Research Article

Jackie Harjani, Belen López*, and Kishin Sadarangani

Existence and uniqueness of mild solutions for a fractional differential equation under Sturm-Liouville boundary conditions when the data function is of Lipschitzian type

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Abstract: In this article, we present a sufficient condition about the length of the interval for the existence and uniqueness of mild solutions to a fractional boundary value problem with Sturm-Liouville boundary conditions when the data function is of Lipschitzian type. Moreover, we present an application of our result to the eigenvalues problem and its connection with a Lyapunov-type inequality.

Keywords: fractional differential equation, boundary value problem, mild solution

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1 Introduction

In the book [1], the authors presented the following result.

Theorem 1. (Theorem 7.7 of [1]) *Suppose that $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition with respect to the second variable, i.e., there exists $k > 0$ such that*

$$|f(t, x) - f(t, y)| \le k|x - y|,$$

for any $t \in [a, b]$ and $x, y \in \mathbb{R}$.

If $b - a < \frac{2\sqrt{2}}{k}$, then the following boundary value problem

$$\begin{cases} y''(t) = -f(t, y(t)), & a < t < b, \\ y(a) = A, & y(b) = B, \end{cases} \tag{1}$$

where $A, B \in \mathbb{R}$ has a unique continuous mild solution.

In [2], the author studied the same question for the following fractional boundary value problem

$$\begin{cases} D^{\alpha}_{a^+} y(t) = -f(t, y(t)), & a < t < b, \\ y(a) = 0, & y(b) = B, \end{cases} \tag{2}$$

where $1 < \alpha \le 2$, $B \in \mathbb{R}$ and $D^{\alpha}_{a^+}$ denotes the fractional Riemann-Liouville derivative.

*** Corresponding author: Belen López,** Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain, e-mail: blopez@dma.ulpgc.es
Jackie Harjani: Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain, e-mail: jharjani@dma.ulpgc.es
Kishin Sadarangani: Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain, e-mail: ksadaran@dma.ulpgc.es

The main result appearing in [2] is the following.

Theorem 2. *Assume that* $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ *is continuous and satisfies*

$$|f(t,x)-f(t,y)|\le K|x-y|,$$

for any $t\in[a,b]$ *and* $x,y\in\mathbb{R}$, *where* $K>0$.

If $b-a<\Gamma(\alpha)^{\frac{1}{\alpha}}\frac{\alpha^{\frac{\alpha+1}{\alpha}}}{K^{\frac{1}{\alpha}}(\alpha-1)^{\frac{\alpha-1}{\alpha}}}$, *then problem* (2) *has a unique continuous mild solution.*

Recently, in [3,4], the authors considered the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}_{a^+}y(t)=-f(t,y(t)), & a<t<b,\\ y(a)=A, & y(b)=B, \end{cases} \tag{3}$$

where $1<\alpha<2$, $A,B\in\mathbb{R}$ and ${}^cD^{\alpha}_{a^+}$ denotes the Caputo fractional derivative, and they obtained a similar result to Theorem 2.

Motivated by the aforementioned papers, we study the existence and uniqueness of mild solutions for the following fractional differential equation with Sturm-Liouville-type boundary conditions

$$\begin{cases} {}^cD^{p}_{a^+}y(t)=-f(t,y(t)), & a<t<b,\\ \alpha y(a)-\beta y'(a)=0, & \gamma y(b)+\delta y'(b)=0, \end{cases} \tag{4}$$

where $1<p\le 2$, $\alpha,\beta,\gamma,\delta\ge 0$ and $\Delta=\alpha\gamma(b-a)+\alpha\delta+\beta\gamma>0$.

Moreover, we apply our result to the eigenvalues problem and present a Lyapunov-type inequality.

This kind of problem appears in a great number of papers in the literature (see [5–9] and references therein, among others).

The rest of the article is organized as follows. In Section 2, we recall some basic facts about fractional calculus and present an auxiliary lemma which will be used later. Section 3 contains the main result of the article, and in Section 4 we present some applications of our result.

2 Background

We start this section presenting some basic concepts about fractional calculus. This material can be found in [10].

Definition 1. Suppose $p>0$ and $f\in C[a,b]$. The Riemann-Liouville fractional integral of order p of f is given by

$$(I^p_{a^+}f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{p-1}f(s)\,\mathrm{d}s.$$

Definition 2. Suppose $n-1<p<n$ and $f\in C^{(n)}[a,b]$. The Caputo fractional derivate of order p is defined as

$$({}^cD^p_{a^+}f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-s)^{n-\alpha-1}f^{(n)}(s)\,\mathrm{d}s.$$

In order to transform problem (4) into an integral equation, we need the following result which appears in Lemma 1 of [11].

Lemma 1. *Let $h \in C[a, b]$, $1 < p \leq 2$, $\alpha, \beta, \gamma, \delta \geq 0$ and $\Delta = \alpha\gamma(b-a) + \alpha\delta + \beta\gamma > 0$. Then, the unique solution $y \in C^2[a, b]$ to the fractional boundary value problem*

$$\begin{cases} {}^cD^p_{a^+}y(t) = -h(t), & a < t < b, \\ \alpha y(a) - \beta y'(a) = 0, & \gamma y(b) + \delta y'(b) = 0, \end{cases} \tag{5}$$

is given by

$$y(t) = \int_a^b G(t, s)h(s)\,\mathrm{d}s,$$

where

$$G(t, s) = \frac{(b-s)^{p-2}[\gamma(b-s) + \delta(p-1)]}{\Delta\Gamma(p)} H(t, s),$$

where $H(t, s)$ is the function defined on $[a, b] \times [a, b]$ as

$$H(t, s) = \begin{cases} \beta + \alpha(t-a) - \dfrac{\Delta(t-s)^{p-1}}{\gamma(b-s)^{p-1} + \delta(p-1)(b-s)^{p-2}}, & a \leq s \leq t \leq b, \\ \beta + \alpha(t-a), & a \leq t \leq s \leq b. \end{cases}$$

Remark 1. In [11], it is proved that

$$|H(t, s)| \leq \max\left\{\alpha\frac{2-p}{p-1}(b-a) - \beta, \beta + \alpha(b-a)\right\},$$

for $(t, s) \in [a, b] \times (a, b)$.

Remark 2. Note that $H(t, s)$ is a continuous function on $[a, b] \times [a, b]$, while $G(t, s)$ satisfies that $\lim_{s \to b^-} G(t, s) = \infty$ for any $t \in [a, b]$ and, consequently, $G(t, s)$ is not a continuous function on $[a, b] \times [a, b]$.

3 Main result

Taking into account Remark 2, we start this section with the following lemma.

Lemma 2. *Suppose that $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $x \in C[a, b]$. Then, the function Tx defined by*

$$(Tx)(t) = \int_a^b G(t, s)f(s, x(s))\,\mathrm{d}s, \quad t \in [a, b],$$

satisfies that $Tx \in C[a, b]$.

Proof. In order to prove that $Tx \in C[a, b]$, we take t_0 and $t_n \in [a, b]$ such that $t_n \to t_0$ when $n \to \infty$ and we have to prove that $(Tx)(t_n) \to (Tx)(t_0)$.

To do this, we fix $\varepsilon > 0$.

Since f is continuous, f is bounded on the compact $[a, b] \times [-\|x\|, \|x\|]$.

Put $M = \sup\{|f(t, y)| : (t, y) \in [a, b] \times [-\|x\|, \|x\|]\}$. Then,

$$\begin{aligned}
|(Tx)(t_n) - (Tx)(t_0)| &= \left| \int_a^b G(t_n, s) f(s, x(s))\mathrm{d}s - \int_a^b G(t_0, s) f(s, x(s))\mathrm{d}s \right| \\
&= \left| \int_a^b (G(t_n, s) - G(t_0, s)) f(s, x(s))\mathrm{d}s \right| \\
&\leq \int_a^b |G(t_n, s) - G(t_0, s)||f(s, x(s))|\mathrm{d}s \\
&\leq M \int_a^b |G(t_n, s) - G(t_0, s)|\mathrm{d}s \\
&= M \int_a^b \frac{(b-s)^{p-2}[\gamma(b-s) + \delta(p-1)]}{\Delta\Gamma(p)} \cdot |H(t_n, s) - H(t_0, s)|\mathrm{d}s.
\end{aligned}$$

Put $G = \gamma\frac{(b-a)^p}{p} + \delta(b-a)^{p-1}$.

From the continuity of H on the compact $[a, b] \times [a, b]$, by Heine's theorem, H is uniformly continuous on $[a, b] \times [a, b]$, and this implies that for $\varepsilon > 0$ given there exists $\delta > 0$ such that if $(t, s), (t', s') \in [a, b] \times [a, b]$ with $|t - t'| < \delta$ and $|s - s'| < \delta$, then $|H(t, s) - H(t', s')| < \frac{\varepsilon \cdot \Delta\Gamma(p)}{M \cdot G}$.

Since $t_n \to t_0$ when $n \to \infty$, for $\delta > 0$, we can find $n_0 \in \mathbb{N}$ such that $|t_n - t_0| < \delta$ for any $n \geq n_0$. Therefore, for $n \geq n_0$, we have

$$\begin{aligned}
|(Tx)(t_n) - (Tx)(t_0)| &\leq M \int_a^b \frac{(b-s)^{p-2}[\gamma(b-s) + \delta(p-1)]}{\Delta\Gamma(p)} |H(t_n, s) - H(t_0, s)|\mathrm{d}s \\
&\leq \frac{M \cdot \varepsilon \cdot \Delta\Gamma(p)}{M \cdot G \cdot \Delta\Gamma(p)} \int_a^b (b-s)^{p-2}[\gamma(b-s) + \delta(p-1)]\mathrm{d}s = \varepsilon.
\end{aligned}$$

This proves that $Tx \in C[a, b]$. □

Definition 3. A function $x \in C[a, b]$ is said to be a mild solution to problem (4) if it is a fixed point of the operator T.

Theorem 3. *Suppose that $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $L > 0$ such that*

$$|f(t, x) - f(t, y)| \leq L|x - y|, \tag{6}$$

for any $t \in [a, b]$ and $x, y \in \mathbb{R}$. If the condition

$$L \cdot \max\left\{\alpha\frac{2-p}{p-1}(b-a) - \beta, \beta + \alpha(b-a)\right\} \cdot \frac{1}{\Delta\Gamma(p)} \cdot \left[\gamma\frac{(b-a)^p}{p} + \delta(b-a)^{p-1}\right] < 1 \tag{7}$$

holds, then the equation $Tx = x$, where T is the operator defined in Lemma 2, has a unique continuous solution, that is, problem (4) *has a unique mild solution.*

Proof. Consider the metric space $C[a, b]$ of real and continuous functions defined on $[a, b]$ with the metric

$$d(f, g) = \max\{|f(t) - g(t)| : t \in [a, b]\}.$$

It is well known that $(C[a, b], d)$ is a complete metric space.

Now, we define Tx for $x \in C[a, b]$, as

$$(Tx)(t) = \int_a^b G(t, s) f(s, x(s)) \mathrm{d}s, \quad t \in [a, b].$$

By Lemma 2, T applies $C[a, b]$ into itself.

In order to apply the Banach contraction mapping theorem, we estimate $d(Tx, Ty)$, for $x, y \in C[a, b]$. To do this, we take $t \in [a, b]$ and we have

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| \int_a^b G(t, s)(f(s, x(s)) - f(s, y(s))) \mathrm{d}s \right| \\ &\le \int_a^b |G(t, s)||f(s, x(s)) - f(s, y(s))| ds \le L \int_a^b |G(t, s)||x(s) - y(s)| \mathrm{d}s \\ &\le L \cdot \sup\{|x(t) - y(t)| : t \in [a, b]\} \int_a^b |G(t, s)| \mathrm{d}s \\ &= L \cdot d(x, y) \int_a^b \frac{(b - s)^{p-2}[\gamma(b - s) + \delta(p - 1)]}{\Delta\Gamma(p)} |H(t, s)| \mathrm{d}s \\ &\le L \cdot d(x, y) \max\left\{ \alpha \frac{2 - p}{p - 1}(b - a) - \beta, \beta + \alpha(b - a) \right\} \cdot \int_a^b (b - s)^{p-2} \frac{[\gamma(b - s) + \delta(p - 1)]}{\Delta\Gamma(p)} \mathrm{d}s \\ &= L \cdot \max\left\{ \alpha \frac{2 - p}{p - 1}(b - a) - \beta, \beta + \alpha(b - a) \right\} \cdot \left[\gamma \frac{(b - a)^p}{p} + \delta(b - a)^{p-1} \right] \frac{1}{\Delta\Gamma(p)} \cdot \mathrm{d}(x, y). \end{aligned}$$

By our assumption, the Banach contraction mapping theorem says us that the equation $Tx = x$ has a unique solution in $C[a, b]$, that is, problem (4) has a unique mild solution.

This completes the proof. □

4 Applications

In this section, we present an application of our result to the eigenvalues problem and a Lyapunov-type inequality.

Consider the fractional Sturm-Liouville eigenvalues problem

$$\begin{cases} {}^cD_{0^+}^p x(t) + \lambda x(t) = 0, & 0 < t < 1, \\ x(0) - x'(0) = x(1) + x'(1) = 0, \end{cases} \tag{8}$$

with $1 < p \le 2$.

The real values of λ for which there exists a non-trivial solution to problem (8) are called eigenvalues associated with problem (8) and the corresponding solutions are called eigenfunctions.

Problem (8) is a particular case of problem (4), where $a = 0$, $b = 1$, $\alpha = \beta = \gamma = \delta = 1$ and $f(t, x) = \lambda x$ which is a continuous function on $[0, 1] \times \mathbb{R}$.

In our case,

$$\Delta = \alpha\gamma(b - a) + \alpha\delta + \beta\gamma = 1 \cdot 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 3 > 0.$$

Moreover, for $t \in [0, 1]$ and $x, y \in \mathbb{R}$, we have

$$|f(t, x) - f(t, y)| = |\lambda| \cdot |x - y|,$$

and, therefore, the constant $L = |\lambda|$.

In this case, inequality (7) appearing in Theorem 3 can be expressed as

$$|\lambda| \max\left\{\frac{2-p}{p-1}-1, 2\right\}\cdot\frac{1}{3\Gamma(p)}\left[\frac{1}{p}+1\right]<1.$$

If this inequality is satisfied, then, by Theorem 3, problem (8) has a unique mild solution.

Since the trivial solution $x(t) = 0$ for $t \in [0, 1]$ satisfies (8) and belongs to $C^2[0, 1]$, it will be the unique solution and, therefore, λ is not an eigenvalue of problem (8).

Summarizing, if $|\lambda| < \frac{3p\Gamma(p)}{(p+1)\max\left\{\frac{3-2p}{p-1}, 2\right\}}$, then λ is not an eigenvalue of problem (8).

In particular, for $p = \frac{3}{2}$, we have that

$$\text{if } |\lambda| < \frac{9/2\cdot\Gamma(3/2)}{5} \cong 0'7976, \text{ then } \lambda \text{ is not an eigenvalue.}$$

Finally, we connect our result with the ones appearing in [11].

The main result in [11] is the following.

Theorem 4. *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^p_{a^+}x(t) + q(t)\cdot x(t) = 0, & a < t < b, \\ \alpha\cdot x(a) - \beta\cdot x'(a) = 0, & \gamma\cdot x(b) + \delta\cdot x'(b) = 0, \end{cases} \tag{9}$$

where $1 < p \le 2$, $\alpha, \beta, \gamma, \delta \ge 0$, $\Delta = \alpha\gamma(b-a) + \alpha\delta + \beta\gamma > 0$ *and* $q : [a, b] \to \mathbb{R}$ *is a continuous function.*

Then, if problem (9) *has a nontrivial solution, then we have the following Lyapunov-type inequality*

$$\int_a^b (b-s)^{p-2}[\gamma(b-s) + \delta(p-1)]\,|q(s)|\mathrm{d}s \ge \frac{\Delta\Gamma(p)}{\max\left(\alpha\frac{2-p}{p-1}(b-a) - \beta, \beta + \alpha(b-a)\right)}.$$

Now, note that problem (9) is a particular case of our problem (4), where $f(t, x) = q(t)x$.

It is clear that under assumption about the continuity of q, the function f is continuous on $[a, b] \times \mathbb{R}$. Moreover, for any $t \in [a, b]$ and $x, y \in \mathbb{R}$, we have

$$|f(t,x) - f(t,y)| = |q(t)x - q(t)y| = |q(t)|\cdot|x-y| \le \|q\|_\infty |x-y|,$$

where $\|q\|_\infty$ denotes the following quantity

$$\|q\|_\infty = \sup\{|q(t)| : t \in [a, b]\}.$$

This says that assumption (6) in Theorem 3 is satisfied with $L = \|q\|_\infty$.

Therefore, by Theorem 3, if

$$\|q\|_\infty \max\left\{\alpha\frac{2-p}{p-1}(b-a) - \beta, \beta + \alpha(b-a)\right\}\cdot\frac{1}{\Delta\Gamma(p)}\left[\gamma\frac{(b-a)^p}{p} + \delta(b-a)^{p-1}\right] < 1,$$

then problem (9) has a unique mild solution. Since the trivial solution satisfies (9) and it belongs to $C^2[a, b]$, it will be the unique solution of problem (9).

Therefore, if problem (9) has a nontrivial solution, then the following inequality

$$\|q\|_\infty \ge \frac{\Delta\Gamma(p)}{\max\left(\alpha\frac{2-p}{p-1}(b-a) - \beta, \beta + \alpha(b-a)\right)\left[\gamma\frac{(b-a)^p}{p} + \delta(b-a)^{p-1}\right]}$$

holds.

Now, if we put

$$k = \max\left\{\alpha\frac{2-p}{p-1}(b-a)-\beta, \beta+\alpha(b-a)\right\}\cdot\frac{1}{\Delta\Gamma(p)}\left[\gamma\frac{(b-a)^p}{p}+\delta(b-a)^{p-1}\right],$$

then we have the following result.

Theorem 5.
(a) *If* $\|q\|_\infty < \frac{1}{k}$, *then problem* (9) *has a unique trivial solution.*
(b) *If problem* (9) *has a nontrivial continuous solution, then* $\|q\|_\infty \geq \frac{1}{k}$.

This result matches with the ones appearing in [11].

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