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A Classification Scheme for Fuzzy Cellular Automata with Applications to ECA

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Using methods from the theory of fuzzy cellular automata as formulated over the past decade we present an analytic comprehensive derivation of a classification of fuzzy cellular automata (FCA) into four classes (1-4). Since fuzzy cellular automata include the elementary cellular automata in the limit, this classification can be applied to the original boolean ECA as considered by Wolfram and others [26]. When restricted in this way we can derive, in part, the Wolfram classification scheme.

Keywords: fuzzy cellular automata, FCA, ECA, cellular automata, boolean cellular automata, classification

1 INTRODUCTION

The problem of classifying a collection of mathematical objects is old and of current interest in a wide variety of fields. The starting point is to clearly outline or define what is meant by a *classification* of the set. The well known classification of ECA by Wolfram in [22–26], is based on visual attributes of space time diagrams taken over large sets of random initial conditions. One of the basic problems in classifying ECA is that the classification of the 256 boolean Wolfram ECA is not well defined in a mathematical sense. Much work went into defining the four well-known Wolfram classes (1-4) and this encouraged further classifications in papers that have enriched our

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understanding of CA theory, e.g., [1–3, 7], [10–11]. We recall briefly the structure of Wolfram's classification theory. Starting from a randomly chosen initial configuration a (boolean) ECA has an

1. evolution that leads to a homogeneous stable configuration (Class 1)
2. evolution that leads to periodically repeating patterns (Class 2)
3. evolution that leads to *chaotic* patterns (Class 3)
4. evolution that leads to complex patterns, generated by mobile interacting structures which are relatively long lived (Class 4).

There has been some success in understanding ECA as a result and much controversy has arisen as well. This is compounded by the fact that to the best of our knowledge the details of this classification by Wolfram was never published as such, yet subtle hints were given by Wolfram and collaborators as to which class specific rules may lie [26].

In the theory of fuzzy cellular automata one fuzzifies the disjunctive normal form (DNF) of the ECA by redefining the operations of disjunction, conjunction and negation so as to now build a map from the interval $[0, 1]^3$ into the unit interval. The resulting theory gives a rich theory for the iterations of continuous maps from $[0, 1]^3$ into $[0, 1]$ (see [5, 13–14, 20]). We summarize this construction of FCA herein for completeness.

The main question in this paper deals with the classification problem for FCA and its role in the classification problem of ECA alluded to above. Our approach to the problem of the classification of boolean ECA is via FCA; it avoids the visual attributes of the dynamics of corresponding transition (space-time) diagrams and is based solely on easily verifiable initial conditions and observations based upon the corresponding FCA. But first, we must redefine the notion of rule-equivalence, as we feel that equivalent rules (called here F-equivalent) should belong to the same class, regardless of the notion of *Class* under consideration. Then we observe that F-equivalent rules are necessarily equivalent (in the Wolfram sense). Once this is achieved we define the various classes axiomatically with the curious fact that exactly *four* classes, no more, no less, result from the definitions almost all of which contain rules that Wolfram placed in similarly numbered classes. One cannot expect a perfect class correspondence though and one of the known exceptions is ECA 110 which in our theory is Class 3, while it is Class 4 in Wolfram's classification. This in itself should deserve further study.

In this sense then, we can derive an algorithm to determine the Class of an ECA (via its related FCA) and this has been done in the case of all 256 rule names in this paper (see the Appendices). The result is that most of the rules known to have been classified as being in, say, Class X by Wolfram are also in Class X in our classification.

After reviewing the basic properties of elementary and fuzzy cellular automata we turn to the classification problem, state the theorems and their consequences, give a few consequences, and prove the theorems at the very end. Finally we exhibit the tables that derive from these theorems for reference purposes.

2 ELEMENTARY CELLULAR AUTOMATA

For the simplest introduction to this subject we refer to Wolfram's latest book [26]. Since this is a subject of enormous current interest we need only sketch the main definitions as it is our intention to move quickly to the case of continuous cellular automata, the ones considered here being called fuzzy cellular automata (or FCA). In general, a cellular automaton (CA) can be thought of a regular uniform lattice, finite or infinite, of cells where each cell contains a discrete variable or value. The state of the automaton at time t for the lattice-site i, denoted by x_i^t, is completely determined by its values in a neighborhood of this cell. More precisely, the evolution of a CA is defined in terms of an expression of the form

$$x_i^{t+1} = g(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t), \tag{1}$$

where g is sometimes called the *local rule* (or rule-table or local function) defining the CA. A (k, r)-CA is one where the lattice variables take one of k possible values and whose evolution is of the form (1). In this paper we are considering only (2, 1)-CA's, usually called elementary cellular automata (ECA for short), and their fuzzy counterparts. Since the neighborhood of each cell consists of two other such cells and each one of these three cells can contain at most two values (say, 0 and 1 by convention) it follows that there can be at most 2^{2^3} such CA, that is 256 ECA's (these are also referred to as boolean CA's). Since g maps the set of values in $\{0, 1\}^3$ to the two-point set $\{0, 1\}$, we describe the map g by looking at its values on the basic points: $(000, 001, 010, 011, 100, 101, 110, 111) \mapsto (r_0, r_1, \ldots, r_7)$ where each $r_i = 0$ or 1. As is done traditionally, we name each rule according to the numerical value of the binary string $(r_7, r_6, \ldots, r_0)$. That is, we name the rule via the value of the sum:

$$\textit{Rule Name} = \sum_{i=0}^{7} r_i 2^i.$$

Since we are dealing with two values, we may express each such local rule in a disjunctive normal form (DNF) using the binary operators *and* and *or*, [14]. That is, we can write the local rule as an expansion of ORs and of ANDs of the 3-tuples which generate a 1 under the given local rule, i.e., we can always write

$$g(x_1, x_2, x_3) = \vee_{i|r_i=1} \wedge_{j=1}^{3} x_j^{d_{ij}}, \tag{2}$$

where d_{ij} is the j-th digit from left to right of the binary representation of i and where x^0 represents $\neg x$ (the negation of x).

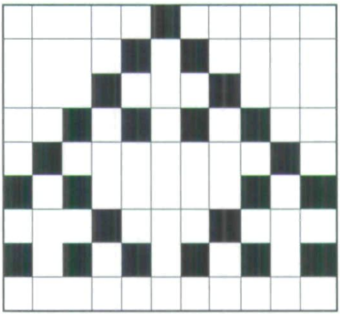
Example 1. By way of an example consider rule 218. Since $218 = 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^1$ its representation as a binary mapping:

$$(000, 001, 010, 011, 100, 101, 110, 111) \mapsto (0, 1, 0, 1, 1, 0, 1, 1)$$

gives the following DNF,

$$g_{218}(x_1, x_2, x_3) = (\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge \neg x_3) \\ \vee (x_1 \wedge x_2 \wedge \neg x_3) \vee (x_1 \wedge x_2 \wedge x_3). \tag{3}$$

An application of this rule to any triple $(0, 1, 0), (0, 0, 1)$, etc. starting from a single "1" seed value (or even finitely many random initial states) against a background of zeros shows that its evolution continues indefinitely left and right and may be computed for any finite number of time states. By setting the cells to small colored blocks, black for 1 and white for 0, we can visualize its dynamics as it is normally done. For example, the grid below shows the bi-color distribution of cells for FCA 90 with, e.g., initial seed 0.75. Note that it is identical to ECA 90's evolution for a single seed. This result is a consequence of the fact that the "look-up" tables for the two rules are identical, see [5], [14], [17].



3 INTRODUCTION TO FUZZY CELLULAR AUTOMATA (FCA)

The properties of binary cellular automata have their origins in works by Von Neumann [21] and Wolfram [26], [25]. We turn our attention to using elementary binary cellular automata in order to define a new class of continuous cellular automata and we do this by a process we call *fuzzification*. That is, we first "fuzzify" the automaton by removing the *binary* restriction and allowing the cell values to be any real number in the interval $[0, 1]$. This is done by altering the operations in the DNF (2) above of each of the rules for binary cellular automata as follows: $(x \vee y)$ becomes $(x + y)$, $(x \wedge y)$ becomes $(x \cdot y)$ and $(\neg x)$ becomes $(1 - x)$. Since each $x_j^{d_{ij}} \in [0, 1]$ in the DNF we know that the product $\prod_{j=1}^{3} x_j^{d_{ij}}$ is also a positive number in [0, 1]. The rule sum

$$g(x, y, z) = \sum_{i=0}^{7} r_i \cdot \prod_{j=1}^{3} x_j^{d_{ij}}$$

is thus maximized when $r_i = 1$ and minimized when $r_i = 0$ for all $i = 0, 1, \ldots, 7$. These values correspond to fuzzy rules $g_0(x, y, z) = 0$ and $g_{255}(x, y, z) = 1$ respectively. Since each FCA is essentially a partial sum of FCA 255 (=1) it follows that every corresponding local function is bounded below by 0 and above by 1. For other basic properties of these rules we refer the reader to [5] and [12]-[18]. For a more elaborate introduction to these FCA we refer the interested reader to [14], [17].

For example, let us again consider Rule 218. From above, we recall that the DNF of this rule is given by (3) above. We can fuzzify this using the identifications $x \vee y = x + y$ etc. defined above to find:

$$\begin{aligned} g_{218}(x, y, z) &= (1 - x)(1 - y)z + (1 - x)yz \\ &\quad + x(1 - y)(1 - z) + xy(1 - z) + xyz \\ &= x + z - 2xz + xyz \end{aligned}$$

We may then choose a seed value $\alpha \in [0, 1]$ and examine the evolution of the automaton over several discrete time steps. Let us choose, for example, $\alpha = 0.5$. This gives an evolution similar to rule 218 above with the number 0.5 scattered about. The space-time diagram is very similar to the one found in the discrete (or boolean) case which does not lead to anything of much interest. In fact, we can show that for any fuzzy rule the space-time diagram produced by an arbitrary seed $\alpha \in (0, 1)$ approaches the boolean space-time diagram as $\alpha \to 1^-$. However, we may also consider the dynamics of a fuzzy cellular automaton with several seeds on a background of zeros. For example, we now

time	state										
0	0	0	0	0	0.25	0.5	0.75	0	0	0	0
1	0	0	0	0.25	0.5	0.718	0.5	0.75	0	0	0
2	0	0	0.25	0.5	0.699	0.679	0.660	0.5	0.75	0	0
3	0	0.25	0.5	0.687	0.737	0.749	0.724	0.667	0.5	0.75	0
4	0.25	0.5	0.679	0.753	0.786	0.794	0.778	0.741	0.666	0.5	0.75
...					...						

TABLE 1
Fuzzy rule 218 running on the 3 seeds $(0.25, 0.5, 0.75)$

choose three consecutive seeds of 0.25,0.5,0.75. This generates a much more interesting evolution as seen in Table 1 below. The dynamics of its space-time diagram are discussed as a part of a more general scheme in [14].

It is important to note that the above method of fuzzification is not unique. We may choose to transform the binary expressions into functions on the interval using alternative fuzzy logics (cf., [17],[20]).

4 A FUZZY BASED CLASSIFICATION SCHEME

Prior to the formulation of another classification we must seek to better understand the notion of *equivalence* of ECA. This is done by defining an equivalent notion for the corresponding FCA, dubbed *F-equivalence*. Whatever the notion of "equivalence" being used, in our view, it is natural to assume *at least* that whenever "two FCA are equivalent" they should

1) belong to the same class,
2) have the same fixed points and
3) these fixed points should be of the same type (i.e., either attracting or repelling).

Terminology: In the sequel an FCA (or fuzzy rule), as understood here, will be called even (resp. odd) if its rule name is even (resp. odd). In addition, the symbol $g_n(x, y, z)$ will normally denote the local rule corresponding to FCA n. Whenever the rule name of an FCA is not explicitly mentioned we refer to its local function simply as $g(x, y, z)$.

Definition 1. *Consider two FCA with local rules $f(x, y, z), g(x, y, z)$. We say that these FCA are F-equivalent if*

$$f(x, y, z) = g(z, y, x), \tag{4}$$

for all $x, y, z \in [0, 1]$.

The reason for this definition is the next crucial result.

Lemma 1. *F-equivalent FCA's have the same fixed points and these fixed points are of the same type.*

It turns out that this alternate notion of equivalence carries over into the boolean case and gives a sharpening of Wolfram's notion of equivalence. Recall that in Wolfram's theory of equivalent rules (also adopted by many others), the allowable transformations include changing white squares into dark squares, and reflecting the (single-seed-generated) space-time diagram about the central line formed by the cells x_0^t, for $t = 0, 1, 2, \ldots$. Our Definition 1 essentially says that the former transformation is not allowed, i.e., rules obtained by changing the color of squares are not necessarily F-equivalent.

Indeed, whenever two rules are F-equivalent they are also Wolfram equivalent (or W-equivalent for short), but the reverse implication generally fails as we see next.

Example 2. The ECA 136, 192, 238, 252 are W-equivalent [[26],p.883]. However, they are not F-equivalent since the fixed points of the respective FCA are not of the same type. Specifically, 1 is repelling and 0 is attracting for FCA 136 and 192 while, for FCA 238 and 252, 1 is an attracting and 0 is a repelling fixed point. Of course, one could put all these rules in the same Wolfram Class by definition thereby satisfying 1) but one cannot avoid the negation of 3).

Example 3. An FCA may be equivalent to itself only, since it may be that (4) holds only for $g = f$. Examples of such are FCA 18, 90, 104. On the other hand, FCA 172 and 228 are F-equivalent, as are FCA 110 and 124.

Remark 1. An algorithm for determining the (unique) F-equivalent FCA of a given FCA follows: We write the rule name in base 2 and then write every exponent appearing there in base 2 as well. Since every such exponent is necessarily a number between 0 and 7, the base 2 representation of such a number consists of at most three (ordered) boolean numbers. Next, we interchange the first and the third such number and then substitute these numbers back into the original representation as the *new* exponents. The resulting number is the rule name of the F-equivalent FCA. An example should clarify this construction.

Example 4. Given FCA 192, we write $192 = 2^6 + 2^7$. Now $6 = (110)_2$ while $7 = (111)_2$. Interchanging the first and third numbers in $(110)_2$ gives us $(011)_2$, while the other remains unchanged. Thus, the *new* exponents are given by writing $(011)_2 = 3$ and $(111)_2 = 7$ as base 10 numbers and substituting these in lieu of the original exponents. This gives $2^3 + 2^7 = 136$, which is the F-equivalent FCA.

The method outlined in Remark 1 also applies to "self F-equivalent" FCA.

Example 5. For FCA 19, we have $19 = 2^4 + 2^1 + 2^0$. Since $4 = (100)_2$, $1 = (001)_2$ and $0 = (000)_2$ we see that interchanging the first and the third boolean entries does not change the resulting set of boolean numbers. Thus the new exponents remain unchanged as well. Hence the F-equivalent FCA is FCA 19, i.e., FCA 19 is self F-equivalent.

We now turn to the main question in this paper i.e., we produce a classification of ECA starting from their fuzzy counterparts.

Definition 2. *Let $g(x, y, z)$ be the local rule of an FCA. Its G-function is the function whose domain is $[0, 1]$ and whose values are given by*

$$G(x) = g(0, 0, x) + g(0, x, 0) + g(x, 0, 0).$$

The classification referred to is based on the following result regarding the G-function. The proof of Theorem 1 can be produced on a case-by-case basis if so desired (using the Tables in the Appendix).

Theorem 1. *For a given FCA with local function g, its G-function*

$$G : [0, 1] \to [0, 3].$$

In addition,

1) *If the rule is even then, for every $x \in [0, 1]$, either $G(x) = 0$, $G(x) = x$, $G(x) = 2x$, or $G(x) = 3x$, and exactly one of these must occur.*
2) *If the rule is odd then, for every $x \in [0, 1]$, either $G(x) = 3$, $G(x) = 3-x$, $G(x) = 3 - 2x$, or $G(x) = 3 - 3x$, and exactly one of these must occur.*

This theorem indicates that for any FCA with an even rule name, the function $G(x)/x$ is a constant on $(0, 1]$. A similar result applies in the case of FCA with odd rule names: In this case for every such rule name there is an i $(1 \le i \le 4)$ such that for every $x \in (0, 1]$, $G(x) + (i - 1)x = 3$. These quantities are now used to *define* the four classes of FCA (ECA):

Definition 3. *Let $0 \le n \le 255$. We say that FCA n is of Class i, where i is an integer $1 \le i \le 4$, if for every $x \in [0, 1]$,*

$$G(x) = 3\left(\frac{1 + (-1)^{n+1}}{2}\right) + (-1)^n (i - 1)\, x. \tag{5}$$

Remark 2. By Theorem 1 it follows that i can only be equal to $1, 2, 3, 4$ and so all the even rules may be grouped together into exactly four classes. The same is true for all the odd rules, however we do not place these in the same corresponding classes as the even rules at this time (see the Tables in the Appendix).

Example 3. Referring to Example 2 above, we can easily see that FCA 136 and 192 are both Class 1 (in our classification); they both have the fixed points 0, 1, and 0 is attracting while 1 is repelling in both cases. So, not only are these FCA F-equivalent, but they belong to the same class! This is a particular case of the following general result.

Theorem 2. *An FCA has only one F-equivalent FCA.*

Indeed, more is true.

Theorem 3. *F-equivalent FCA's must belong to the same Class.*

Remark 3. In Theorem 2 we do not exclude the possibility that the F-equivalent FCA is the FCA itself. Since fuzzy rules (or FCA) include boolean rules (or

ECA) in the limit as the seed values approach 1 (i.e., FCA include ECA) it is natural to use this definition so as to redefine the various *classes* of ECA. Theorem 2 shows that an FCA in a given class can have only one F-equivalent counterpart in the same Class! This is probably the best that one can hope for in terms of a suitable classification of FCA. It basically says that equivalent FCA's should be in the same Class and have the same fixed points and the same type of fixed points. In addition, elements in the same Class can be distinguished from each other using the notion of F-equivalence, their fixed points and their respective types of fixed points. Of course, Theorem 3 is the result we sought, in the end.

5 DISCUSSION

The FCA 0, 32, 72, 104, 128, 160, 200 and 232 are all Class 1 FCA (see the Appendix). Next, FCA 4, 36, 76, 108, 132, 164, 204, 236 are examples of Class 2 FCA. Furthermore, FCA 18, 50, 90, 122, 146, 178, 218, 250 are examples of Class 3 FCA, while finally FCA 22, 54, 94, 126, 150, 182, 222, 254 are examples of Class 4 FCA (cf., [26], p. 232). We note that all of the rules listed above are in the same ECA classes as defined by Wolfram [[26], p. 232]. In contrast, Wolfram [[26], p.252] places ECA 110 in Class 4 (which is our Class 3, since $G(x) = 2x$ here). The Wolfram classification actually *moves rules around* the various classes depending upon the type of initial conditions, although one would think that an FCA's "Class" should be independent of initial strings.

A glance at Table 6 reveals interesting features of *exceptional* FCA, i.e., those FCA whose fixed points fill all of the interval [0, 1] (given by FCA 170, 172, 184, 202, 204, 216, 226, 228, 240). We see that these FCA are all Class 2. We note, in passing, that the dynamics of these exceptional FCA are studied in [13], [18].

Next, we observe that regardless whether the rule name is even or odd there are three times as many FCA in Class 2 and 3 than there are in Class 1 and 4 (see Tables 3 and 4). In fact, the 8 *additive* FCA (i.e., those whose corresponding ECA are additive in the usual sense, [[26], p. 962]) are spread in the same ratio: There is one additive FCA in each of Class 1 and 4 and three additive FCA in each of Class 2 and 3. Note that FCA n has zero for a fixed point if and only if n is even, yet if n is odd 1 is a fixed point of these only if $n > 128$.

In addition, from Tables 4, 5 and 6 we see that, except for the exceptional FCA where every number is a fixed point, only 10 specific numbers can be fixed points of the 128 even FCA, namely

$$0, 1, 2 - \sqrt{2}, \frac{1}{2}(2 - \sqrt{2}), \frac{1}{3}(3 - \sqrt{3}), \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \phi, 1 - \phi$$

where $\phi = (\sqrt{5} - 1)/2$ is the golden number. Of these 10 numbers only $(3 - \sqrt{3})/3$ is a fixed point of one and only one FCA: FCA 22. Furthermore, another fact is that, of these 10 numbers, half of them are rational and the other half are irrational. For the case of the 128 remaining odd FCA, only 18 numbers can be fixed points 3 of which are rational, namely $1, 1/2, 1/3$ (cf., Tables 8, 9, 10).

We observe that the only rules having the golden number as a (attracting) fixed point are FCA 110, 122 and 124 (all Class 3). Thus, although ECA 124 is W-equivalent to ECA 110, ECA 122 does not enter the picture at all. The fuzzy scenario given here however seems to indicate that FCA 122 is quite similar to FCA 110 (and its F-equivalent cousin, FCA 124). It can only be hoped that the properties of FCA 122 (or ECA 122) are similar to those of FCA 110 (or ECA 110) but this requires further investigation.

6 PROOFS

Proof. (Lemma 1) Let f, g be the local rules of the given FCA's. Since they are F-equivalent, (4) and the definition of the fixed points together imply that the fixed points x are given by $x = f(x, x, x) = g(x, x, x)$. The result follows. □

Proof. (Theorem 2) Consider an FCA with rule function f. We know already (either by the Tables in the Appendix or by more general means) that there is at least one FCA with rule function g such that (4) holds. Assume there is another distinct FCA with rule function h, say, with this property. Since the relation of these FCA is an actual equivalence relation we have that $f(x, y, z) = g(z, y, x) = h(z, y, x)$ from which $g(z, y, x) = h(z, y, x)$ for all $x, y, z \in [0, 1]$. Thus, the rule functions g and h are identical and this is a contradiction. Hence there can be at most one FCA that is F-equivalent to a given one. It follows by Theorem 3 that these FCA must be in the same Class. That there exists at least one such FCA is clear. □

Proof. (Theorem 3) We show that F-equivalent FCA's must be in the same Class as defined in Definition 3. For let f, g be the local rules of the given FCA and G_f, G_g their associated G-functions (see Definition 2). Since there holds (4) we see that, for every $x \in [0, 1]$, $f(x, 0, 0) = g(0, 0, x)$, $f(0, x, 0) = g(0, x, 0)$ and $f(0, 0, x) = g(x, 0, 0)$. It follows that $G_f(x) = G_g(x)$ for every $x \in [0, 1]$. But since these G-functions satisfy the conclusion of Theorem 1-(1) we see that, since they are equal, the FCA they represent must therefore be in the same Class. □

7 CONCLUSION

We present a new axiomatic definition of equivalence of FCA that leads to a new definition of the same notion for ECA. As a result we are led naturally to a new analytically derived classification of FCA (and so ECA) into four classes within each of which we find many of the original Wolfram ECA. The impact of this FCA classification on the asymptotics of space-time diagrams will be considered in a sequel to this paper.

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$n=0$	0	1	$1-x-y+xy-z+xz+yz-xyz$
2	$z-xz-yz+xyz$	3	$1-x-y+xy$
4	$y-xy-yz+xyz$	5	$1-x-z+xz$
6	$y-xy+z-xz-2yz+2xyz$	7	$1-x-yz+xyz$
8	$yz-xyz$	9	$1-x-y+xy-z+xz+2yz-2xyz$
10	$z-xz$	11	$1-x-y+xy+yz-xyz$
12	$y-xy$	13	$1-x-z+xz+yz-xyz$
14	$y-xy+z-xz-yz+xyz$	15	$1-x$
16	$x-xy-xz+xyz$	17	$1-y-z+yz$
18	$x-xy+z-2xz-yz+2xyz$	19	$1-y-xz+xyz$
20	$x+y-2xy-xz-yz+2xyz$	21	$1-xy-z+xyz$
22	$x+y-2xy+z-2xz-2yz+3xyz$	23	$1-xy-xz-yz+2xyz$
24	$x-xy-xz+yz$	25	$1-y-z+2yz-xyz$
26	$x-xy+z-2xz+xyz$	27	$1-y-xz+yz$
28	$x+y-2xy-xz+xyz$	29	$1-xy-z+yz$
30	$x+y-2xy+z-2xz-yz+2xyz$	31	$1-xy-xz+xyz$
32	$xz-xyz$	33	$1-x-y+xy-z+2xz+yz-2xyz$
34	$z-yz$	35	$1-x-y+xy+xz-xyz$
36	$y-xy+xz-yz$	37	$1-x-z+2xz-xyz$
38	$y-xy+z-2yz+xyz$	39	$1-x+xz-yz$
40	$xz+yz-2xyz$	41	$1-x-y+xy-z+2xz+2yz-3xyz$
42	$z-xyz$	43	$1-x-y+xy+xz+yz-2xyz$
44	$y-xy+xz-xyz$	45	$1-x-z+2xz+yz-2xyz$
46	$y-xy+z-yz$	47	$1-x+xz-xyz$
48	$x-xy$	49	$1-y-z+xz+yz-xyz$
50	$x-xy+z-xz-yz+xyz$	51	$1-y$
52	$x+y-2xy-yz+xyz$	53	$1-xy-z+xz$
54	$x+y-2xy+z-xz-2yz+2xyz$	55	$1-xy-yz+xyz$
56	$x-xy+yz-xyz$	57	$1-y-z+xz+2yz-2xyz$
58	$x-xy+z-xz$	59	$1-y+yz-xyz$
60	$x+y-2xy$	61	$1-xy-z+xz+yz-xyz$
62	$x+y-2xy+z-xz-yz+xyz$	63	$1-xy$
64	$xy-xyz$	65	$1-x-y+2xy-z+xz+yz-2xyz$
66	$xy+z-xz-yz$	67	$1-x-y+2xy-xyz$
68	$y-yz$	69	$1-x+xy-z+xz-xyz$
70	$y+z-xz-2yz+xyz$	71	$1-x+xy-yz$
72	$xy+yz-2xyz$	73	$1-x-y+2xy-z+xz+2yz-3xyz$
74	$xy+z-xz-xyz$	75	$1-x-y+2xy+yz-2xyz$
76	$y-xyz$	77	$1-x+xy-z+xz+yz-2xyz$
78	$y+z-xz-yz$	79	$1-x+xy-xyz$
80	$x-xz$	81	$1-y+xy-z+yz-xyz$
82	$x+z-2xz-yz+xyz$	83	$1-y+xy-xz$
84	$x+y-xy-xz-yz+xyz$	85	$1-z$
86	$x+y-xy+z-2xz-2yz+2xyz$	87	$1-xz-yz+xyz$
88	$x-xz+yz-xyz$	89	$1-y+xy-z+2yz-2xyz$
90	$x+z-2xz$	91	$1-y+xy-xz+yz-xyz$
92	$x+y-xy-xz$	93	$1-z+yz-xyz$
94	$x+y-xy+z-2xz-yz+xyz$	95	$1-xz$
96	$xy+xz-2xyz$	97	$1-x-y+2xy-z+2xz+yz-3xyz$
98	$xy+z-yz-xyz$	99	$1-x-y+2xy+xz-2xyz$
100	$y+xz-yz-xyz$	101	$1-x+xy-z+2xz-2xyz$

TABLE 2
Continued

102	$y + z - 2yz$	103	$1 - x + xy + xz - yz - xyz$
104	$xy + xz + yz - 3xyz$	105	$1 - x - y + 2xy - z + 2xz + 2yz - 4xyz$
106	$xy + z - 2xyz$	107	$1 - x - y + 2xy + xz + yz - 3xyz$
108	$y + xz - 2xyz$	109	$1 - x + xy - z + 2xz + yz - 3xyz$
110	$y + z - yz - xyz$	111	$1 - x + xy + xz - 2xyz$
112	$x - xyz$	113	$1 - y + xy - z + xz + yz - 2xyz$
114	$x + z - xz - yz$	115	$1 - y + xy - xyz$
116	$x + y - xy - yz$	117	$1 - z + xz - xyz$
118	$x + y - xy + z - xz - 2yz + xyz$	119	$1 - yz$
120	$x + yz - 2xyz$	121	$1 - y + xy - z + xz + 2yz - 3xyz$
122	$x + z - xz - xyz$	123	$1 - y + xy + yz - 2xyz$
124	$x + y - xy - xyz$	125	$1 - z + xz + yz - 2xyz$
126	$x + y - xy + z - xz - yz$	127	$1 - xyz$
128	xyz	129	$1 - x - y + xy - z + xz + yz$
130	$z - xz - yz + 2xyz$	131	$1 - x - y + xy + xyz$
132	$y - xy - yz + 2xyz$	133	$1 - x - z + xz + xyz$
134	$y - xy + z - xz - 2yz + 3xyz$	135	$1 - x - yz + 2xyz$
136	yz	137	$1 - x - y + xy - z + xz + 2yz - xyz$
138	$z - xz + xyz$	139	$1 - x - y + xy + yz$
140	$y - xy + xyz$	141	$1 - x - z + xz + yz$
142	$y - xy + z - xz - yz + 2xyz$	143	$1 - x + xyz$
144	$x - xy - xz + 2xyz$	145	$1 - y - z + yz + xyz$
146	$x - xy + z - 2xz - yz + 3xyz$	147	$1 - y - xz + 2xyz$
148	$x + y - 2xy - xz - yz + 3xyz$	149	$1 - xy - z + 2xyz$
150	$x + y - 2xy + z - 2xz - 2yz + 4xyz$	151	$1 - xy - xz - yz + 3xyz$
152	$x - xy - xz + yz + xyz$	153	$1 - y - z + 2yz$
154	$x - xy + z - 2xz + 2xyz$	155	$1 - y - xz + yz + xyz$
156	$x + y - 2xy - xz + 2xyz$	157	$1 - xy - z + yz + xyz$
158	$x + y - 2xy + z - 2xz - yz + 3xyz$	159	$1 - xy - xz + 2xyz$
160	xz	161	$1 - x - y + xy - z + 2xz + yz - xyz$
162	$z - yz + xyz$	163	$1 - x - y + xy + xz$
164	$y - xy + xz - yz + xyz$	165	$1 - x - z + 2xz$
166	$y - xy + z - 2yz + 2xyz$	167	$1 - x + xz - yz + xyz$
168	$xz + yz - xyz$	169	$1 - x - y + xy - z + 2xz + 2yz - 2xyz$
170	z	171	$1 - x - y + xy + xz + yz - xyz$
172	$y - xy + xz$	173	$1 - x - z + 2xz + yz - xyz$
174	$y - xy + z - yz + xyz$	175	$1 - x + xz$
176	$x - xy + xyz$	177	$1 - y - z + xz + yz$
178	$x - xy + z - xz - yz + 2xyz$	179	$1 - y + xyz$
180	$x + y - 2xy - yz + 2xyz$	181	$1 - xy - z + xz + xyz$
182	$x + y - 2xy + z - xz - 2yz + 3xyz$	183	$1 - xy - yz + 2xyz$
184	$x - xy + yz$	185	$1 - y - z + xz + 2yz - xyz$
186	$x - xy + z - xz + xyz$	187	$1 - y + yz$
188	$x + y - 2xy + xyz$	189	$1 - xy - z + xz + yz$
190	$x + y - 2xy + z - xz - yz + 2xyz$	191	$1 - xy + xyz$
192	xy	193	$1 - x - y + 2xy - z + xz + yz - xyz$
194	$xy + z - xz - yz + xyz$	195	$1 - x - y + 2xy$
196	$y - yz + xyz$	197	$1 - x + xy - z + xz$
198	$y + z - xz - 2yz + 2xyz$	199	$1 - x + xy - yz + xyz$
200	$xy + yz - xyz$	201	$1 - x - y + 2xy - z + xz + 2yz - 2xyz$
202	$xy + z - xz$	203	$1 - x - y + 2xy + yz - xyz$

TABLE 2
Continued

204	y	205	$1-x+xy-z+xz+yz-xyz$
206	$y+z-xz-yz+xyz$	207	$1-x+xy$
208	$x-xz+xyz$	209	$1-y+xy-z+yz$
210	$x+z-2xz-yz+2xyz$	211	$1-y+xy-xz+xyz$
212	$x+y-xy-xz-yz+2xyz$	213	$1-z+xyz$
214	$x+y-xy+z-2xz-2yz+3xyz$	215	$1-xz-yz+2xyz$
216	$x-xz+yz$	217	$1-y+xy-z+2yz-xyz$
218	$x+z-2xz+xyz$	219	$1-y+xy-xz+yz$
220	$x+y-xy-xz+xyz$	221	$1-z+yz$
222	$x+y-xy+z-2xz-yz+2xyz$	223	$1-xz+xyz$
224	$xy+xz-xyz$	225	$1-x-y+2xy-z+2xz+yz-2xyz$
226	$xy+z-yz$	227	$1-x-y+2xy+xz-xyz$
228	$y+xz-yz$	229	$1-x+xy-z+2xz-xyz$
230	$y+z-2yz+xyz$	231	$1-x+xy+xz-yz$
232	$xy+xz+yz-2xyz$	233	$1-x-y+2xy-z+2xz+2yz-3xyz$
234	$xy+z-xyz$	235	$1-x-y+2xy+xz+yz-2xyz$
236	$y+xz-xyz$	237	$1-x+xy-z+2xz+yz-2xyz$
238	$y+z-yz$	239	$1-x+xy+xz-xyz$
240	x	241	$1-y+xy-z+xz+yz-xyz$
242	$x+z-xz-yz+xyz$	243	$1-y+xy$
244	$x+y-xy-yz+xyz$	245	$1-z+xz$
246	$x+y-xy+z-xz-2yz+2xyz$	247	$1-yz+xyz$
248	$x+yz-xyz$	249	$1-y+xy-z+xz+2yz-2xyz$
250	$x+z-xz$	251	$1-y+xy+yz-xyz$
252	$x+y-xy$	253	$1-z+xz+yz-xyz$
254	$x+y-xy+z-xz-yz+xyz$	255	1

TABLE 2
Rule Names n, and their Local Fuzzy Rules $g_n(x, y, z)$.

Class 1	Class 2	Class 3	Class 4
0	2, 4, 10	6, 14, 18	22
8	12, 16, 24	20, 26, 28	30
32	34, 36, 42	38, 46, 50	54
40	44, 48, 56	52, 58, 60	62
64	66, 68, 74	70, 78, 82	86
72	76, 80, 88	84, 90, 92	94
96	98, 100, 106	102, 110, 114	118
104	108, 112, 120	116, 122, 124	126
128	130, 132, 138	134, 142, 146	150
136	140, 144, 152	148, 154, 156	158
160	162, 164, 170	166, 174, 178	182
168	172, 176, 184	180, 186, 188	190
192	194, 196, 202	198, 206, 210	214
200	204, 208, 216	212, 218, 220	222
224	226, 228, 234	230, 238, 242	246
232	236, 240, 248	244, 250, 252	254

TABLE 3
Classification of even numbered rules according to the values of the G-function

Class 1	Class 2	Class 3	Class 4
23	7, 15, 19	3, 5, 11	1
31	21, 27, 29	13, 17, 25	9
55	39, 47, 51	35, 37, 43	33
63	53, 59, 61	45, 49, 57	41
87	71, 79, 83	67, 69, 75	65
95	85, 91, 93	77, 81, 89	73
119	103, 111, 115	99, 101, 107	97
127	117, 123, 125	109, 113, 121	105
151	135, 143, 147	131, 133, 139	129
159	149, 155, 157	141, 145, 153	137
183	167, 175, 179	163, 165, 171	161
191	181, 187, 189	173, 177, 185	169
215	199, 207, 211	195, 197, 203	193
223	213, 219, 221	205, 209, 217	201
247	231, 239, 243	227, 229, 235	225
255	245, 251, 253	237, 241, 249	233

TABLE 4
Classification of odd numbered rules according to the values of the G-function

Rule Name	F-Equiv. Rule	Real Fix. Point(s)	Class
0	0	$0a$	1
2	16	$0r$	2
4	4	$0r$	2
6	20	$0r, 1-\sqrt{2}/2\ a$	3
8	64	$0a$	1
10	80	$0r$	2
12	68	$0r$	2
14	84	$0r, 1-\phi\ a$	3
16	2	$0r$	2
18	18	$0r, 1-\sqrt{2}/2\ a$	3
20	6	$0r, 1-\sqrt{2}/2\ a$	3
22	22	$0r, 1-\sqrt{3}/3\ a$	4
24	66	$0r$	2
26	82	$0r, 1-\phi\ a$	3
28	70	$0r, 1-\phi\ a$	3
30	86	$0r, 1/2\ a$	4
32	32	$0a$	1
34	48	$0r$	2
36	36	$0r$	2
38	52	$0r, 1-\phi\ a$	3
40	96	$0a$	1
42	112	$0a$	2
44	100	$0a$	2
46	116	$0r, 1/2\ a$	3
48	34	$0r$	2
50	50	$0r, 1-\phi\ a$	3
52	38	$0r, 1-\phi\ a$	3
54	54	$0r, 1/2\ a$	4
56	98	$0a$	2
58	114	$0r, 1/2\ a$	3
60	102	$0r, 1/2\ a$	3
62	118	$0r, 2-\sqrt{2}\ a$	4
64	8	$0a$	1
66	24	$0r$	2
68	12	$0r$	2
70	28	$0r, 1-\phi\ a$	3
72	72	$0a$	1
74	88	$0a$	2
76	76	$0a$	2
78	92	$0r, 1/2\ a$	3
80	10	$0r$	2
82	26	$0r, 1-\phi\ a$	3
84	14	$0r, 1-\phi\ a$	3
86	30	$0r, 1/2\ a$	4
88	74	$0a$	2
90	90	$0r, 1/2\ a$	3
92	78	$0r, 1/2\ a$	3
94	94	$0r, 2-\sqrt{2}\ a$	4
96	40	$0a$	1
98	56	$0a$	2
100	44	$0a$	2

TABLE 5
Classification of even numbered rules, their F-equivalent rule, their fixed points, and their type. Here $\phi = (\sqrt{5}-1)/2$ is the golden number. The notation xr, $y\ a$ in Col. 3 means that x is a repelling fixed point and y is an attracting fixed point.

Rule Name	F-Equiv. Rule	Real Fix. Point(s)	Class
102	60	$0r, 1/2\ a$	3
104	104	$0a$	1
106	120	$0r$	2
108	108	$0r$	2
110	124	$0r, \phi\ a$	3
112	42	$0a$	2
114	58	$0r, 1/2\ a$	3
116	46	$0r, 1/2\ a$	3
118	62	$0r, 2-\sqrt{2}\ a$	4
120	106	$0r$	2
122	122	$0r, \phi\ a$	3
124	110	$0r, \phi\ a$	3
126	126	$0r, 2/3\ r$	4
128	128	$0a, 1r$	1
130	144	$0r, 1r$	2
132	132	$0r, 1r$	2
134	148	$0r, 1/3\ a$	3
136	192	$0a, 1r$	1
138	208	$0r, 1r$	2
140	196	$0r, 1r$	2
142	212	$0r, 1/2\ a$	3
144	130	$0r, 1r$	2
146	146	$0r, 1/3\ a$	3
148	134	$0r, 1/3\ a$	3
150	150	$0r, 1/2\ a$	4
152	194	$0r, 1r$	2
154	210	$0r, 1/2\ a$	3
156	198	$0r, 1/2\ a$	3
158	214	$0r, 2/3\ a$	4
160	160	$0a, 1r$	1
162	176	$0r, 1r$	2
164	164	$0r, 1r$	2
166	180	$0r, 1/2\ a$	3
168	224	$0a, 1r$	1
170	240	$0r$, all x	2
172	228	$0r$, all x	2
174	244	$0r, 1r$	3
176	162	$0r, 1r$	2
178	178	$0r, 1/2\ a$	3
180	166	$0r, 1/2\ a$	3
182	182	$0r, 2/3\ a$	4
184	226	$0r$, all x	2
186	242	$0r, 1r$	3
188	230	$0r, 1r$	3
190	246	$0r, 1r$	4
192	136	$0a, 1r$	1
194	152	$0r, 1r$	2
196	140	$0r, 1r$	2
198	156	$0r, 1/2\ a$	3
200	200	$0a, 1r$	1

TABLE 6
Classification of even numbered rules, their F-equivalent rule, their fixed points, and their type. Here $\phi = (\sqrt{5}-1)/2$ is the golden number. The notation xr, $y\ a$ in Col. 3 means that x is a repelling fixed point and y is an attracting fixed point.

Rule Name	F-Equiv. Rule	Real Fix. Point(s)	Class
202	216	$0r$, all x	2
204	204	$0r$, all x	2
206	220	$0r$, $1r$	3
208	138	$0r$, $1r$	2
210	154	$0r$, $1/2\ a$	3
212	142	$0r$, $1/2\ a$	3
214	158	$0r$, $2/3\ a$	4
216	202	$0r$, all x	2
218	218	$0r$, $1r$	3
220	206	$0r$, $1r$	3
222	222	$0r$, $1r$	4
224	168	$0a$, $1r$	1
226	184	$0r$, all x	2
228	172	$0a$, all x	2
230	188	$0r$, $1r$	3
232	232	$0a$, $1/2\ r$	1
234	248	$0r$, $1a$	2
236	236	$0r$, $1a$	2
238	252	$0r$, $1a$	3
240	170	$0r$, all x	2
242	186	$0r$, $1r$	3
244	174	$0r$, $1r$	3
246	190	$0r$, $1r$	4
248	234	$0r$, $1a$	2
250	250	$0r$, $1a$	3
252	238	$0r$, $1a$	3
254	254	$0r$, $1a$	4

TABLE 7

Classification of even numbered rules, their F-equivalent rule, their fixed points, and their type. Here $\phi = (\sqrt{5} - 1)/2$ is the golden number. The notation xr, $y\ a$ in Col. 3 means that x is a repelling fixed point and y is an attracting fixed point.

Rule Name	F-Equiv. Rule	Real Fix. Point(s)	Class
1	1	$0.317672196^{1}\ r$	4
3	17	$0.381966012^{2}\ r$	3
5	5	$0.381966012^{2}\ r$	3
7	21	$0.445041868^{3}\ r$	2
9	65	$0.352201128^{4}\ a$	4
11	81	$0.430159708^{5}\ a$	3
13	69	$0.430159708^{5}\ a$	3
15	85	$1/2\ r$	2
17	3	$0.381966012^{2}\ r$	3
19	19	$0.445041868^{3}\ r$	2
21	7	$0.445041868^{3}\ r$	2
23	23	$1/2\ r$	1
25	67	$0.430159708^{5}\ a$	3
27	83	$1/2\ r$	2
29	71	$1/2\ r$	2
31	87	$0.554958133^{6}\ r$	1
33	33	$0.352201128^{4}\ a$	4
35	49	$0.430159708^{5}\ a$	3
37	37	$0.430159708^{5}\ a$	3
39	53	$1/2\ r$	2
41	97	$0.405585523^{7}\ a$	4
43	113	$1/2\ a$	3
45	101	$1/2\ a$	3
47	117	$0.569840292^{8}\ a$	2
49	35	$0.430159708^{5}\ a$	3
51	51	$1/2\ r$	2
53	39	$1/2\ r$	2
55	55	$0.554958133^{6}\ r$	1
57	99	$1/2\ a$	3
59	115	$0.569840292^{8}\ a$	2
61	103	$0.569840292^{8}\ a$	2
63	119	$\phi^{9}\ r$	1
65	9	$0.352201128^{4}\ a$	4
67	25	$0.430159708^{5}\ a$	3
69	13	$0.430159708^{5}\ a$	3
71	29	$1/2\ r$	2
73	73	$0.405585523^{7}\ a$	4
75	89	$1/2\ a$	3
77	77	$1/2\ a$	3
79	93	$0.569840292^{8}\ a$	2
81	11	$0.430159708^{5}\ a$	3
83	27	$1/2\ r$	2
85	15	$1/2\ r$	2
87	31	$0.554958133^{6}\ r$	1
89	75	$1/2\ a$	3
91	91	$0.569840292^{8}\ a$	2
93	79	$0.569840292^{8}\ a$	2
95	95	$\phi^{9}\ r$	1
97	41	$0.405585523^{7}\ a$	4
99	57	$1/2\ a$	3
101	45	$1/2\ a$	3

TABLE 8
Classification of odd numbered rules, their F-equivalent rule, their fixed points, and their type. The notation xr, $y\ a$ in Col. 3 means that x is a repelling fixed point and y is an attracting fixed point. Here $\phi = (\sqrt{5}-1)/2$ is the golden number. Actual values are at the very end of the article.

Rule Name	F-Equiv. Rule	Real Fix. Point(s)	Class
103	61	$0.569840292^{8}\ a$	2
105	105	$1/2\ a$	4
107	121	$0.594414477^{10}\ a$	3
109	109	$0.594414477^{10}\ a$	3
111	125	$0.647798872^{11}\ a$	2
113	43	$1/2\ a$	3
115	59	$0.569840292^{8}\ a$	2
117	47	$0.569840292^{8}\ a$	2
119	63	$\phi^{9}\ r$	1
121	107	$0.594414477^{10}\ a$	3
123	123	$0.647798872^{11}\ a$	2
125	111	$0.647798872^{11}\ a$	2
127	127	$0.682327802^{12}\ r$	1
129	129	$1r, 1/3\ r$	4
131	145	$1r,\ (\sqrt{2}-1),\ a$	3
133	133	$1r,\ (\sqrt{2}-1)\ a$	3
135	149	$1r, 1/2\ r$	2
137	193	$1r,\ 0.381966012^{2}\ a$	4
139	209	$1r, 1/2\ a$	3
141	197	$1r, 1/2\ a$	3
143	213	$1r,\ \phi\ a$	2
145	131	$1r,\ (\sqrt{2}-1)\ a$	3
147	147	$1r,\ 1/2\ r$	2
149	135	$1r,\ 1/2\ r$	2
151	151	$1r,\ \sqrt{3}/3\ a$	1
153	195	$1r,\ 1/2\ a$	3
155	211	$1r,\ \phi\ a$	2
157	199	$1r,\ \phi\ a$	2
159	215	$1r,\ \sqrt{2}/2\ a$	1
161	161	$1r,\ (1-\phi)\ a$	4
163	177	$1r,\ 1/2\ a$	3
165	165	$1r,\ 1/2\ a$	3
167	181	$1r,\ \phi\ a$	2
169	225	$1r,\ 1/2\ a$	4
171	241	$1r$	3
173	229	$1r$	3
175	245	$1r$	2
177	163	$1r,\ 1/2\ a$	3
179	179	$1r,\ \phi\ a$	2
181	167	$1r,\ \phi\ a$	2
183	183	$1r,\ \sqrt{2}/2\ a$	1
185	227	$1r$	3
187	243	$1r$	2
189	231	$1r$	2
191	247	$1r$	1
193	137	$1r,\ 0.381966012^{2}\ a$	4
195	153	$1r,\ 1/2\ a$	3
197	141	$1r, 1/2\ a$	3
199	157	$1r,\ \phi\ a$	2
201	201	$1r, 1/2\ a$	4

TABLE 9

Classification of odd numbered rules, their F-equivalent rule, their fixed points, and their type. Here $\phi = (\sqrt{5}-1)/2$ is the golden number. The notation xr, $y\ a$ in Col. 3 means that x is a repelling fixed point and y is an attracting fixed point.

Rule Name	F-Equiv. Rule	Real Fix. Point(s)	Class
203	217	$1r$	3
205	205	$1r$	3
207	221	$1r$	2
209	139	$1r, 1/2\ a$	3
211	155	$1r,\ \phi\ a$	2
213	143	$1r,\ \phi\ a$	2
215	159	$1r,\ \sqrt{2}/2\ a$	1
217	203	$1r$	3
219	219	$1r$	2
221	207	$1r$	2
223	223	$1r$	1
225	169	$1r,\ 1/2\ a$	4
227	185	$1r$	3
229	173	$1r$	3
231	189	$1r$	2
233	233	$1a$	4
235	249	$1a$	3
237	237	$1a$	3
239	253	$1a$	2
241	171	$1r$	3
243	187	$1r$	2
245	175	$1r$	2
247	191	$1r$	1
249	235	$1a$	3
251	251	$1a$	2
253	239	$1a$	2
255	255	$1a$	1

TABLE 10
Classification of odd numbered rules, their F-equivalent rule, their fixed points, and their type. Here $\phi = (\sqrt{5} - 1)/2$ is the golden number. The notation xr, $y\ a$ in Col. 3 means that x is a repelling fixed point and y is an attracting fixed point.

Notation: (cf., Tables 8, 9, 10)

[1]: $-\frac{1}{6}\sqrt[3]{108+12\sqrt{93}}+2\,\frac{1}{\sqrt[3]{108+12\sqrt{93}}}+1$

[2]: $1-\phi$

[3]: $-\frac{1}{12}\sqrt[3]{-28+84\,i\sqrt{3}}-\frac{7}{3}\,\frac{1}{\sqrt[3]{-28+84\,i\sqrt{3}}}+\frac{1}{3}$

$-\frac{1}{2}\,i\sqrt{3}\left(\frac{1}{6}\sqrt[3]{-28+84\,i\sqrt{3}}-\frac{14}{3}\,\frac{1}{\sqrt[3]{-28+84\,i\sqrt{3}}}\right)$

[4]: $-\frac{1}{6}\sqrt[3]{26+6\sqrt{33}}+\frac{4}{3}\,\frac{1}{\sqrt[3]{26+6\sqrt{33}}}+\frac{2}{3}$

[5]: $-\frac{1}{6}\sqrt[3]{44+12\sqrt{69}}+\frac{10}{3}\,\frac{1}{\sqrt[3]{44+12\sqrt{69}}}+\frac{2}{3}$

[6]: $-1/12\sqrt[3]{28+84\,i\sqrt{3}}-7/3\,\frac{1}{\sqrt[3]{28+84\,i\sqrt{3}}}+2/3$

$-1/2\,i\sqrt{3}\left(1/6\sqrt[3]{28+84\,i\sqrt{3}}-14/3\,\frac{1}{\sqrt[3]{28+84\,i\sqrt{3}}}\right)$

[7]: $-1/18\sqrt[3]{188+36\sqrt{93}}+\frac{22}{9}\,\frac{1}{\sqrt[3]{188+36\sqrt{93}}}+5/9$

[8]: $1/6\sqrt[3]{44+12\sqrt{69}}-10/3\,\frac{1}{\sqrt[3]{44+12\sqrt{69}}}+1/3$

[9]: $\phi=(\sqrt{5}-1)/2$

[10]: $1/18\sqrt[3]{188+36\sqrt{93}}-\frac{22}{9}\,\frac{1}{\sqrt[3]{188+36\sqrt{93}}}+4/9$

[11]: $1/6\sqrt[3]{26+6\sqrt{33}}-4/3\,\frac{1}{\sqrt[3]{26+6\sqrt{33}}}+1/3$

[12]: $1/6\sqrt[3]{108+12\sqrt{93}}-2\,\frac{1}{\sqrt[3]{108+12\sqrt{93}}}$