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On fields having the extension property

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Abstract

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Let T be an intermediate over the field K. We say that K has the Extension Property if every automorphism of $K(T)$ extends an automorphism of K. Fields with this property play a crucial role in the study of homogeneity conditions in spaces of orderings of function fields.

In this paper we introduce a new extension property (Generalized Extension Property) and we show it has a better algebraic behaviour. We also rescue most known results about fields with the extension property and determine a new class of fields verifying the generalized extension property.

Introduction

A field has the Extension Property if every automorphism of its rational fraction field is an extension of some automorphism of the base field. These fields play a crucial role in the study of homogeneity conditions in the space of orderings of a field [2].

Gamboa [1] has studied several classes of fields with this property, in particular, he shows that the following families have the Extension Property:

(i) Algebraic extensions of the field $\mathbb{Q}$ of rational numbers.

(ii) Euclidean, algebraically closed and pythagorean fields.

(iii) Fields with an unique ordering such that the cardinal of $\mathrm{Hom}(F, F)$ is smaller than the cardinal of F. A particular case are fields with an unique archimedean ordering since in this case $\mathrm{Hom}(F, F) = \mathrm{Id}$.

(iv) Finite fields.

(v) n-fields, (i.e. fields where the equation $x^n - ax - 1 = 0$ has a root for any element a of the field, for some fixed n.

(vi) Henselian fields whose residual field is real closed.

(vii) Fields F whose characteristic is zero and such that the index of $\dot{F}^2$ in $\dot{F}$ is finite ($\dot{F} = F - \{0\}$). This generalizes the case of Euclidean fields where $[\dot{F} : \dot{F}^2] = 2$.

(viii) Fields F with an ordering P contained in $\mathbb{Q} + F^2$, e.g., the field of formal series $\mathbb{Q}((x))$.

In this paper we shall introduce an apparently stronger type of extension property. It will simplify some techniques and will broaden the results.

1. Definition and first results

In what follows K and L are given fields and $K(T)$, $L(T)$ represent their rational fractions fields in one single variable.

Definition 1.1. K has the *Extension Property* (in short: K is E.P.) if for every automorphism φ of $K(T)$ we have that $\varphi(K) \subseteq K$.

Definition 1.2. K has the *Generalized Extension Property* (in short: K is G.E.P.) if for every field L and every field homomorphism $\varphi : K \to L(T)$ we have that $\varphi(K) \subseteq L$.

Obviously, every G.E.P. field is an E.P. field, but the following question remains open:

Question. Is every E.P. field a G.E.P. field too?

The following properties hold (proofs are analogous to those in [1]):

Proposition 1.3. (i) *Every prime field is G.E.P.*

(ii) *Every algebraic extension of a G.E.P. field is G.E.P. (In particular, finite fields are G.E.P.)*

(iii) *n-fields are G.E.P. (In particular, pythagorean fields and algebraically closed fields, which are 2-fields, are G.E.P.)*

(*iv*) *Fields F whose characteristic is zero and such that the index of $\dot{F}^2$ in $\dot{F}$ is finite, are G.E.P. In particular, euclidean and among them, real closed fields, are G.E.P.* □

2. A new class of G.E.P. fields

Now we present the main result.

Lemma 2.1. *Let K be a field and let $a \in K$ such that the question $x^n - a = 0$ has roots in K for infinitely many values of $n \in \mathbb{N}$. Then, for every homomorphism $\varphi : K \to L(T)$, it is true that $\varphi(a) \in L$.*

Proof. Let $b \in K$. We write $\varphi(b) = f_b/g_b$, where $f_b, g_b \in L[T]$ are coprime, and define

$$d(b) = \deg(f_b) + \deg(g_b) \in \mathbb{Z}\,.$$

We observe that $\varphi(b) \in L$ if and only if $d(b) = 0$.

If $\varphi(a) \not\in L$, then $d(a) > 0$, so we choose a natural number $r > d(a)$ and an element $\alpha \in K$ such that $\alpha^r = a$. Thus $\varphi(a) = \varphi(\alpha)^r$, so $d(\alpha) = d(a)/r$. On the other hand, we know that $0 < d(a) < r$. Therefore, we arrive at $0 < d(\alpha) < 1$, which is absurd. □

Theorem 2.2. *Let K be a henselian field. Then it is G.E.P.*

Proof. Let L be a field and $\varphi : K \to L(T)$ a homomorphism. Let A be a henselian valuation ring of K whose maximal ideal is $\mathfrak{M}$. First of all we shall prove:

$$\text{for every } m \in \mathfrak{M}\,, \quad \varphi(m) \in L\,.$$

Indeed, let $a = 1 + m$, and let us consider the polynomial $f_n(x) = x^n - a$ for every $n \in \mathbb{N}$ coprime with the characteristic of $\mathbf{k} = A/\mathfrak{M}$. The image $\bar{f}_n$ of f_n under the canonical epimorphism $A[x] \to \mathbf{k}[x]$ is $\bar{f}_n = x^n - 1$, for $a + \mathfrak{M} = 1 + \mathfrak{M}$. The element $1 \in \mathbf{k}$ is a simple root of $\bar{f}_n$ because its derivative $(\bar{f}_n)'(1) = n$ that is different from zero.

A being a henselian ring, the polynomial f_n has a root in A, therefore in K, for all such n. Now we apply Lemma 2.1 and conclude that $\varphi(a) \in L$, and since $\varphi(m) = \varphi(a) - 1$, we get $\varphi(m) \in L$.

Now we can prove that $\varphi(A) \subset L$.

Let $u \in A - \mathfrak{M}$ and let $m \in \mathfrak{M}$, $m \neq 0$. Then $m' = m \cdot u \in \mathfrak{M}$. We know that $\varphi(m) \in L$ and $\varphi(m') \in L$ too, so $\varphi(u) = \varphi(m')/\varphi(m)$ is an element of L. To finish, if $b \in K - A$, then $b^{-1} \in \mathfrak{M}$, so $\varphi(b^{-1}) \in L$. Thus $\varphi(b) = \varphi(b^{-1})^{-1} \in L$. □

Corollary 2.3. *Let K be a complete valued field. Then K is G.E.P.*

Proof. If K is archimedian, then it is isomorphic to the field $\mathbb{R}$ of real numbers or to the field $\mathbb{C}$ of complex numbers via Ostrowski's theorem [3]. Therefore, by Proposition 1.3(iii) and (iv), K is G.E.P. If K is not archimedian, it is henselian [4], so by Theorem 2.2 it is G.E.P. □

Remarks. (1) We note that Theorem 2.2 is essentially stronger than Proposition 3.9 in [1], for we suppress the hypothesis of real closedness of the residue field.

(2) Purely transcendental extension fields of a field are not G.E.P. Thus, Corollary 2.3 shows that complete valued fields (e.g. p-adic ones, series fields, ...) are not purely transcendental extensions of some other field.

3. Subfields that are G.E.P.

Proposition 3.1. *Let K be a field and let $(K_i)_{i\in I}$ be a family of G.E.P. subfields. Let K_I be the subfield generated by $\bigcup_{i\in I} K_i$. Then K_I is G.E.P.*

Proof. Let $\varphi : K_I \to L(T)$ be given. For every $i \in I$, $\varphi|_{K_i}(K_i) \subset L$, implying $K_i \subset \varphi^{-1}(L)$.

Then, $\bigcup_{i\in I} K_i \subset \varphi^{-1}(L)$ and therefore $\varphi(K_I) \subset L$. □

From the above proposition and since prime fields are G.E.P., one has the following:

Corollary 3.2. *Let K be a field. Then there exists a maximum G.E.P. subfield in K, which we shall denote by* Gep(K). □

We get also the following corollary:

Corollary 3.3. *Let K be a field and $M \subset K$ a subfield. Then*:

(i) Gep(M) $\subset$ Gep(K),
(ii) *K is G.E.P. if and only if* Gep(K) $= K$,
(iii) Gep($K(T_1 \cdots T_n)$) $=$ Gep(K),
(iv) Gep($K(T_1 \cdots T_n)$) $= K$ *if and only if K is G.E.P.*,
(v) *there is no G.E.P. subfield of $K(T_1 \cdots T_n)$ containing (strictly) K*,
(vi) Gep(K) *is algebraically closed in K.*

Proof. We show only (iii). Using (i) and induction on n, everything reduces to proving that if a field $M \subset K(T)$ is not contained in K, then it can not be G.E.P.

Let $M \subset K(T)$ and let $\xi \in M - K$. We take $L = K(T)$ and consider the restriction to M, $\varphi : M \to L(U)$ of the homomorphism $\psi : K(T) \to L(U) = K(T, U)$ defined as follows: For every $k \in K$, $\psi(k) = k$, and $\psi(T) = U$. There exists some polynomials $f(T), g(T)$ in $K(T)$ such that $\xi = f(T)/g(T)$, so $\varphi(\xi) = \psi(\xi) = f(U)/g(U) \not\subset L$. Therefore, M is not G.E.P.

The other results are obvious. □

4. Fields that are not G.E.P.

After (v) in Corollary 3.3, it is natural to consider whether the only fields that are not G.E.P. are those which are purely transcendental extensions of some other field. The answer is negative:

Proposition 4.1. *There exist fields that are neither G.E.P. nor purely transcendental extensions of other fields.*

Proof. If every non-G.E.P. field were a purely transcendental extension of other field, we could find by Corollary 3.2(v) that, in particular, every subfield of $\mathbb{C}(X_1 \cdots X_n)$ containing $\mathbb{C}$ should be of type $\mathbb{C}(T_1 \cdots T_r)$, where $X_1, \ldots, X_n$ and $T_1, \ldots, T_r$ are sets of algebraically independent variables.

This is the conjecture known as 'Lüroth's problem', which is, nevertheless, false: In [5], one can see that the rational function field L of the algebraic set

$$V = \{(x, y, z, t) \in \mathbb{C}^4 : 1 + x^3 + y^3 + z^3 + t^3 = 0\}$$

is a subfield contradicting the above conjecture, for V is an unirational but not a birational variety. This ends our proof. □

5. Open problems

Within the frame of the question whether every E.P. field is also a G.E.P. field, we can ask the following:

Which G.E.P. field properties are valid for E.P. fields?

We know that any given field admits a G.E.P. algebraic extension. Does there exist a finite G.E.P. extension? Does there exist a minimal algebraic extension which is G.E.P.?

We formulate again an unsolved problem for E.P. fields: If K has unique ordering, is K G.E.P.?

References

[1] J.M. Gamboa, Some new results on ordered fields, J. Algebra 110 (1987) 1–12.
[2] J.M. Gamboa and T. Recio, Ordered fields with the dense orbits property, J. Pure Appl. Algebra 30 (1983) 237–246.
[3] N. Jacobson, Basic Algebra II (Freeman, New York, 1980).
[4] S. Lang, Algebra (Aguilar, Madrid, 1971).
[5] I.R. Shafarevich, Basic Algebraic Geometry (Springer, Berlin, 1977).