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Orthogonal Projections of the Identity: Spectral Analysis and Applications to Approximate Inverse Preconditioning*

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Abstract. Many strategies for constructing different structures of sparse approximate inverse preconditioners for large linear systems have been proposed in the literature. In a more general framework, this paper analyzes the theoretical effectiveness of the optimal preconditioner (in the Frobenius norm) of a linear system over an arbitrary subspace of $M_n(\mathbb{R})$. For this purpose, the spectral analysis of the Frobenius orthogonal projections of the identity matrix onto the linear subspaces of $M_n(\mathbb{R})$ is performed. This analysis leads to a simple, general criterion: The effectiveness of the optimal approximate inverse preconditioners (parametrized by any vectorial structure) improves at the same time as the smallest singular value (or the smallest eigenvalue's modulus) of the corresponding preconditioned matrices increases to 1.

Key words. Frobenius norm, orthogonal projection, eigenvalues, singular values, approximate inverse preconditioning

AMS subject classifications. 15A12, 15A18, 15A60, 65F35

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1. Introduction.

When a physical phenomenon is modeled by a partial differential equation (PDE), the discretization of this PDE by any adequate method (finite differences, finite elements, finite volumes, meshless, etc.) generally leads to a large system of linear equations,

$$Ax = b, \quad A \in \mathbb{R}^{n\times n}, \quad x, b \in \mathbb{R}^{n\times 1}, \tag{1.1}$$

in which the matrix A is nonsingular and sparse, i.e., has relatively few nonvanishing elements.

The solution of these linear systems is usually performed by iterative methods based on Krylov subspaces (see, e.g., [1, 20, 27, 32]). To improve the convergence of these Krylov methods, system (1.1) can be preconditioned with an adequate preconditioning matrix N, transforming it into either of the equivalent problems

$$NAx = Nb, \tag{1.2}$$

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$$ANy = b, \quad x = Ny, \tag{1.3}$$

that is, the left and right preconditioned systems, respectively. In this paper we address only the case of right-hand side preconditioners (1.3), but analogous results can be obtained for the left-hand side preconditioners (1.2).

Often, the preconditioning of system (1.1) is performed in order to get a preconditioned matrix AN as close as possible to the identity in some sense, and the preconditioner N is called an approximate inverse of A. In some cases, the preconditioners are parametrized by prescribed sparsity patterns, but we consider here a more general case of linear parametrization where the preconditioners belong to an arbitrary subspace S of $M_n(\mathbb{R})$ [29]. The closeness of AN to I may be measured by using a suitable matrix norm like, for instance, the Frobenius norm $\|\cdot\|_F$. In this way, the problem of obtaining the best preconditioner N (with respect to the Frobenius norm) of system (1.1) in the subspace S of $M_n(\mathbb{R})$ is reduced to the minimization problem

$$\min_{M \in S} \|AM - I\|_F = \|AN - I\|_F \tag{1.4}$$

and the preconditioned matrix AN can be obtained by orthogonal projection of the identity onto the subspace AS. Here, and from now on, orthogonality is with respect to the Frobenius inner product $\langle \cdot, \cdot \rangle_F$, and we shall refer to the solution N to problem (1.4) as the "optimal" preconditioner of system (1.1) over the subspace S. So, throughout this paper, the term "optimal" means that the approximate inverse N is the matrix that minimizes the Frobenius norm on $AN - I$ over a certain subspace S of $M_n(\mathbb{R})$, but the preconditioner N is not necessarily optimal in any other sense of the word.

The search for approximate inverse preconditioners and, in general, the study of preconditioning strategies for large linear systems is at present one of the most relevant research areas in numerical linear algebra. In [5] we find a very complete survey about this question, and some of the most recent works in this area can be found, for instance, in [2, 3, 7, 8, 9, 14, 22, 23, 26, 30, 34] and in the references contained therein. Some of the first methods for constructing sparse approximate inverses that are best approximations in the Frobenius norm can be found in [4, 31] (see [1, p. 335], [5], [6] for a more detailed historical review of this question). Other posterior approaches in this sense are described, for instance, in [10, 13, 19, 21] and in the references therein. Recently, explicit expressions for both the preconditioner N defined by (1.4) and $\|AN - I\|_F$, valid for any subspace S of $M_n(\mathbb{R})$, and applications of these formulas to the computation of sparse approximate inverses were presented in [29]. Moreover, in that work the general parametrization from arbitrary subspaces is illustrated with a natural generalization of the normal equations related to the system (1.1). The effect of ordering on the optimal sparse approximate inverse preconditioners has been studied from both theoretical and experimental points of view in [17]. Among other Frobenius norm minimization preconditioners not extracted from sparse matrix subspaces, we must mention here the preconditioners for structured matrices obtained by orthogonal projections onto unitary matrix algebras (like, for instance, circulant preconditioners for Toeplitz matrices); see, e.g., [12, 15, 33] and the references therein.

With a more general and simple formulation, problem (1.4) can be written as

$$\min_{P \in T} \|P - I\|_F = \|Q - I\|_F, \tag{1.5}$$

where T is an arbitrary subspace of $M_n(\mathbb{R})$. Note that, in our context, problems (1.4) and (1.5) are indeed equivalent, since A is nonsingular, and for $T = AS$ problem (1.5) becomes problem (1.4) with solution $Q = AN$, so that the results we obtain for (1.5) will be, in particular, valid for (1.4).

The results about preconditioning contained in this work are limited to preconditioners of the approximate inverse type obtained from Frobenius norm minimization (over an arbitrary subspace of $M_n(\mathbb{R})$). In this context, the main goal of this paper is to establish a simple, unified criterion that allows us to estimate the theoretical effectiveness of the optimal preconditioners N, for all possible subspaces S of $M_n(\mathbb{R})$. For this purpose, the remainder of the paper has been organized as follows. In section 2, we establish some spectral properties of the solution Q to problem (1.5), i.e., the orthogonal projection of the identity onto an arbitrary subspace T of $M_n(\mathbb{R})$. In section 3, we apply these general results to the effectiveness analysis of the approximate inverse preconditioners N defined by problem (1.4). Finally, in section 4 we present our conclusions.

2. Orthogonal Projections of the Identity. The following two lemmas characterize the best approximation to the identity matrix from an arbitrary subspace T of $M_n(\mathbb{R})$.

LEMMA 2.1. *Let T be a vector subspace of $M_n(\mathbb{R})$. Then the solution to problem* (1.5) *satisfies*

$$\operatorname{tr}\left(QP^T\right) = \operatorname{tr}(P) \quad \forall P \in T, \tag{2.1}$$

$$\|Q - I\|_F^2 = n - \operatorname{tr}(Q). \tag{2.2}$$

Moreover,

$$\|Q\|_F^2 = \operatorname{tr}(Q). \tag{2.3}$$

Proof. Using the orthogonal projection theorem, we have

$$\langle Q - I, P\rangle_F = 0 \ \ \forall P \in T, \tag{2.4}$$

which is equivalent to (2.1). Next, from (2.4), we get

$$\|Q - I\|_F^2 = \langle Q - I, Q\rangle_F + \langle I - Q, I\rangle_F = n - \operatorname{tr}(Q),$$

and finally, (2.3) is (2.1) for $P = Q$. □

Remark 1. From (2.2) and (2.3) we get the following bounds for the trace of the orthogonal projections of the identity:

$$0 \leq \operatorname{tr}(Q) \leq n. \tag{2.5}$$

Different explicit expressions can be given for Q, but for the purpose of this paper the following simple expression is enough.

LEMMA 2.2. *Let T be a vector subspace of $M_n(\mathbb{R})$ of dimension d and let $\{P_i\}_{i=1}^d$ be an orthogonal basis of T. Then the solution to problem* (1.5) *is*

$$Q = \sum_{i=1}^{d} \frac{\operatorname{tr}(P_i)}{\|P_i\|_F^2} P_i, \tag{2.6}$$

$$\|Q - I\|_F^2 = n - \sum_{i=1}^{d} \frac{[\operatorname{tr}(P_i)]^2}{\|P_i\|_F^2}. \tag{2.7}$$

Proof. Representing Q by its expansion with respect to the orthonormal basis $\{\frac{P_i}{\|P_i\|_F}\}_{i=1}^d$ of T, we obtain

$$Q = \sum_{i=1}^{d} \left\langle I, \frac{P_i}{\|P_i\|_F} \right\rangle_F \frac{P_i}{\|P_i\|_F} = \sum_{i=1}^{d} \frac{\operatorname{tr}(P_i)}{\|P_i\|_F^2} P_i$$

and from (2.2) and (2.6) we get (2.7). □

Now we develop the spectral analysis of the orthogonal projection Q. In the following, we shall denote by $\{\lambda_i\}_{i=1}^n$ and $\{\sigma_i\}_{i=1}^n$ the sets of eigenvalues and singular values, respectively, of Q arranged, as usual, in nonincreasing order. So, denoting by r the rank of Q, that is, the number of its nonzero singular values, and by m the number of its nonzero eigenvalues, we have

$$\begin{gathered} |\lambda_1| \geq \cdots \geq |\lambda_m| > |\lambda_{m+1}| = \cdots = |\lambda_n| = 0, \\ \sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0, \end{gathered} \tag{2.8}$$

and the well-known inequality [25, p. 181]

$$0 \leq m \leq r \leq n, \quad Q = 0_{n\times n} \text{ iff } 0 = m = r, \quad Q \text{ nonsingular iff } m = r = n. \tag{2.9}$$

LEMMA 2.3. *Let T be a vector subspace of $M_n(\mathbb{R})$ and Q the solution to problem* (1.5) *with* $\operatorname{rank}(Q) = r$ $(1 \leq r \leq n)$. *Then*

$$\sum_{i=1}^{r} \sigma_i^2 = \sum_{i=1}^{r} \lambda_i. \tag{2.10}$$

Proof. Using (2.3) and taking into account that

$$\|Q\|_F^2 = \operatorname{tr}\left(QQ^T\right) = \sum_{i=1}^{r} \sigma_i^2, \qquad \operatorname{tr}(Q) = \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{r} \lambda_i,$$

the proof is straightforward. □

LEMMA 2.4. *Let T be a vector subspace of $M_n(\mathbb{R})$ and Q the solution to problem* (1.5) *with* $\operatorname{rank}(Q) = r$ $(1 \leq r \leq n)$. *Then*

$$\sum_{i=1}^{r} \lambda_i^2 \leq \sum_{i=1}^{r} |\lambda_i|^2 \leq \sum_{i=1}^{r} \sigma_i^2 = \sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} |\lambda_i| \leq \sum_{i=1}^{r} \sigma_i. \tag{2.11}$$

Proof. First, note that the above chain of inequalities makes sense because Q is a real matrix, and then $\sum_{i=1}^{r} \lambda_i^2 = \operatorname{tr}\left(Q^2\right)$, $\sum_{i=1}^{r} \lambda_i = \operatorname{tr}(Q)$ are real numbers. The central equality is (2.10). While this equality is a consequence of the orthogonal projection condition of Q, the four inequalities in (2.11) are valid for any square real matrix. Indeed, the first and third inequalities hold because of the triangle inequality for modulus. The second and fourth inequalities hold because of the additive Weyl's inequalities [25, p. 176], [35]

$$\sum_{i=1}^{k} |\lambda_i|^p \leq \sum_{i=1}^{k} \sigma_i^p \qquad (p > 0, \;\; k = 1, \ldots, n)$$

for $p = 2$, 1 and $k = r$. □

From the last lemma we immediately obtain the following spectral property of the orthogonal projections of the identity.

THEOREM 2.5. *The smallest nonzero singular value and the smallest nonzero eigenvalue's modulus of the orthogonal projection of the identity onto any subspace of* $M_n(\mathbb{R})$ *are never greater than* 1.

Proof. From (2.11) we get

$$\sum_{i=1}^{r} \sigma_i^2 \leq \sum_{i=1}^{r} \sigma_i \ \Rightarrow \ \sigma_r \leq 1$$

and

$$\sum_{i=1}^{r} |\lambda_i|^2 \leq \sum_{i=1}^{r} |\lambda_i| \ \Leftrightarrow \ \sum_{i=1}^{m} |\lambda_i|^2 \leq \sum_{i=1}^{m} |\lambda_i| \ \Rightarrow \ |\lambda_m| \leq 1. \qquad \square$$

Nothing can be asserted about the rest of the nonzero singular values and the eigenvalue's modulus of Q, which can be greater than, equal to, or less than unity, as the following counterexample shows.

Example 1. Let us fix $k \in \mathbb{R}$ and consider the following subspace of $M_2(\mathbb{R})$:

$$T_k = span\left\{\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

The solution Q_k to problem (1.5) for subspace T_k can be obtained by using formula (2.6) as follows:

$$Q_k = \frac{\operatorname{tr}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}}{\left\|\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right\|_F^2} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \frac{k+1}{k^2+1} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we get

$$\begin{cases} k=-2: & |\lambda_1| = \sigma_1 = \frac{2}{5} < 1, \quad |\lambda_2| = \sigma_2 = \frac{1}{5} \leq 1, \\ k=1: & |\lambda_1| = \sigma_1 = 1, \quad |\lambda_2| = \sigma_2 = 1 \leq 1, \\ k=2: & |\lambda_1| = \sigma_1 = \frac{6}{5} > 1, \quad |\lambda_2| = \sigma_2 = \frac{3}{5} \leq 1. \end{cases}$$

The next lemma provides us with lower and upper bounds on $\|Q-I\|_F$, involving σ_r and λ_m.

LEMMA 2.6. *Let* T *be a vector subspace of* $M_n(\mathbb{R})$ *and* Q *the solution to problem* (1.5) *with* $\operatorname{rank}(Q) = r$ $(1 \leq r \leq n)$. *Let* m *be the number of nonzero eigenvalues of* Q *and suppose that* $m \geq 1$. *Then*

$$n - r + (1-\sigma_r)^2 \leq \|Q-I\|_F^2 \leq n - r + r\left(1-\sigma_r^2\right), \tag{2.12}$$

$$n - m + (1-|\lambda_m|)^2 \leq \|Q-I\|_F^2 \leq n - m + m\left(1-|\lambda_m|^2\right). \tag{2.13}$$

Proof. The left-hand inequalities are valid for any square real matrix (the orthogonal projection condition of Q is not required here). Indeed, from (2.11) we have

$$\|Q-I\|_F^2 = \|Q\|_F^2 - 2\mathrm{tr}\,(Q) + n = \sum_{i=1}^{r}\sigma_i^2 - 2\sum_{i=1}^{r}\lambda_i + n \geq \sum_{i=1}^{r}\sigma_i^2 - 2\sum_{i=1}^{r}\sigma_i + n$$

$$= n - r + \sum_{i=1}^{r}(1-\sigma_i)^2 \geq n - r + (1-\sigma_r)^2 . \tag{2.14}$$

Next, from Schur's inequality [1, p. 631] for matrix $Q - I$, we get

$$\|Q-I\|_F^2 \geq \sum_{i=1}^{n}|\lambda_i - 1|^2 = n - m + \sum_{i=1}^{m}|1-\lambda_i|^2 \geq n - m + (1-|\lambda_m|)^2 .$$

To prove the right-hand inequalities (a consequence of the orthogonal projection condition of Q), we use (2.2), (2.3), and (2.11):

$$\|Q-I\|_F^2 = n - \mathrm{tr}\,(Q) = n - \|Q\|_F^2 = n - \sum_{i=1}^{r}\sigma_i^2 \leq n - \sum_{i=1}^{r}|\lambda_i|^2 = n - \sum_{i=1}^{m}|\lambda_i|^2 .$$

Then we get, on one hand,

$$\|Q-I\|_F^2 = n - r + r - \sum_{i=1}^{r}\sigma_i^2 \leq n - r + r\left(1-\sigma_r^2\right)$$

and, on the other hand,

$$\|Q-I\|_F^2 \leq n - m + m - \sum_{i=1}^{m}|\lambda_i|^2 \leq n - m + m\left(1-|\lambda_m|^2\right). \qquad \square$$

3. Applications to the Approximate Inverse Preconditioning. In the context of the preconditioning (1.3) of system (1.1), it is obvious that the preconditioner N must be nonsingular in order to obtain a nonsingular preconditioned matrix AN (recall that we have assumed that the coefficient matrix A is nonsingular). So, in order to apply the results of section 2 to the special case $Q = AN$, from now on $\{\lambda_i\}_{i=1}^n$ and $\{\sigma_i\}_{i=1}^n$ will denote the sets of eigenvalues and singular values, respectively, of matrix AN and, according to (2.9), equation (2.8) is now rewritten as

$$|\lambda_1| \geq \cdots \geq |\lambda_n| > 0, \quad \sigma_1 \geq \cdots \geq \sigma_n > 0.$$

3.1. The Matrix Residual Norm. Assuming that matrix AN is nonsingular, Theorem 2.5 assures in this special case that σ_n, $|\lambda_n| \in \,]0,1]$. The following theorem establishes the tight relation between the matrix residual norm $\|AN - I\|_F$ and the closeness of σ_n ($|\lambda_n|$) to unity.

THEOREM 3.1. *Let $A \in M_n(\mathbb{R})$ be nonsingular and S a vector subspace of $M_n(\mathbb{R})$ such that the solution N to problem* (1.4) *is nonsingular. Then*

$$(1-|\lambda_n|)^2 \leq (1-\sigma_n)^2 \leq \|AN-I\|_F^2 \leq n\left(1-|\lambda_n|^2\right) \leq n\left(1-\sigma_n^2\right). \tag{3.1}$$

Proof. Using (2.12) and (2.13) for $Q = AN$, $m = r = n$, and the well-known inequality $\sigma_n \leq |\lambda_n|$ [25, p. 191], the proof is straightforward. $\square$

Remark 2. Theorem 3.1 states that $\|AN - I\|_F$ decreases to 0 at the same time as the smallest singular value (or the smallest eigenvalue's modulus) of the preconditioned matrix AN increases to 1. In other words, we get a good approximate inverse N of A when σ_n ($|\lambda_n|$) is sufficiently close to 1. Of course, the optimal theoretical situation corresponds to the case

$$\sigma_n = 1 \Leftrightarrow N = A^{-1} \ \left(\text{i.e., } A^{-1} \in S\right) \Leftrightarrow \lambda_n = 1.$$

3.2. Theoretical Effectiveness of the Optimal Preconditioners. Usually, in works about preconditioning, the theoretical effectiveness analysis of the new proposed preconditioners is performed. Essential properties for the convergence of most iterative methods are the clustering at 1 of eigenvalues and singular values, the condition number, and the departure from normality of the preconditioned linear system (see, e.g., [18, 21]). In this subsection we analyze the effectiveness of the preconditioners N defined by problem (1.4) showing that we can reduce the analysis of the four above parameters to the distance from σ_n ($|\lambda_n|$) to unity. Indeed, we shall prove that if any of these two values is close to 1, then the preconditioner N has good behavior with respect to all these four points.

Let us illustrate in an intuitive way the basic idea that explains this fact. Obviously, for a square matrix, the closeness to 1 of its smallest singular value does not imply, in general, the clustering at 1 of the whole set of its singular values: σ_n can be close to 1, while the rest of the singular values can be very far from 1. However, this cannot happen for the full-rank orthogonal projections Q of the identity (as the nonsingular preconditioned matrices AN). Indeed, using (2.3) and (2.5) for $Q = AN$, we get

$$\sum_{i=1}^{n} \sigma_i^2 = \|AN\|_F^2 = \operatorname{tr}(AN) \le n \quad (\sigma_1 \ge \cdots \ge \sigma_n > 0,\ \sigma_n \le 1) \tag{3.2}$$

and then the closeness to 1 of σ_n implies here the clustering at 1 of singular values. In the extreme case, if $\sigma_n = 1$, then all the singular values necessarily equal 1. Moreover, it is well known that the spectral condition number coincides with the ratio $\frac{\sigma_1}{\sigma_n}$ of the largest to the smallest singular value [24, p. 442], and hence from (3.2) we also get the following intuitive conclusion: The closer σ_n is to 1, the closer σ_1 is to σ_n, and therefore the closer the spectral condition number is to 1.

Now we rigorously expose these intuitive ideas in the next theorem (a consequence of Theorem 3.1), which summarizes the behavior of the above-mentioned four parameters in relation to σ_n and λ_n. For estimating the AN's departure from normality we use a scale-invariant measure of nonnormality (i.e., invariant under multiplication of the matrix by a constant) as the Henrici number [11]

$$He\,(AN) = \frac{\left\| AN\,(AN)^T - (AN)^T\,AN \right\|_F}{\left\| (AN)^2 \right\|_F},$$

and to obtain alternative estimations using other measures of nonnormality, one can use the inequalities that compare different measures with the numerator of $He\,(AN)$ (see, e.g., [16]).

THEOREM 3.2. *Let $A \in M_n(\mathbb{R})$ be nonsingular and S a vector subspace of $M_n(\mathbb{R})$ such that the solution N to problem* (1.4) *is nonsingular. Then*

$$\frac{1}{n}\sum_{i=1}^{n}|1-\lambda_i|^2 \le 1-|\lambda_n|^2 \le 1-\sigma_n^2, \tag{3.3}$$

$$\frac{1}{n}\sum_{i=1}^{n}(1-\sigma_i)^2 \le 1-|\lambda_n|^2 \le 1-\sigma_n^2, \tag{3.4}$$

$$\kappa_2(AN) \le \frac{\left[n-(n-1)\sigma_n^2\right]^{\frac{1}{2}}}{\sigma_n} \le \frac{\left[n-(n-1)|\lambda_n|^{2q}\right]^{\frac{1}{2}}}{|\lambda_n|^q} \quad \textit{for some } q \ge 1, \tag{3.5}$$

$$He(AN) \le \frac{\left[2n\left(1-|\lambda_n|^4\right)\right]^{\frac{1}{2}}}{|\lambda_n|^2} \le \frac{\left[2n\left(1-\sigma_n^4\right)\right]^{\frac{1}{2}}}{\sigma_n^2}. \tag{3.6}$$

Proof. To prove (3.3) (clustering of eigenvalues) it is sufficient to use Schur's inequality [1, p. 631] for matrix $AN-I$ and (3.1):

$$\sum_{i=1}^{n}|\lambda_i-1|^2 \le \|AN-I\|_F^2 \le n\left(1-|\lambda_n|^2\right) \le n\left(1-\sigma_n^2\right).$$

To prove (3.4) (clustering of singular values) we use (2.14) for $Q=AN$, $r=n$ and (3.1):

$$\sum_{i=1}^{n}(1-\sigma_i)^2 \le \|AN-I\|_F^2 \le n\left(1-|\lambda_n|^2\right) \le n\left(1-\sigma_n^2\right).$$

To prove the first inequality of (3.5) (condition number) we just use (3.2):

$$\sum_{i=1}^{n}\sigma_i^2 \le n \;\Rightarrow\; \sigma_1^2 \le n-\sum_{i=2}^{n}\sigma_i^2 \le n-(n-1)\sigma_n^2.$$

Hence

$$\kappa_2(AN) = \frac{\sigma_1}{\sigma_n} \le \frac{\left[n-(n-1)\sigma_n^2\right]^{\frac{1}{2}}}{\sigma_n}.$$

For the second inequality, note that we can suppose that $|\lambda_n| \ne 1$ since, otherwise, from (3.1) we have that AN is the identity matrix and then the three quantities in (3.5) are obviously equal to 1. Hence, using Theorem 2.5 we have

$$0<\sigma_n \le |\lambda_n| < 1 \;\Rightarrow\; \exists\, q \ge 1 \text{ s.t. } \quad |\lambda_n|^q \le \sigma_n,$$

which concludes the proof of (3.5).

To prove (3.6) (measure of nonnormality), using the inequality (see, e.g., [16, 28])

$$\left\|AN(AN)^T-(AN)^T AN\right\|_F \le \left[2\left(\|AN\|_F^4-\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^2\right)\right]^{\frac{1}{2}},$$

Schur's inequality [1, p. 631] for matrix $(AN)^2$, and (3.2), we have

$$He\,(AN) \leq \left[2\frac{\|AN\|_F^4 - \left(\sum\limits_{i=1}^{n} |\lambda_i|^2\right)^2}{\left\|(AN)^2\right\|_F^2} \right]^{\frac{1}{2}} \leq \left[2\frac{n^2 - \left(\sum\limits_{i=1}^{n} |\lambda_i|^2\right)^2}{\sum\limits_{i=1}^{n} |\lambda_i|^4} \right]^{\frac{1}{2}}$$

$$\leq \left[2\frac{n^2 - \left(n\,|\lambda_n|^2\right)^2}{n\,|\lambda_n|^4} \right]^{\frac{1}{2}} = \frac{\left[2n\left(1 - |\lambda_n|^4\right)\right]^{\frac{1}{2}}}{|\lambda_n|^2} \leq \frac{\left[2n\left(1 - \sigma_n^4\right)\right]^{\frac{1}{2}}}{\sigma_n^2}. \qquad \square$$

Remark 3. Theorem 3.2 has shown that when σ_n (or $|\lambda_n|$) increases to 1, all four classical parameters for the convergence of iterative methods improve their behavior. More precisely, the left-hand sides of (3.3), (3.4), and the Henrici number $He\,(AN)$ decrease to 0 while the spectral condition number $\kappa_2\,(AN)$ decreases to 1.

Remark 4. Of course, the purpose of the last two theorems is to provide theoretical results (σ_n and $|\lambda_n|$ are not known and in practice they will almost always be very small) which reduce the effectiveness analysis of the preconditioners N to an unique critical quantity (σ_n or $|\lambda_n|$).

4. Conclusions. The spectral analysis of the Frobenius orthogonal projections of the identity has determined the following results for the optimal preconditioners N of a linear system, $Ax = b$, defined by (1.4): The smallest singular value σ_n and the smallest eigenvalue's modulus $|\lambda_n|$ of the preconditioned matrix AN are never greater than 1 (Theorem 2.5) and they increase to 1 at the same time as the matrix residual norm $\|AN - I\|_F$ decreases to zero (Theorem 3.1). Moreover, when σ_n ($|\lambda_n|$) is close to 1, the preconditioner N improves on the four classical parameters for the convergence of iterative methods: clustering of eigenvalues and singular values, condition number, and departure from normality of the preconditioned linear system (Theorem 3.2). In this way, the closeness to 1 of σ_n ($|\lambda_n|$) has been identified as a simple, general criterion for determining the theoretical effectiveness of the Frobenius norm-based approximate inverse preconditioners N parametrized by any vectorial structure (not restricted to sparsity patterns).

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