformation of a sheet of material in the plane of the sheet.

1.3   deformation of an elastic solid

A deformation of an elastic solid through which a mass point of the solid with co-ordinates X_1, X_2, X_3 in the undeformed state moves to a point with co-ordinates x_1, x_2, x_3 in the deformed state and the deformation is defined by

$$x_i = x_i (X_1, X_2, X_3) , \quad i = 1, 2, 3$$

Notes

1. A **homogeneous deformation** is one in which the relationships between the co-ordinates in the undeformed and deformed states reduce to

$$x_i = \sum_{j=1}^{3} f_{ij} X_j , \quad i = 1, 2, 3$$

where the f_{ij} are constants.

2. An **inhomogeneous deformation** is one in which the incremental changes in the undeformed and deformed co-ordinates are related by

$$\mathrm{d}x_i = \sum_{j=1}^{3} f_{ij} \mathrm{d}X_j , \quad i = 1, 2, 3$$

where $f_{ij} = \partial x_i / \partial X_j$, $i, j = 1, 2, 3$, and where the f_{ij} are the functions of the coordinates x_j.

3. The f_{ij} in notes 1 and 2 are **deformation gradients**.

1.4   deformation gradient tensor for an elastic solid

Recommended symbol: F

The tensor whose components are deformation gradients in an elastic solid.

Notes

1. The components of F are denoted f_{ij}.

2. See 1.3 for the definitions of f_{ij}.

1.5   deformation of a viscoelastic liquid or solid

A deformation of a viscoelastic liquid or solid through which a mass point of the viscoelastic liquid or solid with co-ordinates x_1', x_2', x_3' at time t' moves to a point with co-ordinates x_1, x_2, x_3 at time t such that there are functions g_i, $i = 1,2,3$, where

$$g_i (x_1', x_2', x_3', t') = g_i (x_1, x_2, x_3, t).$$

Notes

1. t' often refers to some past time and t to the present time.

2. The relationships between the total differentials of the functions g_i define how particles of the material move relative to each other. Thus, if two particles are at small distances dx_1', dx_2', dx_3' apart at time t' and dx_1, dx_2, dx_3 at time t then

$$\sum_{j=1}^{3} g_{ij}' dx_j' = \sum_{j=1}^{3} g_{ij} dx_j$$

where $g_{ij}'(x_1', x_2', x_3', t') = \dfrac{\partial g_i(x_1', x_2', x_3', t')}{\partial x_j'}$

and $g_{ij}(x_1, x_2, x_3, t) = \dfrac{\partial g_i(x_1, x_2, x_3, t)}{\partial x_j}$

$i, j = 1,2,3.$

3. The matrix with elements g_{ij} is denoted G and the matrix with elements g'_{ij} is denoted G'.

4. A **homogeneous deformation** is one in which the functions g_i are linear functions of the x_j, $i, j = 1,2,3$. As a result, the g_{ij} and G are functions of t only and the equations which define the deformation become

$$\sum_{j=1}^{3} g_{ij}'(t')x_j' = \sum_{j=1}^{3} g_{ij}(t)x_j$$

5. **Homogeneous deformations** are commonly used or assumed in the methods employed for characterising the mechanical properties of viscoelastic polymeric liquids and solids.

1.6 deformation gradients in a viscoelastic liquid or solid

Recommended symbol: f_{ij}

If two mass points of a liquid are at a small distance dx_1', dx_2', dx_3' apart at time t' then the deformation gradients are the rates of change of dx_i' with respect to dx_j, $i, j = 1,2,3$.

Note

$$f_{ij} = \partial x_i' / \partial x_j, \quad i, j = 1, 2, 3$$

1.7 deformation gradient tensor for a viscoelastic liquid or solid

Recommended symbol: F

The tensor whose components are deformation gradients in a viscoelastic liquid or solid.

Notes

1. The components of F are denoted f_{ij}

2. See 1.6 for the definition of f_{ij}.

3. By matrix multiplication, $F = (G')^{-1}G$ where the matrices G and G' are those defined in 1.5.

1.8   strain tensor

A symmetric tensor that results when a deformation gradient tensor is factorised into a rotation tensor followed or preceded by a symmetric tensor

Notes

1. A strain tensor is a measure of the relative displacement of the mass points of a body.

2. The deformation gradient tensor F may be factorised as
$$F = R\,U = V R,$$
where R is an orthogonal matrix representing a rotation and U and V are strain tensors which are symmetric.

3. Alternative strain tensors are often more useful.
   For example:
   the **Cauchy** tensor, $C = U^2 = F^T F$
   the **Green** tensor, $B = V^2 = F F^T$
   the **Finger** tensor, C^{-1}
   the **Piola** tensor, B^{-1}
   'T' denotes transpose and '-1' denotes inverse. B is most useful for solids and C and C^{-1} for viscoelastic liquids and solids.

4. If the 1,3; 3,1; 2,3; 3,2; 3,3 elements of a strain tensor are equal to zero then the strain is termed **plane strain**

1.9   Cauchy tensor

Recommended symbol: C

The strain tensor for a viscoelastic liquid or solid, whose elements are

$$c_{ij} = \sum_{k=1}^{3} \frac{\partial x_k'}{\partial x_i} \cdot \frac{\partial x_k'}{\partial x_j},$$

where x_i' and x_i are co-ordinates of a particle at times t' and t, respectively.

Notes

1. See 1.5 for the definition of x_i' and x_i

2. See 1.8 for the definition of a *strain tensor*.

1.10   Green tensor

Recommended symbol: B

The strain tensor for an elastic solid, whose elements are

$$b_{ij} = \sum_{k=1}^{3} \frac{\partial x_i}{\partial X_k} \cdot \frac{\partial x_j}{\partial X_k}.$$

where X_i and x_i are co-ordinates in the undeformed and deformed states, respectively.

Notes

1. See 1.3 for the definition of X_i and x_i.

2. See 1.8 for the definition of a *strain tensor*

3. For small strains, B may be expressed by the equation

$$B = I + 2\varepsilon \ ,$$

where I is the unit matrix of order three and ε is the small-strain tensor. The components of ε are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \ ,$$

with $u_k = x_k - X_k$, $k = 1,2,3$, the displacements due to the deformation.

1.11 Finger tensor

Recommended symbol: C^{-1}

The strain tensor, for a viscoelastic liquid or solid, whose elements are

$$c_{ij}^{-1} = \sum_{k=1}^{3} \frac{\partial x_i}{\partial x'_k} \cdot \frac{\partial x_j}{\partial x'_k} \ .$$

where x'_i and x_i are co-ordinates of a particle at times t' and t, respectively.

Notes

1. See 1.5 for the definition of x'_i and x_i.

2. See 1.8 for the definition of a *strain tensor*.

1.12 rate-of-strain tensor

Recommended symbol: D

The time derivative of a strain tensor for a viscoelastic liquid or solid in homogeneous deformation at reference time, t.

Unit: s^{-1}

Notes

1. For an *inhomogeneous deformation*, the material derivative has to be used to find time derivatives of strain.

2. $D = \lim_{t' \to t}\left(\frac{\partial U}{\partial t'}\right) = \lim_{t' \to t}\left(\frac{\partial V}{\partial t'}\right)$, where U and V are defined in 1.8, note 2.

3. The elements of D are

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \ ,$$

where the v_k are the components of the velocity v at x and time, t.

1.13 vorticity tensor

Recommended symbol: W

The derivative, for a viscoelastic liquid or solid in homogeneous deformation, of the rotational part of the deformation-gradient tensor at reference time, t.

Unit: s^{-1}

Notes

1. For an *inhomogeneous deformation* the material derivative has to be used.

2. $W = \lim\limits_{t' \to t}\left(\dfrac{\partial R}{\partial t'}\right)$, where R is defined in **1.8, note 2**.

3. The elements of W are

$$w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) ,$$

where the v_k are the components of the velocity v at x and time t.

1.14 Rivlin-Ericksen tensors

Recommended symbol: A_n

The Rivlin-Ericksen tensor of order n, for a viscoelastic liquid or solid in homogeneous deformation, is the nth time derivative of the Cauchy strain tensor at reference time, t.

Unit: s^{-n}

Notes

1. For an *inhomogeneous deformation* the material derivatives have to be used.

2. $A_n = \lim\limits_{t' \to t}\left(\dfrac{\partial^n C}{\partial t'^n}\right)$, where C is defined in **1.9**.

$A_0 = I$, where I is the unit matrix of order three.

4. $A_1 = \dot{F}^T + \dot{F} = 2D$, where F is the *deformation-gradient tensor* (see **1.7**),

$\dot{F} = \lim\limits_{t' \to t}\left(\dfrac{\partial F}{\partial t'}\right)$, 'T' denotes transpose and D is the *rate-of-strain tensor* (see **1.12**).

5. In general, $A_{n+1} = \dot{A}_n + \dot{F}^T A_n + A_n \dot{F}$, $n = 0,1,2,\ldots$.

2. DEFORMATIONS USED EXPERIMENTALLY

All deformations used in conventional measurements of mechanical properties are interpreted in terms of *homogeneous deformations*.

2.1 general orthogonal homogeneous deformation of an elastic solid

A deformation, such that a mass point of the solid with co-ordinates X_1, X_2, X_3 in the undeformed

state moves to a point with co-ordinates x_1, x_2, x_3 in the deformed state, with

$$x_i = \lambda_i X_i \ , \ i = 1, 2, 3,$$

where the λ_i are constants.

Notes

1. The relationships between the x_i and X_i for orthogonal homogeneous deformations are a particular case of the general relationships given in 1.3, provided the deformation does not include a rotation and the co-ordinate axes are chosen as the principal directions of the deformation.

2. The λ_i are effectively **deformation gradients**, or, for finite deformations, the **deformation ratios** characterising the deformation.

3. For an incompressible material

$$\lambda_1 \lambda_2 \lambda_3 = 1 \ .$$

4. The λ_i are elements of the deformation gradient tensor F (see 1.4) and the resulting *Cauchy* and *Green* tensors C and B (see 1.9 and 1.10) are

$$C = B = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

2.2 uniaxial deformation of an elastic solid

An orthogonal, homogeneous deformation in which, say.

$$\lambda_1 = \lambda$$
and $$\lambda_2 = \lambda_3$$

Notes

1. See 2.1 for the definition of λ_i, $i = 1,2,3$.

2. For an incompressible material

$$\lambda_2 = \lambda_3 = 1/\lambda^{\frac{1}{2}}$$

2.3 uniaxial deformation ratio
deformation ratio

Recommended symbol: λ

The quotient of the length (l) of a sample under uniaxial tension or compression and its original length (l_0)

$$\lambda = l/l_0$$

Notes

1. In tension λ (>1) may be termed the **extension ratio.**

2. In compression λ (<1) may be termed the **compression ratio**.

3. λ is equivalent to λ_1 in **2.1** and **2.2**.

2.4 uniaxial strain
engineering strain

Recommended symbol: ε

The change in length of a sample in uniaxial tensile or compressive deformation divided by its initial length

$$\varepsilon = (l_1 - l_0) / l_0$$

where l_0 and l_1 are, respectively, the initial and final lengths.

Notes

1. $\varepsilon = \lambda - 1$, where λ is the *uniaxial deformation ratio* (see **2.3**).

2. $\varepsilon > 0$ is referred to as **(uniaxial) tensile strain**.

3. $\varepsilon < 0$ is referred to as **(uniaxial) compressive strain**.

2.5 Hencky strain

Recommended symbol: ε_H

The integral over the total change in length of a sample of the incremental strain in uniaxial tensile deformation

$$\varepsilon_H = \int_{l_0}^{l_1} dl \, / \, l = \ln(l_1/l_0)$$

l_0, l_1 and l are, respectively, the initial, final and instantaneous lengths.

Notes

1. See *uniaxial strain* (**2.4**).

2. The same equation can be used to define a quantity ε_H (< 0) in compression.

2.6 Poisson's ratio

Recommended symbol: μ

In a sample under small uniaxial deformation, the negative quotient of the lateral strain (ε_{lat}) and the longitudinal strain (ε_{long}) in the direction of the uniaxial force

$$\mu = -\left(\frac{\varepsilon_{lat}}{\varepsilon_{long}}\right)$$

Notes

1. **Lateral strain** ε_{lat} is the strain normal to the uniaxial deformation.
   $\varepsilon_{lat} = \lambda_2 - 1 = \lambda_3 - 1$ (see **2.2** and **2.4**).

2. For an isotropic, incompressible material, $\mu = 0.5$. It should be noted that, in materials referred to as incompressible, volume changes do in fact occur in deformation, but they may be neglected.

3. For an anisotropic material, μ varies with the direction of the uniaxial deformation.

4. *Poisson's ratio* is also sometimes called the **lateral contraction ratio** and is sometimes used in cases of non-linear deformation. The present definition will not apply in such cases.

2.7 pure shear of an elastic solid

An orthogonal, homogeneous deformation in which

$$\lambda_1 = \lambda$$
$$\lambda_2 = 1/\lambda$$
$$\lambda_3 = 1$$

Note

See 2.1 for the definition of λ_i, $i = 1,2,3$.

2.8 simple shear of an elastic solid

A homogeneous deformation, such that a mass point of the solid with co-ordinates X_1, X_2, X_3 in the undeformed state moves to a point with co-ordinate x_1, x_2, x_3 in the deformed state, with

$$x_1 = X_1 + \gamma X_2$$

$$x_2 = X_2$$

$$x_3 = X_3$$

where γ is constant.

Notes

1. The relationships between the x_i and X_i, $i = 1,2,3$, in simple shear are a particular case of the general relationships given in 1.3.

2. γ is known as the **shear** or **shear strain**.

3. The *deformation gradient tensor* for the simple shear of an elastic solid (see 1.4) is

$$F = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the *Cauchy* (*C*) and *Green* (*B*) *strain tensors* (see 1.9 and 1.10) are

$$C = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.9 bulk compression

Recommended symbol: χ

The fractional decrease in volume (*V*) caused by a hydrostatic pressure

$$\chi = - \Delta V / V.$$

Note

Also referred to as **volume compression, isotropic compression** and **bulk compressive strain.**

2.10 general homogeneous deformation or flow of a viscoelastic liquid or solid

A flow or deformation such that a particle of the viscoelastic liquid or solid with co-ordinate vector X' at time t' moves to a point with co-ordinate vector X at time t with

$$G'X' = GX$$

where G' and G are tensors defining the type of deformation or flow and are functions of time only.

Notes

1. The definition is equivalent to that given in 1.5 note 4. Accordingly, the elements of G' and G are denoted $g'_{ij}(t')$ and $g_{ij}(t)$ and those of X' and X, (x'_1, x'_2, x'_3) and (x_1, x_2, x_3).

2. For an incompressible material
$$\det G = 1$$
where $\det G$ is the determinant of G.

3. Deformations and flows used in conventional measurements of properties of viscoelastic liquids and solids are usually interpreted assuming incompressibility.

2.11 homogeneous orthogonal deformation or flow of an incompressible viscoelastic liquid or solid

A deformation or flow, as defined in 2.10, such that

$$G = \begin{pmatrix} g_{11}(t) & 0 & 0 \\ 0 & g_{22}(t) & 0 \\ 0 & 0 & g_{33}(t) \end{pmatrix}.$$

Notes

1. The g_{ii} are defined in 1.5, notes 2 to 4.

2. If $g_{22} = g_{33} = 1/g_{11}^{1/2}$ the elongational deformation or flow is **uniaxial**.

3. The *Finger strain tensor* for an homogeneous orthogonal deformation or flow of incompressible, viscoelastic liquid or solid (see **1.11**) is

$$C^{-1} = \begin{pmatrix} \left(\dfrac{g_{11}'(t')}{g_{11}(t)}\right)^2 & 0 & 0 \\ 0 & \left(\dfrac{g_{22}'(t')}{g_{22}(t)}\right)^2 & 0 \\ 0 & 0 & \left(\dfrac{g_{33}'(t')}{g_{33}(t)}\right)^2 \end{pmatrix}$$

2.12 steady uniaxial homogeneous elongational deformation or flow of an incompressible viscoelastic liquid or solid

Uniaxial homogeneous elongational flow in which

$$g_{11}(t) = \exp(-\dot{\gamma}_E t)$$

where $\dot{\gamma}_E$ is a constant, and $g_{22} = g_{33} = 1/g_{11}^{1/2}$.

Notes

1. $g_{11}(t)$, $g_{22}(t)$ and $g_{33}(t)$ are elements of the tensor G defined in 1.5.

2. From the definition of general homogeneous flow (1.5) $(G'X' = GX = \text{constant})$ in the particular case of *steady uniaxial elongation flow*

$$x_1 g_{11}(t) = x_1 \exp\left(-\dot{\gamma}_E t\right) = \text{constant}$$

and differentiation with respect to time gives

$$\dot{\gamma}_E = (1/x_1)(dx_1/dt)$$

Hence, $\dot{\gamma}_E$ is the **elongational or extensional strain rate.**

3. The *Finger strain tensor* for a steady uniaxial homogeneous elongation deformation or flow of an incompressible viscoelastic liquid or solid (see **1.11**) is

$$C^{-1} = \begin{pmatrix} \exp\left(2\dot{\gamma}_E \left(t-t'\right)\right) & 0 & 0 \\ 0 & \exp\left(-\dot{\gamma}_E \left(t-t'\right)\right) & 0 \\ 0 & 0 & \exp\left(-\dot{\gamma}_E \left(t-t'\right)\right) \end{pmatrix}.$$

2.13 homogeneous simple shear deformation or flow of an incompressible viscoelastic liquid or solid

A flow or deformation such that

$$G = \begin{pmatrix} 1 & -\gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\gamma(t)$ is the **shear**.

Notes

1. The general tensor G is defined in 1.5.

2. $\dot{\gamma} = d\gamma(t)/dt$ is the **shear rate**. The unit of $\cdot\dot{\gamma}$ is s^{-1}.

3. If $\gamma(t) = \dot{\gamma}\cdot t$, where $\dot{\gamma}$ is a constant, then the flow has a constant shear rate and is known as **steady (simple) shear flow**.

4. If $\gamma(t) = \gamma_0 \sin 2\pi\nu t$ then the flow is **oscillatory (simple) shear flow** of frequency ν and amplitude γ_0. The unit of ν is Hz.

5. The *Finger strain tensor* for *simple shear flow* (see 1.11) is

$$C^{-1} = \begin{pmatrix} 1+\left(\gamma(t)-\gamma(t')\right)^2 & \gamma(t)-\gamma(t') & 0 \\ \gamma(t)-\gamma(t') & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\gamma(t)-\gamma(t')$ is the amount of *shear* given to the liquid between the times t' and t.
For *steady simple shear* flow C^{-1} becomes

$$C^{-1} = \begin{pmatrix} 1+\dot{\gamma}^2(t-t') & \dot{\gamma}(t-t') & 0 \\ \dot{\gamma}(t-t') & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. STRESSES OBSERVED EXPERIMENTALLY

For a given deformation or flow, the resulting stress depends on the material. However, the *stress tensor* (see 1.2) does take particular general forms for experimentally used deformations (see section 2). The definitions apply to elastic solids, and viscoelastic liquids and solids.

3.1 stress tensor resulting from an orthogonal deformation or flow

Recommended symbol: σ

For an orthogonal deformation or flow the stress tensor is diagonal with

$$\sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

Unit: Pa

Notes

1. See 1.2 for the general definition of σ.

2. If the *strain tensor* is diagonal for all time then the *stress tensor* is diagonal for all time for isotropic materials.

3. For a uniaxial (orthogonal) deformation or flow $\sigma_{22} = \sigma_{33}$.

4. For a **pure shear deformation or flow** the stresses (σ_{11}, σ_{22}, σ_{33}) are usually all different from each other.

5. The *stress tensor* resulting from a *pure shear deformation or flow* is called a **pure shear stress.**

3.2 tensile stress

Recommended symbol: σ

The component σ_{11} of the stress tensor resulting from a tensile uniaxial deformation.

Unit: Pa

Notes

1. The *stress tensor* for a *uniaxial deformation* is given in 3.1.

2. The Ox_1 direction is chosen as the direction of the *uniaxial deformation*.

3.3 compressive stress

Recommended symbol: σ

The component σ_{11} of the stress tensor resulting from a compressive uniaxial deformation.

Unit: Pa

Note

See notes 1 and 2 of 3.2.

3.4   nominal stress
engineering stress

Recommended symbol: σ

The force resulting from an applied tensile or compressive uniaxial deformation divided by the initial cross-sectional area of the sample normal to the applied deformation.

Unit: Pa

Note

The term *engineering* or *nominal stress* is often used in circumstances when the deformation of the body is not infinitesimal and its cross-sectional area changes.

3.5 stress tensor resulting from a simple shear deformation or flow

Recommended symbol: σ

For a simple shear deformation or flow the stress tensor takes the form

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

where σ_{21} is numerically equal to σ_{12}.

Unit: Pa

Notes

1. See 1.2 for the general definition of σ

2. σ_{ii}, i = 1,2,3 are denoted **normal stresses**.

3. σ_{12} is called the **shear stress**.

3.6 first normal-stress difference
first normal-stress function

Recommended symbol: N_1

The difference between the first two normal stresses σ_{11} and σ_{22} in simple shear flow

$$N_1 = \sigma_{11} - \sigma_{22}.$$

Unit: Pa

Notes

1. See 3.5 for the definition of σ_{11} and σ_{22}.

2. For Newtonian liquids (see 4.2 note 3) $N_1 = 0$.

3.7 second normal-stress difference
second normal-stress function

Recommended symbol: N_2

The difference between the second and third normal-stresses ($\sigma_{22} - \sigma_{33}$) in simple shear flow

$$N_2 = \sigma_{22} - \sigma_{33}.$$

Unit: Pa

Notes

1. See 3.5 for the definition of σ_{22} and σ_{33}

2. For *Newtonian liquids* (see 4.2 note 3), $N_2 = 0$.

4. QUANTITIES RELATING STRESS AND DEFORMATION

4.1 constitutive equation for an elastic solid

An equation relating stress and strain in an elastic solid.

Notes

1. For an elastic solid, the constitutive equation may be written

$$\sigma = \frac{2}{I_3^{\frac{1}{2}}}\left(\frac{\partial W}{\partial I_1}B + \frac{\partial W}{\partial I_2}\left(I_1 B - B^2\right) + I_3\frac{\partial W}{\partial I_3}I\right),$$

where B is the *Green strain tensor* (see **1.10**).
I_1, I_2, I_3 are invariants of B,
with $I_1 = \mathrm{Tr}(B)$
$\qquad I_2 = 1/2\,((\mathrm{Tr}(B))^2 - \mathrm{Tr}(B^2))$
$\qquad I_3 = \det(B)$,
where 'Tr' denotes trace and 'det' denotes determinant. (Invariants are independent of the co-ordinate axes used and for symmetric tensors there are three independent invariants.)
W is a function of I_1, I_2, and I_3 and is known as the **stored energy function** and is the increase in energy (stored energy) per unit initial volume due to the deformation.

2. For small deformations, the constitutive equation may be written

$$\sigma = 2G\varepsilon + l\mathrm{Tr}\,(\varepsilon)\,,$$

where G is the shear modulus (see **4.10**), ε is the *small-strain tensor* (see **1.10, note 3**) and l is a **Lamé constant**

3. The *Lamé constant*, (l), is related to the shear modulus (G) and Young's modulus (E) (see **4.7**) by the equation

$$l = G(2G - E)/(E - 3G),$$

4. For an incompressible elastic solid, the constitutive equation may be written

$$\sigma + PI = 2\frac{\partial W}{\partial I_1}B - 2\frac{\partial W}{\partial I_2}B^{-1}\,,$$

where P is the hydrostatic (or isotropic) pressure, $I_3 = 1$ and W is a function of I_1 and I_2, only.

5. For small deformations of an incompressible, inelastic solid, the constitutive equation may be written

$$\sigma + PI = 2G\varepsilon$$

4.2 constitutive equation for an incompressible viscoelastic liquid or solid

An equation relating stress and deformation in an incompressible viscoelastic liquid or solid.

Notes

1. A possible general form of constitutive equation when there is no dependence of stress on amount of strain is

$$\sigma + PI = f(A_1, A_2, \ldots, A_n) \ ,$$

where $A_1, A_2, \ldots$ are the Rivlin-Ericksen tensors (see 1.14)

2. For a *non-Newtonian liquid* (see note 3), a form of the general constitutive equation which may be used is

$$\sigma + PI = \eta A_1^2 + \alpha A_1 + \beta A_2 ,$$

where η is the *viscosity* (see 4.12) and α and β are constants.

3. A **Newtonian liquid** is a liquid for which the constitutive equation may be written

$$\sigma + PI = \eta A_1 = 2\eta D,$$

where D is the *rate-of-strain tensor* (see 1.12). Liquids which do not obey this constitutive equation are termed **non-Newtonian liquids**.

4. For cases where there is a dependence of stress on strain history the following constitutive equation may be used, namely

$$\sigma + PI = 2 \int_{-\infty}^{t} \left(\frac{\partial \Omega}{\partial I_1} C^{-1} - \frac{\partial \Omega}{\partial I_2} C \right) dt' \ .$$

where C is the *Cauchy strain tensor* (see 1.9) and Ω is a function of the invariants I_1, I_2 and I_3 of C^1 and the time interval t-t'. Ω is formally equivalent to the stored-energy function, W, of a solid (see 4.1, note 4).

4.3 modulus

Recommended symbols: general M
                          in bulk compressive deformation K
                          in uniaxial deformation E
                          in shear deformation G

The quotient of stress and strain where the type of stress and strain is defined by the type of deformation employed.

Unit: Pa

Notes

1. The detailed definitions of K, E and G are given in 4.5, 4.7 and 4.10.

2. An **elastic modulus** or **modulus of elasticity** is a modulus of a body which obeys Hooke's law (stress $\propto$ strain).

4.4 compliance

Recommended symbols: general C
in bulk compressive deformation B
in uniaxial deformation D
in shear deformation J

The quotient of strain and stress where the type of strain and stress is defined by the type of deformation employed.

Unit: Pa^{-1}

Notes

1. $C = 1/M$, where M is modulus (see 4.3).

2. The detailed definitions of B, D and J are given in 4.6, 4.8 and 4.11.

4.5 bulk modulus

Recommended symbol: K

The quotient of hydrostatic pressure (P) and bulk compression (χ)

$$K = P/\chi .$$

Unit: Pa

Notes

1. Also known as **bulk compressive modulus**.

2. For the definition of χ, see 2.9.

3. At small deformations, the bulk modulus is related to *Young's modulus* (E) (see 4.7) by

$K = E/(3(1 - 2\mu))$

where μ is *Poisson's ratio* (see 2.6).

4.6 bulk compliance

Recommended symbol: B

The quotient of bulk compression (χ) and hydrostatic pressure (P)

$$B = \chi/P .$$

Unit: Pa^{-1}

Notes

1. Also known as **bulk compressive compliance**.

2. For the definition of χ, see 2.9.

3. $B = 1/K$, where K is the *bulk modulus* (see 4.5)

4.7 Young's modulus

Recommended symbol: E

The quotient of uniaxial stress (σ) and strain (ε) in the limit of zero strain

$$E = \lim_{\varepsilon \to 0} (\sigma/\varepsilon) .$$

Unit: Pa

Notes

1. The stress is a *true stress*, as in 3.2 and 3.3, and not a *nominal stress*, as in 3.4.

2. ε is defined in 2.4.

3. *Young's modulus* may be evaluated using *tensile* or *compressive uniaxial deformation* (see 2.4). If determined using tensile deformation it may be termed **tensile modulus**.

4. For non-Hookean materials (see 4.3), the *Young's modulus* is sometimes evaluated as:

   (i) the **secant modulus** - the quotient of *stress* (σ) and strain at some nominal *strain* (ε) in which case

   $$E = \sigma/\varepsilon$$

   (ii) the **tangent modulus** - the slope of the stress-strain curve at some nominal *strain* (ε'), in which case

   $$E = (d\sigma/d\varepsilon)_{\varepsilon=\varepsilon'}.$$

4.8 uniaxial compliance

Recommended symbol: D

The quotient of uniaxial strain (ε) and uniaxial stress (σ) in the limit of zero strain

$$D = \lim_{\varepsilon \to 0} (\varepsilon / \sigma) .$$

Unit: Pa^{-1}

Notes

1. The stress is a *true stress* as in 3.2 and 3.3. and not a *nominal stress.* as in 3.4.

2. ε is defined in 2.4.

3. Uniaxial compliance may be evaluated using *tensile or compressive uniaxial deformation* (see 2.4). If determined using tensile deformation it may be termed **tensile compliance.**

4. $D = 1/E$, where E is *Young's modulus* (see 4.7).

4.9 extensional viscosity
elongational viscosity

Recommended symbol: η_E

The quotient of the difference between the longitudinal stress (σ_{11}) and the lateral stress (σ_{22}) and the elongational strain rate ($\dot{\gamma}_E$) in steady uniaxial flow

$$\eta_E = (\sigma_{11} - \sigma_{22}) / \dot{\gamma}_E$$

Unit: Pa s

Note

See 3.1 and 2.12 for the definitions of σ_{11}, σ_{22} and $\dot{\gamma}_E$

4.10 shear modulus

Recommended symbol: G

The quotient of shear stress (σ_{12}) and shear strain (γ)

$$G = \sigma_{12}/\gamma.$$

Unit: Pa

Notes

1. See 2.8 for the definitions of γ for an elastic solid and 3.5 for the definition of σ_{12}.

2. The *shear modulus* is related to *Young's modulus* (E) (see 4.7) by the equation

$$G = E/(2(1+\mu))$$

where μ is *Poisson's ratio* (see 2.6).

3. For elastomers, which are assumed incompressible, the modulus is often evaluated in *uniaxial tensile* or *compressive deformation* using $\lambda - \lambda^{-2}$ as the strain function (where λ is the *uniaxial deformation ratio* (see 2.3)). In the limit of zero deformation the *shear modulus* is evaluated as

$$\frac{d\sigma}{d(\lambda - \lambda^{-2})} = \frac{E}{3} = G \text{ (for } \mu = 0.5) \ ,$$

where σ is the *tensile* or *compressive stress* (see 3.2 and 3.3).

4.11 shear compliance

Recommended symbol: J

The quotient of shear strain (γ) and shear stress (σ_{12})

$$J = \gamma/\sigma_{12}$$

Unit: Pa^{-1}

Notes

1. See 2.8 for the definition of γ for an elastic solid and 3.5 for the definition of σ_{12}.

2. $J = 1/G$, where G is the *shear modulus* (see 4.10).

**4.12   shear viscosity
coefficient of viscosity
viscosity**

Recommended symbol: η

The quotient of shear stress (σ_{12}) and shear rate ($\dot{\gamma}$) in steady, simple shear flow

$$\eta = \sigma_{12}/\dot{\gamma}$$

Unit: Pa s

Notes

1. See 3.5 and 2.13 for the definitions of σ_{12} and $\dot{\gamma}$

2. For *Newtonian liquids* (see 4.2 note 3), σ_{12} is directly proportional to $\dot{\gamma}$ and η is constant.

3. For *non-Newtonian liquids* (see 4.2 note 3), when σ_{12} is not directly proportional to $\dot{\gamma}$, η varies with $\dot{\gamma}$. The value of η evaluated at a given value of $\dot{\gamma}$ is termed the **non-Newtonian viscosity**.

4. Some experimental methods, such as capillary flow and flow between parallel plates, employ a range of shear rates. The value of η evaluated at some nominal average value of $\dot{\gamma}$ is termed the **apparent viscosity** and given the symbol η_{app}. It should be noted that *apparent viscosity* is an imprecisely defined quantity.

5. Extrapolation of η or η_{app} for *non-Newtonian liquids* to zero $\dot{\gamma}$ gives the **zero-shear viscosity**, which is given the symbol η_0.

6. Extrapolation of η and η_{app} for *non-Newtonian liquids* to infinite $\dot{\gamma}$ gives the **infinite-shear viscosity**, which is given the symbol η_∞.

4.13   first normal-stress coefficient

Recommended symbol: ψ_1

The quotient of the first normal stress difference (N_1) and the square of the shear rate ($\dot{\gamma}$) in the limit of zero shear rate

$$\psi_1 = \lim_{\dot{\gamma} \to 0} (N_1/\dot{\gamma}^2) .$$

Unit: Pa s^2

Note

See 3.6 and 2.13 for the definitions of N_1 and $\dot{\gamma}$

4.14 second normal-stress coefficient

Recommended symbol: ψ_2

The quotient of the second normal stress difference (N_2) and the square of the shear rate ($\dot{\gamma}$) in the limit of zero shear rate

$$\psi_2 = \lim_{\dot{\gamma} \to 0}\left(N_2/\dot{\gamma}^2\right)$$

Unit: Pa s^2

Note

See 3.7 and 2.13 for the definitions of N_2 and $\dot{\gamma}$.

5. LINEAR VISCOELASTIC BEHAVIOUR

5.1 viscoelasticity

The time-dependent response of a liquid or solid subjected to stress or strain.

Notes

1. Both viscous and elastic responses to stress or strain are required for the description of viscoelastic behaviour.

2. Viscoelastic properties are usually measured as responses to an instantaneously applied or removed constant stress or strain or a **dynamic stress or strain**. The latter is defined as a sinusoidal stress or strain of small amplitude, which may or may not decrease with time.

5.2 linear viscoelastic behaviour

The interpretation of the viscoelastic behaviour of a liquid or solid in *simple shear* or *uniaxial deformation* such that

$$P(D)\sigma = Q(D)\gamma$$

where σ is the shear stress or uniaxial stress, γ is the shear strain or uniaxial strain, and $P(D)$ and $Q(D)$ are polynomials in D, where D is the differential coefficient operator d/dt.

Notes

1. In *linear viscoelastic behaviour*, stress and strain are assumed to be small so that the squares and higher powers of σ and γ may be neglected.

2. See 3.5 and 2.13 for the definitions of σ and γ in *simple shear*.

3. See 3.2 and 2.12 for definitions of σ and γ ($\equiv \gamma_E$) in *uniaxial deformations*.

4. The polynomials $Q(D)$ and $P(D)$ have the forms:

$Q(D) = a(D + q_0) \ldots \ldots (D + q_n)$
(a polynomial of degree n + 1)

$P(D) = (D + p_0) (D + p_1) \ldots (D + p_n)$
(a polynomial of degree n + 1)

and

$P(D) = (D + p_0) (D + p_1) \ldots (D + p_{n-1})$
(a polynomial of degree n)

where
(i) a is a constant
(ii) $q_0 \geq 0, p_0 > 0$ and $p_s, q_s > 0, s = 1, \ldots, n$.
(iii) $q_i < p_i < q_{i+1}$ and $q_n < p_n$ (if p_n exists)
p_i and q_i are related to *relaxation* and *retardation times*, respectively (see 5.6 - 5.9).

5. If $q_0 = 0$, the material is a **liquid**, otherwise it is a **solid**.

6. Given that $Q(D)$ is a polynomial of degree $n + 1$; if $P(D)$ is also of degree $n + 1$ the material shows instantaneous elasticity; if $P(D)$ is of degree n, the material does not show instantaneous elasticity (i.e. elasticity immediately the deformation is applied.)

7. There are definitions of linear viscoelasticity which use integral equations instead of the differential equation in definition 5.2. (See, for example, ref. 11 .) Such definitions have certain advantages regarding their mathematical generality. However, the approach in the present document, in terms of differential equations, has the advantage that the definitions and descriptions of various viscoelastic properties can be made in terms of commonly used mechano-mathematical models (e.g. the Maxwell and Voigt-Kelvin models).

5.3 Maxwell model
Maxwell element

A model of the linear viscoelastic behaviour of a liquid in which

$$(\alpha D + \beta)\sigma = D\gamma$$

where α and β are positive constants, D is the differential coefficient operator d/dt, and σ and γ are the stress and strain in *simple shear* or *uniaxial deformation*..

Notes

1. See 5.2 for a discussion of σ and γ.

2. The relationship defining the *Maxwell model* may be written

$$d\sigma/dt + (\beta/\alpha)\sigma = (1/\alpha)d\gamma/dt$$

3. Comparison with the general definition of *linear viscoelastic behaviour* (5.2) shows that the polynomials $P(D)$ and $Q(D)$ are of order one, $q_0=0$, $p_0=\beta/\alpha$ and $a=1/\alpha$. Hence, a material described by a *Maxwell model* is a *liquid* ($q_0 = 0$) having instantaneous elasticity ($P(D)$ and $Q(D)$ are of the same order).

4. The *Maxwell model* may be represented by a spring and a dashpot filled with a *Newtonian liquid* in series, in which case $1/\alpha$ is the **spring constant** (force = $1/\alpha$)·extension) and $1/\beta$ is the **dashpot constant** (force = $(1/\beta)$·rate of extension).

5.4 Voigt-Kelvin model
Voigt-Kelvin element

A model of the *linear viscoelastic behaviour* of a solid in which

$$\sigma = (\alpha + \beta D)\gamma$$

where α and β are positive constants, D is the differential coefficient operator d/dt, and σ and γ are the stresses and strain in *simple shear* or *uniaxial deformation*..

Notes

1. The Voigt-Kelvin model is also known as the **Voigt model** or **Voigt element**.

2. See 5.2 for a discussion of σ and γ.

3. The relationship defining the *Voigt-Kelvin model* may be written

$$\sigma = \alpha\gamma + \alpha\beta(d\gamma/dt).$$

4. Comparison with the general definition *of linear viscoelastic behaviour* (5.2) shows that the polynomial $P(D)$ is of order zero, $Q(D)$ is of order one, $aq_0 = \alpha$, and $a = \beta$. Hence, a material described by the *Voigt-Kelvin model* is a *solid* ($q_0>0$) without instantaneous elasticity ($P(D)$ is a polynomial of order one less than $Q(D)$).

5. The **Voigt-Kelvin model** may be represented by a spring and a dashpot filled with a Newtonian liquid in parallel, in which case α is the **spring constant** (force = α·extension) and β is the **dashpot constant** (force = β·rate of extension).

5.5 standard linear viscoelastic solid

A model of the linear viscoelastic behaviour of a solid in which

$$(\alpha_1 + \beta_1 D)\sigma = (\alpha_2 + \beta_2 D)\gamma$$

where α_1, β_1, α_2 and β_2 are positive constants, D is the differential coefficient operator d/dt, and σ and γ are the stress and strain in *simple shear* or *uniaxial deformation*.

Notes

1. See 5.2 for a discussion of σ and γ.

2. The relationship defining *the standard linear viscoelastic solid* may be written
$$\alpha_1\sigma + \beta_1(d\sigma/dt) = \alpha_2\gamma + \beta_2(d\gamma/dt)$$

3. Comparison with the general definition of a *linear viscoelastic behaviour* (5.2) shows that the polynomial $P(D)$ and $Q(D)$ are of order one, $q_0 = \alpha_2/\beta_2$, $a = \beta_2/\beta_1$ and $p_0 = $

α_1/α_2. Hence, the *standard linear viscoelastic solid* is a solid ($aq_0>0$) having instantaneous elasticity ($P(D)$ and $Q(D)$ are of the same order).

4. The *standard linear viscoelastic solid* may be represented by:

(i) a *Maxwell model* (of spring constant h_2 and dashpot constant k_2) in parallel with a spring (of spring constant h_1) in which case $\alpha_1 = h_2$, $\beta_1 = k_2$, $\alpha_2 = h_1 h_2$ and $\beta_2 = h_1 k_2 + h_2 k_2$.

(ii) a *Voigt-Kelvin model* (of spring constant h_2 and dashpot constant k_2) in series with a spring (of spring constant h_1) in which case $\alpha_1 = h_1 + h_2$, $\beta_1 = k_2$, $\alpha_2 = h_1 h_2$ and $\beta_2 = h_1 k_2$.

5. The *standard linear viscoelastic solid* can be used to represent both *creep* (see 5.9) and *stress relaxation* (see 5.7) in materials in terms of single retardation and relaxation times, respectively.

5.6 relaxation time

Recommended symbol: τ

A time characterising the response of a viscoelastic liquid or solid to the instantaneous application of a constant strain.

Unit: s

Notes

1. The response of a material to the instantaneous application of a constant strain is termed *stress relaxation* (see 5.7).

2. The relaxation time of a *Maxwell element* (5.3) is $\tau = 1/p_0 = \alpha/\beta$.

3. The *relaxation time* of a *standard linear viscoelastic solid* (5.5) is $\tau = 1/p_0 = \beta_1/\alpha_1$.

4. Generally, a *linear viscoelastic material* has a spectrum of relaxation times, which are the reciprocals of p_i, $i = 0, 1, ..., n$ in the polynomial $P(D)$ (see 5.2).

5. The **relaxation spectrum** (spectrum of relaxation times) describing *stress relaxation* in polymers may be considered as arising from a group of *Maxwell elements* in parallel (see 5.7).

5.7 stress relaxation

The change in stress with time after the instantaneous application of a constant strain.

Notes

1. The applied strain is of the form $\gamma = 0$ for $t < 0$ and $\gamma = \gamma_0$ for $t > 0$ and is usually a *uniaxial extension* or a *simple shear* (see 5.2).

2. For *linear viscoelastic behaviour*, the stress takes the form

$$\sigma(t) = (c + \overline{\psi}(t))\gamma_0$$

c is a constant that is non-zero if the material has instantaneous elasticity and $\overline{\psi}(t)$ is the **relaxation function**.

3. $\overline{\psi}(t)$ has the form

$$\overline{\psi}(t) = \sum_{i=0}^{n} \beta_i e^{-p_i t}$$

where the β_i are functions of the p_i and q_i of the polynomials $P(D)$ and $Q(D)$ defining the *linear viscoelastic material* (see **5.1**).

4. The relaxation times of the material are $1/p_i$ (see **5.6**).

5.8 retardation time

Recommended symbol: τ

A time characterising the response of a viscoelastic material to the instantaneous application of a constant stress.

Unit: s

Notes:

1. The response of a material to the instantaneous application of a constant stress is termed *creep* (see **5.9**).

2. The *retardation time* of a *Voigt-Kelvin element* is $\tau = 1/q_0 = \beta/\alpha = $ (dashpot constant)/(spring constant).

3. The *retardation time* of a *standard linear viscoelastic solid* (**5.5**) is $\tau = 1/q_0 = \beta_2/\alpha_2$.

4. Generally, a *linear viscoelastic material* has a spectrum of *retardation times*, which are reciprocals of q_i, $i = 0, 1, \ldots, n$ in the polynomial $Q(D)$ (see **5.2**).

5. The **retardation spectrum** (spectrum of retardation times) describing *creep* in polymers may be considered as arising from a group of *Voigt-Kelvin elements* in series (see **5.9**).

5.9 creep

The change in strain with time after the instantaneous application of a constant stress.

Notes

1. The applied stress is of the form $\sigma = 0$ for $t < 0$ and $\sigma = \sigma_0$ for $t > 0$ and is usually a *uniaxial stress* or a *simple shear stress* (see **5.2**).

2. For *linear viscoelastic behaviour*, the *strain* usually takes the form

$$\gamma(t) = (a + bt + \psi(t))\sigma_0$$

a is a constant that is non-zero if the material has instantaneous elasticity and b is a

constant that is non-zero if the material is a liquid. $\psi(t)$ is the **creep function**. In addition,

$$J(t) = \gamma(t)/\sigma_0$$

is sometimes called the **creep compliance**.

3. The **creep function** has the form

$$\psi(t) = \sum_i A_i e^{-q_i t}$$

where the summation runs from $i = 0$ to n for a *solid* and 1 to n for a *liquid*. The A_i are functions of the p_i and q_i of the polynomials $P(D)$ and $Q(D)$ defining the *linear viscoelastic material* and the q_i are the q_i of the polynomial $Q(D)$ (see 5.1).

4. The *retardation times* of the material are $1/q_i$ (see 5.8).

5. Creep is sometimes described in terms of non-linear viscoelastic behaviour, leading, for example, to evaluation of recoverable shear and steady-state recoverable shear compliance. The definitions of such terms are outside the scope of this document.

5.10 forced oscillation

The deformation of a material by the application of a small sinusoidal strain (γ) such that

$$\gamma = \gamma_0 \cos \omega t$$

where γ_0 and ω are positive constants.

Notes

1. γ may be in *simple shear* (see 2.8 and 2.13) or *uniaxial deformation* (often denoted ε, see 2.4).

2. γ_0 is the **strain amplitude**.

3. ω is the **angular velocity** of the circular motion equivalent to a sinusoidal frequency ν, with $\omega = 2\pi\nu$. The unit of ω is rad s^{-1}.

4. For *linear viscoelastic behaviour*, a sinusoidal stress (σ) results from the sinusoidal strain with

$$\sigma = \sigma_0 \cos(\omega t + \delta) = \sigma_0 \cos\delta \cdot \cos\omega t - \sigma_0 \sin\delta \cdot \sin\omega t .$$

σ_0 is the **stress amplitude**. δ is the **phase angle** or **loss angle** between stress and strain.

5. Alternative descriptions of the sinusoidal stress and strain in a viscoelastic material under forced oscillations are:

(i) $\gamma = \gamma_0 \sin \omega t$
  $\sigma = \sigma_0 \sin(\omega t + \delta)$
  $\quad = \sigma_0 \sin\delta \cdot \cos\omega t +$
  $\quad\quad \sigma_0 \cos\delta \cdot \sin\omega t$

$$(ii) \quad \gamma = \gamma_0 \cos(\omega t - \delta) \qquad\qquad \sigma = \sigma_0 \cos \omega t$$
$$= \gamma_0 \cos \delta \cdot \cos \omega t +$$
$$\gamma_0 \sin \delta \cdot \sin \omega t$$

5.11 loss factor
loss tangent

Recommended symbol: tan δ

The tangent of the phase angle difference (δ) between stress and strain during forced oscillations.

Notes

1. tan δ is calculated using

$$\gamma = \gamma_0 \cos \omega t \text{ and } \sigma = \sigma_0 \cos(\omega t + \delta). \text{ (see 5.10)}.$$

2. tan δ is also equal to the ratio of *loss* to *storage modulus* (see **5.12 and 5.13**).

3. A plot of tan δ versus temperature or frequency is known as a **loss curve**.

5.12 storage modulus

Recommended symbol: general M'
                                         in simple shear deformation G'
                                         in uniaxial deformation E'

The ratio of the amplitude of the stress in phase with the strain ($\sigma_0 \cos \delta$) to the amplitude of the strain (γ_0) in the forced oscillation of a material

$$M' = (\sigma_0 \cos \delta)/\gamma_0.$$

Unit: Pa

Note

See **5.10** for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \cos \omega t$ and $\sigma = \sigma_0 \cos(\omega t + \delta)$.

5.13 loss modulus

Recommended symbols: General M''
                                        in simple shear deformation G''
                                         in uniaxial deformation in E''

The ratio of the amplitude of the stress 90° out of phase with the strain ($\sigma_0 \sin \delta$) to the amplitude of the strain (γ_0) in the forced oscillation of a material

$$M'' = (\sigma_0 \sin \delta)/\gamma_0$$

Unit: Pa

Note

See **5.10** for the definition of a *forced oscillation* in which $\gamma = \gamma_0\cos \omega t$ and $\sigma = \sigma_0 \cos (\omega t + \delta)$.

5.14 absolute modulus

Recommended symbols: general $|M^*|$
in simple shear deformation $|G^*|$
in uniaxial deformation $|E^*|$

The ratio of the amplitude of the stress (σ_0) to the amplitude of the strain (γ_0) in the forced oscillation of a material

$$|M^*| = \sigma_0/\gamma_0 \ .$$

Unit: Pa

Notes

1. See **5.10** for the definition of a *forced oscillation* in which $\gamma = \gamma_0\cos \omega t$ and $\sigma = \sigma_0 \cos (\omega t + \delta)$.

2. The absolute modulus is related to the *storage modulus* (**5.12**) and the *loss modulus* (**5.13**) by the relationship

$$|M^*| = \left(\frac{\sigma_0^2 \cos^2 \delta}{\gamma_0^2} + \frac{\sigma_0^2 \sin^2 \delta}{\gamma_0^2} \right) = (M'^2 + M''^2)^{1/2} \ .$$

5.15 complex modulus

Recommended symbols: general M^*
in simple shear deformation G^*
in uniaxial deformation E^*

The ratio of complex stress (σ^*) to complex strain (γ^*) in the forced oscillation of material

$$M^* = \sigma^*/\gamma^* \ .$$

Unit: Pa

Notes

1. See **5.10** for the definition of a *forced oscillation* in which $\gamma = \gamma_0\cos \omega t$ and $\sigma = \sigma_0\cos (\omega t + \delta)$.

2. The **complex strain** $\gamma^* = \gamma_0 e^{i\omega t} = \gamma_0 (\cos \omega t + i \sin \omega t)$, where $i = \sqrt{-1}$, so that the real part of the complex strain is that actually applied to the material.

3. The **complex stress** $\sigma^* = \sigma_0 e^{i(\omega t + \delta)} = \sigma_0 (\cos(\omega t + \delta) + i \sin(\omega t + \delta))$, so that the real part of the complex strain is that actually experienced by the material.

4. The *complex modulus* is related to the *storage* and *loss moduli* through the relationships

$$M^* = \sigma^*/\gamma^* = \sigma_0 e^{i\delta}/\gamma_0 = (\sigma_0/\gamma_0)(\cos\delta + i \sin \delta) = M' + iM''.$$

5. For linear viscoelastic behaviour interpreted in terms of *complex stress* and *strain* (see notes 2, 3)

$$P(D)\sigma^* = Q(D)\gamma^*$$

(see 5.2). Further as $D\sigma^* = d\sigma^*/dt = i\omega\sigma^*$ and $D\gamma^* = i\omega\gamma,^*$

$$M^* = \sigma^*/\gamma^* = Q(i\omega)/P(i\omega)$$

5.16 storage compliance

Recommended symbols: general C'
in simple shear deformation J'
in uniaxial deformation D'

The ratio of the amplitude of the strain in phase with the stress ($\gamma_0 \cos \delta$) to the amplitude of the stress (σ_0) in the forced oscillation of a material

$$C' = (\gamma_0 \cos \delta)/\sigma_0 .$$

Unit: Pa^{-1}

Note

See 5.10, note 5 for the definition of a *forced oscillation* in which
$\gamma = \gamma_0 \cos (\omega t - \delta)$ and $\sigma = \sigma_0 \cos \omega t$.

5.17 loss compliance

Recommended symbols: general C''
in simple shear deformation J''
in uniaxial deformation D''

The ratio of the amplitude of the strain $90°$ out of phase with the stress ($\gamma_0 \sin \delta$) to the amplitude of the stress (σ_0) in the forced oscillation of a material

$$C'' = (\gamma_0 \sin \delta)/\sigma_0 .$$

Unit: Pa^{-1}

Note

See 5.10 for the definition of a *forced oscillation* in which
$\gamma = \gamma_0 \cos(\omega t - \delta)$ and $\sigma = \sigma_0 \cos \omega t$.

5.18 absolute compliance

Recommended symbols: general $|C^*|$
in simple shear deformation $|J^*|$
in uniaxial deformation $|D^*|$

The ratio of the amplitude of the strain (γ_0) to the amplitude of the stress (σ_0) in the forced oscillation of a material

$$|C^*| = \gamma_0/\sigma_0 .$$

Unit: Pa^{-1}

Notes

1. See 5.10 note 5 for the definition of a *forced oscillation* in which
   $\gamma = \gamma_0\cos(\omega t - \delta)$ and $\sigma = \sigma_0\cos \omega t$.

2. The absolute compliance is related to the *storage compliance* (5.16) and the *loss compliance* (5.17) by the relationship

$$|C^*| = \left(\frac{\gamma_0^2 \cos^2 \delta}{\sigma_0^2} + \frac{\gamma_0^2 \sin^2 \delta}{\sigma_0^2} \right)^{1/2} = (C'^2 + C''^2)^{1/2} .$$

3. The absolute compliance is the reciprocal of the *absolute modulus* (5.14).

$$|C^*| = 1/|M^*|.$$

5.19 complex compliance

Recommended symbols: general C^*
                          in simple shear deformation J^*
                          in shear deformation D^*

The ratio of complex strain (γ^*) to complex stress (σ^*) in the forced oscillation of a material

$$C^* = \gamma^*/\sigma^* .$$

Unit: Pa^{-1}

Notes

1. See 5.10 for the definition of a *forced oscillation* in which
   $\gamma = \gamma_0 \cos (\omega t - \delta)$ and $\sigma = \sigma_0 \cos \omega t$.

2. The **complex strain** $\gamma^* = \gamma_0 e^{i(\omega t-\delta)} = \gamma_0 (\cos(\omega t-\delta) + i \sin (\omega t-\delta))$, where $i = \sqrt{-1}$, so that the real part of the complex strain is that actually experienced by the material.

3. The **complex stress** $\sigma^* = \sigma_0 e^{i\omega t} = \sigma_0 (\cos \omega t + i \sin \omega t)$, so that the real part of the complex stress is that actually applied to the material.

4. The complex compliance is related to the *storage* and *loss compliances* (5.16 and 5.17) through the relationships

$$C^* = \gamma^*/\sigma^* = \gamma_0 e^{-i\delta}/\sigma_0 = (\gamma_0/\sigma_0)(\cos\delta - i \sin\delta) = C' - iC''$$

5. The complex compliance is the reciprocal of the *complex modulus*

$$C^* = 1/M^* .$$

5.20 dynamic viscosity

Recommended symbol: η'

The ratio of the stress in phase with the rate of strain ($\sigma_0 \sin \delta$) to the amplitude of the rate of strain ($\omega\gamma_0$) in the forced oscillation of a material

$$\eta' = (\sigma_0 \sin \delta)/(\omega\gamma_0) \ .$$

Unit: Pa s

Notes

1. See 5.10, note 5 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \sin \omega t$
   and $\sigma = \sigma_0 \sin(\omega t + \delta)$, so that $\dot{\gamma} = \omega\gamma_0 \cos \omega t$ and $\sigma = \sigma_0 \sin\delta \cdot \cos\omega t + \sigma_0 \cos\delta \cdot \sin\omega t$.

2. See 5.2, note 6: $\eta' = M''/\omega$ may be used for evaluating the *dynamic viscosity*. The same expression is often used to evaluate the *shear viscosity*. The latter use of this expression is not recommended.

5.21 out-of-phase viscosity

Recommended symbol: η''

The ratio of the stress 90° out of phase with the rate of strain ($\sigma_0 \cos \delta$) to the amplitude of the rate of strain ($\omega\gamma_0$) in the forced oscillation of a material

$$\eta'' = (\sigma_0 \cos \delta)/(\omega\gamma_0) \ .$$

Unit: Pa s

Notes

1. See 5.10, note 5 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \sin \omega t$
   and $\sigma = \sigma_0 \sin(\omega t + \delta)$, so that $\dot{\gamma} = \omega\gamma_0 \cos \omega t$ and
   $\sigma = \sigma_0 \sin\delta \cdot \cos\omega t + \sigma_0 \cos\delta \cdot \sin\omega t$.

2. See 5.22, note 6: $\eta'' = M'/\omega$ may be used to evaluate the *out-of-phase viscosity*.

5.22 complex viscosity

Recommended symbol: η^*

The ratio of complex stress (σ^*) to complex rate of strain ($\dot{\gamma}^*$) in the forced oscillation of a material

$$\eta^* = \sigma^*/\dot{\gamma}^* \ .$$

Unit: Pa s

Notes

1. See 5.10, note 5 for the definition of a *forced oscillation* in which $\gamma = \gamma_0 \sin \omega t$ and $\sigma = \sigma_0 \cos (\omega t + \delta)$ and the rate of strain $\dot{\gamma} = \omega_0 \cos \omega t$.

2. The complex rate of strain $\dot{\gamma}^* = i\,\omega\gamma_0 e^{i\omega t} = i\,\omega\gamma_0 (\cos \omega t + i \sin \omega t)$, where $i = \sqrt{-1}$.

3. The complex stress $\sigma^* = \sigma_0 e^{i(\omega t + \delta)} = \sigma_0 (\cos (\omega t + \delta) + i \sin (\omega t + \delta))$

4. The *complex viscosity* may alternatively be expressed as

$$\eta^* = \sigma^*/\dot{\gamma}^* = (\sigma_0 e^{i\delta})/(i\,\omega\gamma_0) = M^*/i\,\omega$$

where M^* is the *complex modulus* (see 5.15).

5. The *complex viscosity* is related to the *dynamic* and *out-of-phase viscosities* through the relationships

$$\eta^* = \sigma^*/\dot{\gamma}^* = \sigma_0 (\cos \delta + i \sin \delta)/(i\,\omega\gamma_0) = \eta' - i\,\eta''.$$

6. The *dynamic* and *out-of-phase viscosities* are related to the *storage* and *loss moduli* (5.11 and 5.13) by the relationships $\eta^* = \eta' - i\,\eta'' = M^*/i\,\omega = (M' + i\,M'')/i\,\omega$, so that $\eta' = M''/\omega$ and $\eta'' = M'/\omega$.

6. OSCILLATORY DEFORMATIONS AND STRESSES USED EXPERIMENTALLY

There are three modes of **free** and **forced** oscillatory deformations which are commonly used experimentally, **torsional oscillations, uniaxial extensional oscillations** and **flexural oscillations**.

The oscillatory deformations and stresses can be used for solids and liquids. However, the apparatuses employed to measure them are usually designed for solid materials. In principle, they can be modified for use with liquids.

Analyses of the results obtained depend on the shape of the specimen, whether or not the distribution of mass in the specimen is accounted for and the assumed model used to represent the linear viscoelastic properties of the material. The following terms relate to analyses which generally assume small deformations, specimens of uniform cross-section, non-distributed mass and a Voigt-Kelvin solid (see 5.4). These are the conventional assumptions.

6.1 free oscillation

The oscillatory deformation of a material specimen with the motion generated without the continuous application of an external force.

Note

> For any real sample of material the resulting oscillatory deformation is one of decaying amplitude.

6.2 damping curve

The decreased deformation of a material specimen versus time when the specimen is subjected to a free oscillation.

Notes

1. See **6.1** for the definition of a *free oscillation*.

2.· The term 'damping curve' is sometimes used to describe a *loss curve* (see **5.11**).

3. A *damping curve* is usually obtained using a **torsion pendulum**, involving the measurement of decrease of the axial, torsional displacement of a specimen of uniform cross-section of known shape, with the torsional displacement initiated using a torsion bar of known moment of inertia.

4. *Damping curves* are conventionally analysed in terms of the *Voigt-Kelvin* solid (see **5.4**) giving a decaying amplitude and a single frequency.

5. Given the properties of a *Voigt-Kelvin* solid, a damping curve is described by the equation

$$X = A\exp(-\beta t)\cdot\sin(\omega t - \phi),$$

where X is the displacement from equilibrium (for torsion $X = \theta$, the angular displacement), t is time, A is the amplitude, β is the *decay constant* (see **6.3**), ω is the *angular velocity* corresponding to the *decay frequency* (see **5.10** and **6.4**) and ϕ is the phase angle.

6.3 decay constant

Recommended symbol: β ·

The exponential coefficient of the time-dependent decay of a damping curve, assuming Voigt-Kelvin behaviour.

Unit: s^{-1}

Notes

1. See *damping curve* (**6.2**) and the equation therefor (**6.2**, note 5).

2. See *Voigt-Kelvin* solid (**5.4**).

3. For small damping, β is related to the *loss modulus* (M''), see **5.13**, through the equation

$$M'' = 2\beta\omega/H.$$

ω is the angular velocity corresponding to the *decay frequency* (see **5.10** and **6.4**). H depends on the cross-sectional shape of the specimen and the type of deformation. (For example, for the axial torsion of a circular rod of radius a and length l using a *torsion pendulum* (see **6.2**, note 3) with a torsion bar of moment of inertia I

$$H = \pi a^4/(2Il)$$

and $M'' \cong G''$, the loss modulus in simple shear.)

6.4 decay frequency

Recommended symbol: ν

The frequency of a damping curve assuming Voigt-Kelvin behaviour.

Unit: Hz

Notes

1. See *damping curve* (6.2) and the equation therefor (6.2, note 5).

2. See *Voigt-Kelvin* solid (5.4).

3. $\nu = \omega/2\pi$, where ω is the *angular velocity* corresponding to ν (see 5.10).

4. For small damping, the *storage modulus* (M'), see 5.12, may be evaluated from ω through the equation

$$M' = \omega^2/H,$$

where H is discussed in 6.3, note 3. Again, for torsion, $M' \cong G'$, the *storage modulus* in simple shear.

6.5 logarithmic decrement

Recommended symbol: Λ

Natural logarithm of the ratio of the displacement of a damping curve separated by one period of the displacement.

Notes

1. *Voigt-Kelvin* behaviour (see 5.4) is assumed so that the displacement decays with a single period T, where

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega}$$

with ν the frequency and ω is the angular velocity corresponding to ν (see 6.4).

2. The *logarithmic decrement* can be used to evaluate the *decay constant* β (see 6.3). From the equation for the *damping curve* of a *Voigt-Kelvin* solid (see 6.2, note 5).

$$\Lambda = \ln (X_n/X_{n+1}) = \beta(t_{n+1} - t_n) = \beta \cdot T,$$

where X_n and t_n are the displacement and time at a chosen point (usually near a maximum) in the n-th period of the decay, and X_{n+1} and t_{n+1} are the corresponding displacement and time one period later.

3. Λ can also be defined using displacements k periods apart, with

$$\Lambda = (1/k)\ln (X_n/X_{n+k}).$$

4. For small damping, Λ is related to the *loss tangent*, $\tan \delta$ (see 5.11) by

$$\tan \delta = M''/M' = 2\beta/\omega = 2\Lambda/T\omega = \Lambda/\pi$$

(See **6.3** and **6.4** for expressions for M' and M'').

6.6 forced uniaxial extensional oscillations

The uniaxial extensional deformation of a material specimen of uniform cross-sectional area along its long axis by the continuous application of a sinusoidal force of constant amplitude.

Notes

1. For a specimen of negligible mass, the linear-viscoelastic interpretation of the resulting deformation gives

$$(A/L)Q(D)l = P(D)f_0 \cos \omega t$$

where $P(D)$ and $Q(D)$ are the polynomials in $D(=d/dt)$ characterising the *linear-viscoelastic behaviour* (see **5.2**), A is the cross-sectional area of the specimen, L its original length, l is here the change in length, f_0 the amplitude of the applied force of *angular velocity* ω (see **5.10**, note 3) and t the time.

2. For a *Voigt-Kelvin* solid (see **5.4**), with $P(D)=1$ and $Q(D)=\alpha+\beta D$, where α is the spring constant and β the dashpot constant, the equation describing the deformation becomes

$$(A/L)\beta(dl/dt) + (A/L)\alpha l = f_0 \cos \omega t$$

or, in terms of *stress* and *strain*,

$$\alpha\varepsilon + \beta\frac{d\varepsilon}{dt} = \sigma_0 \cos\omega t$$

where $\varepsilon = l/L$ is the *uniaxial strain* (see **2.4** and **5.10**) and $\sigma_0 = f_0/A$ is the amplitude of the *stress* (see **5.10**). The solution of the equation is

$$\varepsilon = \frac{\sigma_0}{\left(\alpha^2 + \beta^2\omega^2\right)^{1/2}} \cos(\omega t - \delta) = \varepsilon_0 \cos(\omega t - \delta)$$

where δ is the *phase angle* (see **5.10**) with $\tan\delta = \beta\omega/\alpha$.

3. From **5.14**, the *absolute modulus* in *uniaxial deformation*

$$|E^*| = \sigma_0/\varepsilon_0 = (\alpha^2+\beta^2\omega^2)^{1/2}$$

where $\alpha = E'$, $\beta\omega = E''$ and $\tan\delta = E''/E'$ equal to the *loss tangent* (see **5.11**).

4. If one end of the specimen is fixed in position and a mass m is attached to the moving end, the *linear-viscoelastic* interpretation of the resulting deformation gives

$$m \cdot P(D)(d^2l/dt^2)+(A/L)Q(D)l = P(D)f_0\cos \omega t$$

where the symbols have the same meaning as in note 1.

5. For a *Voigt-Kelvin* solid (cf. note 2), the equation in note 4 describing the deformation becomes

$$m(d^2l/dt^2)+(A/L)\beta(dl/dt)+(A/L)\alpha \cdot l = f_0 \cdot \cos \omega t$$

with the solution

$$\varepsilon = \frac{\sigma_0 \cdot \left(A/(Lm)\right)}{\left[\left(\dfrac{A_\infty}{Lm} - \omega^2\right)^2 + \omega^2\left(\dfrac{A\beta}{Lm}\right)^2\right]^{1/2}} \cos(\omega t - \theta) = \varepsilon_0 \cos(\omega t - \theta)$$

where $\tan\theta = \dfrac{\left((A\beta)/(Lm)\right) \cdot \omega}{\left((A\alpha)/(Lm)\right) - \omega^2}$ and symbols have the same meaning as in notes 1 and 2.

6. The amplitude of the strain ε_0 is maximal when

$$\omega^2 = A\alpha/(Lm) = \omega_R^2$$

giving the value of the angular velocity (ω_R) of the *resonance frequency* of the specimen (see 6.12) in forced uniaxial extensional oscillation.

7. Notes 2 and 5 show that application of a sinusoidal uniaxial force to a *Voigt-Kelvin* solid of negligible mass, with or without added mass, results in an out-of-phase sinusoidal uniaxial extensional oscillation of the same frequency.

6.7 forced flexural oscillation

The flexural deformation (bending) of a material specimen of uniform cross-sectional area perpendicular to its long axis by the continuous application of a sinusoidal force of constant amplitude.

Notes

1. There are three modes of flexure in common use.

   (i) Application of the flexural force at one end of the specimen with the other end clamped.

   (ii) Application of the flexural force at the centre of the specimen with the two ends clamped (**three-point bending or flexure**).

   (iii) Application of the flexural force at the centre of the specimen with the two ends resting freely on supports (also known as **three-point bending or flexure**).

2. For specimens *without mass*, the linear-viscoelastic interpretation of the resulting deformations follows a differential equation of the same form as that for *a uniaxial extensional forced oscillation* (see 6.6, note 1), namely

$$(HJ/L^3)Q(D)y = P(D)f_0\cos\omega t$$

where $P(D)$, $Q(D)$, f_0, ω and t have the same meaning as for a *forced uniaxial extensional oscillation* (see 6.6, note 1) and H is a constant. The length of the specimen is $2L$. For mode of flexure (i) $H=3$, for (ii) $H=24$ and for (iii) $H=6$ (see note 1). J is the **second moment of area** of the specimen, defined by

$$J = \int_A q^2 \, dA$$

where dA is an element of the cross-sectional area (A) of the specimen and q is the distance of that element from the **neutral axis or plane** of the specimen, lying centrally in the specimen and defined by points which experience neither compression nor extension during the flexure. For a specimen of circular cross-section $J=\pi r^2/4$, where r is the radius, and for one of rectangular cross-section $J=4ab^3/3$, where $2a$ and $2b$ are the lateral dimensions with flexure along the b dimension. Finally, y is the *flexural deflection* (see 6.9) of the specimen at the point of application of the force, of either the end (mode of flexure (i)) or the middle (modes of flexure (ii) and (iii)).

3. For a *Voigt-Kelvin* solid, the equation describing the deformation becomes

$$(HJ/L^3)\alpha \cdot y + (HJ/L^3)\beta(dy/dt) = f_0\cos\omega t$$

with solution

$$y = \frac{f_0 L^3}{HJ\left(\alpha^2 + \beta^2\omega^2\right)^{1/2}}\cos(\omega t - \delta)$$

where δ is the *phase angle* with

$$\tan\delta = \beta\omega/\alpha$$

equal to the *loss tangent* (see 5.11).

4. Unlike the strain in *forced uniaxial extensional oscillations*, those in *forced flexural deformations* are not homogeneous. In the latter modes of deformation, the strains vary from point-to-point in the specimen. Hence, the equation defining the displacement y in terms of the amplitude of applied force (f_0) cannot be converted into one defining strain in terms of amplitude of stress.

5. If a mass m is attached to the specimen at the point of application of the force, the linear-viscoelastic interpretation of the resulting deformation gives

$$m \cdot P(D) \, (d^2y/dt^2) + (HJ/L^3)Q(D)y = P(D)f_0\cos\omega t$$

(cf. 6.6, note 4).

6. For a *Voigt-Kelvin* solid (cf. note 3 and 6.6, note 5), the equation describing the deformation becomes

$$m(d^2y/dt^2) + (HJ/L^3)\beta(dy/dt) + (HJ/L^3)\alpha \cdot y = f_0\cos\omega t$$

with the solution

$$y = \frac{f_0/m}{\left(\left(\frac{HJ\alpha}{L^3 m} - \omega^2\right)^2 + \omega^2\left(\frac{HJ\beta}{L^3 m}\right)^2\right)^{1/2}}\cos(\omega t - \delta)$$

where $\tan\delta = \dfrac{\left(HJ\beta/\left(L^3 m\right)\right)\omega}{\left(\left(HJ\alpha/\left(L^3 m\right)\right) - \omega^2\right)}$

7. The *flexural deflection* y (see **6.9**) is maximal when

$$\omega^2 = HJ\alpha/(L^3 m) = \omega_R^2$$

giving the value of the **angular velocity** (ω_R) of the **resonance frequency** of the specimen (see **6.6**, note 6) in forced flexural oscillations.

8. Notes 3 and 6 show that the application of the defined sinusoidal flexural forces (i), (ii) and (iii) (note 1) to a *Voigt-Kelvin* solid of negligible mass, with or without added mass at the points of application of the forces, results in out-of-plane sinusoidal flexural oscillations of the same frequency.

6.8 flexural force

Recommended symbol: f_0

The amplitude of the force applied to a material specimen to cause a forced flexural oscillation.

Unit: N

Notes

1. See **6.7** for the definition and interpretation of *forced flexural oscillation*.

2. A related quantity is the **flexural stress** which is somewhat arbitrarily defined as the amplitude of the stress in the convex, outer surface of a material specimen in forced flexural oscillation.

6.9 flexural deflection

Recommended symbol: *y*

The deflection of a specimen subject to a forced flexural oscillation at the point of application of the flexural force.

Unit: m

Notes

1. See **6.7** for the definition and interpretation of *forced flexural oscillations*

2. See **6.8** for the definition of *flexural force*.

6.10 flexural modulus

Recommended symbol: $|E^*|$

The modulus measured using forced flexural oscillations.

Unit: Pa

Notes

1. See **6.7** for the definition and interpretation of *forced flexural oscillations*.

2. For a *Voigt-Kelvin solid* (see 5.4) of negligible mass, the absolute modulus can be evaluated from the ratio of the *flexural force* (f_0) and the amplitude of the *flexural deflection* (y) with

$$f_0/Y_0 = (HJ/L^3)(\alpha^2 + \beta^2\omega^2)^{1/2}$$

where Y_0 is the amplitude of the *flexural deflection* (see 6.7, note 3, 6.8 and 6.9),

$$|E^*| = (\alpha^2 + \beta^2\omega^2)^{1/2}$$

(see 5.14 and 6.6, note 3) and the remaining symbols are as defined in 6.7, note 2.

3. The ratio of the loss to the storage flexural modulus (E''/E') is derived from the *loss tangent* (tan δ) of the *forced flexural oscillation* with

$$\tan \delta = \beta\omega/\alpha = E''/E'$$

(see 5.11 and 6.7, note 3).

4. The flexural modulus has been given the same symbol as the *absolute modulus* in uniaxial deformation (see 5.14) as it becomes equal to that quantity in the limit of zero amplitudes of applied force and deformation. Under real experimental conditions it is often used as an approximation to $|E^*|$.

6.11 resonance curve

Recommended symbol: $A(\nu)$

The curve of the frequency dependence of the amplitude of the displacement of a material specimen subject to forced oscillations in the region of a resonance frequency.

Unit: that of the amplitude A

Notes

1. See 6.6 and 6.7 for the description of modes of forced oscillation commonly used.

2. See 6.12 for the definition of *resonance frequency*.

6.12 resonance frequency

Recommended symbol: ν_R

The frequency at a maximum of a resonance curve

Unit: Hz

Notes

1. See 6.11 for the definition of a *resonance curve*.

2. Material specimens subject to a *forced oscillations* (see 6.6 and 6.7) in general have a spectrum of resonance frequencies.

3. In cases of a single *resonance frequency*, the *resonance frequency* is proportional to the square root of the *storage modulus* (M') of the material (see 5.12).

4. A material specimen which behaves as a *Voigt-Kelvin solid* under forced oscillations with a mass added at the point of application of the applied oscillatory force has a single resonance frequency.

5. Under a *forced uniaxial extensional oscillation* the resonance frequency

$$v_R = \omega_R / 2\pi = \left(\frac{A\alpha}{Lm}\right)^{1/2} \Big/ 2\pi = \left(\frac{AE'}{Lm}\right)^{1/2} \Big/ 2\pi$$

(see 6.6 for the origin of the equation and definitions of symbols). E' is *the storage modulus in uniaxial extension* (see 5.12).

6. Under a *forced flexural oscillation* the resonance frequency

$$v_R = \omega_R / 2\pi = \left(\frac{HJ\alpha}{L^3 m}\right)^{1/2} \Big/ 2\pi = \left(\frac{HJE'}{L^3 m}\right)^{1/2} \Big/ 2\pi$$

(see 6.7 for the origin of the equation and the definition of symbols).

6.13 width of the resonance curve

Recommended symbol: Δv

The magnitude of the difference in frequency between two points on a resonance curve on either side of v_R which have amplitudes equal to $(1/\sqrt{2})A(v_R)$.

Unit: Hz

Notes

1. For a material specimen which behaves as a *Voigt-Kelvin solid* under *forced uniaxial extensional oscillation* with mass added at the point of application of the applied oscillatory force, Δv is proportional to the *loss modulus* (E'') (see 5.13).

$$2\pi\Delta v = \frac{A}{Lm} \cdot \beta = \frac{A}{Lm} \cdot \frac{E''}{\omega_R}$$

In addition (6.6, note 6), the *storage modulus* (E') (see 5.12) may be evaluated from

$$\omega_R^2 = \frac{A}{Lm} \cdot \alpha = \frac{A}{Lm} \cdot E'$$

(see 6.6 for the definition of symbols).

2. For a material specimen which behaves as *Voigt-Kelvin solid* under forced flexural oscillations with added mass at the point of application of the applied oscillatory force, Δv is proportional to the loss modulus (E'') (see 5.13)

$$2\pi\Delta v = \frac{HJ}{L^3 m} \cdot \beta = \frac{HJ}{L^3 m} \cdot \frac{E''}{\omega_R} \ .$$

In addition, the *storage modulus* (E') (see 5.12) may be evaluated form

$$\omega_R^2 = \frac{HJ}{L^3 m} \cdot \alpha = \frac{HJ}{L^3 m} \cdot E'$$

(see 6.7 for the definition of symbols).

3. For the *Voigt-Kelvin* behaviours specified in notes 1 and 2, the ratio of Δv and the resonance frequency (v_R) is equal to the *loss tangent* (tan δ).

Under *forced uniaxial extensional oscillation*

$$\frac{\Delta v}{v_R} = \left(\frac{A}{Lm}\right)\beta\omega_R \cdot \frac{Lm}{A\alpha} = \frac{\beta}{\alpha}\omega_R = \frac{E''}{E'} = \tan\delta \ .$$

Under *forced flexural oscillation*

$$\frac{\Delta v_R}{v_R} = \left(\frac{HJ}{L^3 m}\right)\beta\omega_R \cdot \frac{L^3 m}{HJ\alpha} = \frac{\beta}{\alpha}\omega_R = \frac{E''}{E'} = \tan\delta$$

(see 5.11 for the definition of tan δ).

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width of the resonance curve (6.13)

Young's modulus (4.7)

zero-shear viscosity (4.12)

9. GLOSSARY OF SYMBOLS

$A(\nu)$ resonance curve (6.11)

A_n Rivlin-Ericksen tensors (1.14)

B compliance in bulk compressive deformation (4.4)/
bulk compliance/bulk compressive compliance (4.6)

B Green tensor (1.8, 1.10)

B^{-1} Piola tensor (1.8)

C compliance (general symbol) (4.4)

C' storage compliance (general symbol)(5.16)

C'' loss compliance (general symbol) (5.17)

C^* complex compliance (general symbol) (5.19)

$|C^*|$ absolute compliance (general symbol) (5.18)

C Cauchy tensor (1.8, 1.9)

C^{-1} Finger tensor (1.8, 1.11)

D compliance in uniaxial deformation (4.4)/
uniaxial compliance/tensile compliance (4.8)

D' storage compliance in uniaxial deformation (5.16)

D'' loss compliance in uniaxial deformation (5.17)

D^* complex compliance in uniaxial deformation (5.19)

$|D^*|$ absolute compliance in uniaxial deformation (5.18)

D rate-of-strain tensor (1.12)

E modulus in uniaxial deformation (4.3)/Young's
modulus/tensile modulus/secant modulus/tangent modulus (4.7)

E' storage modulus in uniaxial deformation (5.12)

E'' loss modulus in uniaxial deformation (5.13)

E^* complex modulus in uniaxial deformation (5.15)

$|E^*|$ absolute modulus in uniaxial deformation (5.14)

$|E^*|$ flexural modulus (6.10)

F deformation gradient tensor for an elastic solid (1.4) and for a
viscoelastic liquid or solid (1.7)

f_0 flexural force (6.8)

G modulus in shear deformation (4.3)/shear modulus (4.10)

G' storage modulus in simple shear deformation (5.12)

G'' loss modulus in simple shear deformation (5.13)

G^* complex modulus in simple shear deformation (5.15)

$|G^*|$ absolute modulus in simple shear deformation (5.14)

J compliance in shear deformation (4.7)/shear compliance (4.11)/
creep compliance (5.9)

J second moment of area (in a forced flexural oscillation)(6.7)

J' storage compliance in simple shear deformation (5.16)

J'' loss compliance in simple shear deformation (5.17)

J^* complex compliance in simple shear deformation (5.19)

$|J^*|$ absolute compliance in simple shear deformation (5.18)

K	modulus in bulk compressive deformation (4.3)/bulk modulus/ bulk compressive modulus (4.5)

M	modulus (general symbol) (4.3)
M'	storage modulus (general symbol) (5.12)
M''	loss modulus (general symbol) (5.13)
M^*	complex modulus (general symbol) (5.15)
$|M^*|$	absolute modulus (general symbol) (5.14)

N_1	first normal-stress difference/first normal-stress function (3.6)
N_2	second normal-stress difference/second normal-stress function (3.7)

t	traction (1.1)
$\tan \delta$	loss factor/loss tangent (5.11)

W	stored energy function (4.1)
W	vorticity tensor (1.13)

y	flexural deflection (6.9)

β	decay constant (of a damping curve) (6.2, 6.3)

Δv	width of the resonance curve (6.13)

γ	shear/shear strain (2.8)
$\dot{\gamma}$:	shear rate (2.13)
γ_0	strain amplitude (of a forced oscillation)(5.10)
$\dot{\gamma}_E$	elongational strain rate/extension strain rate (2.12)
γ^*	complex strain (of a forced oscillation) (5.15, 5.19)
$\dot{\gamma}^*$	complex rate of strain (of a forced oscillation) (5.22)
δ	phase angle (of a forced oscillation)/loss angle of a forced oscillation (5.10)

ε	uniaxial strain/engineering strain/(uniaxial) tensile strain/(uniaxial) compressive strain (2.4)
ε	small-strain tensor (1.10)
ε_H	Hencky strain (2.5)
ε_{lat}	lateral strain (2.6)

η	shear viscosity/coefficient of viscosity/viscosity (4.12)
η'	dynamic viscosity (5.20)
η''	out-of-phase viscosity (5.21)
η_{app}	apparent viscosity (4.12)
η_E	extensional viscosity/elongational viscosity (4.9)
η_0	zero shear viscosity (4.12)
η^*	complex viscosity (5.22)
η_∞	infinite-shear viscosity (4.12)

λ uniaxial deformation ratio/deformation ratio/
extension ratio/compression ratio (2.3)

λ_i deformation gradients/deformation ratios;
$$i = 1,2,3 \quad (2.1)$$

Λ logarithmic decrement (of a decay curve) (6.5)

μ Poisson's ratio (2.6)

ν decay frequency (of a damping curve) (6.4)

ν_R resonance frequency (6.12)

σ tensile stress (3.2)

σ compressive stress (3.3)

σ engineering stress (3.4)

σ_{ii} normal stresses; i = 1,2,3 (3.5)

σ_0 stress amplitude (of a forced oscillation) (5.10)

σ_{12} shear stress (3.5)

σ stress/stress tensor (1.2, 3.1, 3.5)

σ^* complex stress (in a forced oscillation)
(5.15, 5.19, 5.22)

τ relaxation time (5.6)/retardation time (5.8)

χ bulk compression/volume compression/isotropic compression/
bulk compressive strain (2.9)

ψ_1 first normal-stress coefficient (4.13)

ψ_2 second normal-stress coefficient (4.14)

$\psi(t)$ creep function (5.9)

$\overline{\psi}(t)$ relaxation function (5.7)

ω angular velocity (of a forced oscillation) (5.10)

ω angular velocity (of a decay frequency) (6.2)

ω_R angular velocity of the resonance frequency
(of a forced flexural oscillation) (6.7)