SOME PROPERTIES OF SUMS INVOLVING PELL NUMBERS

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Abstract. In this note we prove that for all positive integers n, the sum S_{4n+1} of the first $4n+1$ Pell numbers is a perfect square. As a consequence, an identity involving binomial coefficients and Pell numbers is given. Also, sums of an even and odd number of terms of odd order are evaluated and some divisibility properties are obtained.

1. Introduction. It is well-known that the Pell numbers are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, ... where $P_0 = 0$, $P_1 = 1$, and for $n \geq 1$, $P_{n+1} = 2P_n + P_{n-1}$. From its characteristic equation $x^2 - 2x - 1 = 0$, we can write a Binet's formula for Pell numbers. Namely,

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Denoting by S_n the sum of the first n nonzero Pell numbers, and taking into account (1), it is easy to see that

$$S_n = \frac{\alpha^{n+1} + \beta^{n+1} - 2}{4}. \tag{2}$$

Furthermore, adding up the terms of Pell's sequence, we observe that if $n \equiv 1 \pmod 4$, then

$$\{S_{4n+1}\}_{n\geq 0} = \{S_1,\ S_5,\ S_9,\ S_{13},\ \dots\} = \{1,\ 7^2,\ 41^2,\ 239^2,\ \dots\}.$$

In this paper the sequences of sums S_{4n+1} and its square roots are studied and some of its properties are used to obtain an identity involving binomial coefficients and Pell numbers. Furthermore, using the Lucas expression for Pell numbers in terms of binomial coefficients [1], two more identities are given. Finally, the partial sums of the sequence of Pell numbers of odd order are evaluated and some divisibility properties are obtained.

2. Main Results. In what follows, some results for the sequence of sums S_{4n+1} and the sequence of its square roots are given. We start with the following theorem.

Theorem 1. For all $n \geq 0$, the sum S_{4n+1} is a perfect square.

Proof. The result follows immediately from (2) and taking into account that $\alpha\beta = -1$. That is,

$$S_{4n+1} = \frac{\alpha^{4n+2} + \beta^{4n+2} - 2}{4} = \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{2}\right)^2 \tag{3}$$

and the proof is complete.

The preceding sum can be used to obtain the following identity involving binomial coefficients and Pell numbers.

Theorem 2. For all $n \geq 0$, we have

$$\sum_{r=0}^{n} \binom{2n+1}{2r} 2^r = P_{2n} + P_{2n+1}. \tag{4}$$

Proof. To prove (4), we need two lemmas.

Lemma 1. If n is a nonnegative integer, then

$$S_{4n+1} = \left(\sum_{r=0}^{n} \binom{2n+1}{2r} 2^r\right)^2.$$

Proof. In fact, from (3), we have that

$$\frac{1}{2}\left(\alpha^{2n+1} + \beta^{2n+1}\right) = \frac{1}{2}\left((1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}\right)$$

$$= \frac{1}{2}\Bigg(1 + \binom{2n+1}{1}\sqrt{2} + \binom{2n+1}{2}\sqrt{2}^2 + \binom{2n+1}{3}\sqrt{2}^3 + \cdots$$

$$+ 1 - \binom{2n+1}{1}\sqrt{2} + \binom{2n+1}{2}\sqrt{2}^2 - \binom{2n+1}{3}\sqrt{2}^3 + \cdots\Bigg)$$

$$= 1 + \binom{2n+1}{2} 2 + \binom{2n+1}{4} 2^2 + \binom{2n+1}{6} 2^3 + \cdots$$

$$= \sum_{r=0}^{n} \binom{2n+1}{2r} 2^r.$$

Therefore,

$$S_{4n+1} = \left(\sum_{r=0}^{n} \binom{2n+1}{2r} 2^r \right)^2$$

and the proof is complete.

Next, we consider the sequence

$$\{a_n\}_{n\geq 0} = \{S_{4n+1}^{1/2}\}_{n\geq 0} = \{1,\ 7,\ 41,\ 239,\ 1393,\ \dots\},$$

and we have the following lemma.

Lemma 2. For all $n \geq 0$,

$$a_n = P_{2n} + P_{2n+1}.$$

Proof. First, we claim that $\{a_n\}_{n\geq 0}$ is a generalized Fibonacci sequence defined by $a_0 = 1$, $a_1 = 7$ and for $n \geq 1$,

$$a_{n+1} = 6a_n - a_{n-1}.$$

Indeed, from (3), it follows that

$$a_n = \frac{\alpha^{2n+1} + \beta^{2n+1}}{2} \to a_{n-1} = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2},\ a_{n+1} = \frac{\alpha^{2n+3} + \beta^{2n+3}}{2}.$$

Now taking into account that

$$\alpha^2 + \frac{1}{\alpha^2} = \beta^2 + \frac{1}{\beta^2} = 6,$$

we find that

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(\alpha^{2n+3} + \beta^{2n+3}) \\ &= \frac{\alpha^{2n+1}}{2}\left(\alpha^2 + \frac{1}{\alpha^2} - \frac{1}{\alpha^2}\right) + \frac{\beta^{2n+1}}{2}\left(\beta^2 + \frac{1}{\beta^2} - \frac{1}{\beta^2}\right) \\ &= \frac{\alpha^{2n+1}}{2}\left(\alpha^2 + \frac{1}{\alpha^2}\right) + \frac{\beta^{2n+1}}{2}\left(\beta^2 + \frac{1}{\beta^2}\right) - \frac{1}{2}(\alpha^{2n-1} + \beta^{2n-1}) \\ &= \frac{(\alpha^{2n+1} + \beta^{2n+1})}{2}6 - a_{n-1} = 6a_n - a_{n-1}, \end{aligned}$$

as claimed.

On the other hand, from the characteristic equation of the preceding recursion, namely, $x^2 - 6x + 1 = 0$ with $x_1 = \alpha^2$ and $x_2 = \beta^2$, we get

$$a_n = C_1\alpha^{2n} + C_2\beta^{2n}. \tag{5}$$

Setting $n = 0$ and $n = 1$ in the preceding expression, we obtain

$$C_1 = \frac{\alpha+1}{\alpha-\beta} \quad \text{and} \quad C_2 = -\frac{\beta+1}{\alpha-\beta},$$

and (5) then becomes

$$a_n = \frac{(\alpha+1)\alpha^{2n} - (\beta+1)\beta^{2n}}{\alpha-\beta} = \frac{(\alpha^{2n+1} - \beta^{2n+1}) + (\alpha^{2n} - \beta^{2n})}{\alpha-\beta}.$$

Applying (1), we have $a_n = P_{2n+1} + P_{2n}$ and the Lemma is proved.

Theorem 2 then immediately follows from Lemma 1 and Lemma 2 and we are done.

Theorem 3. The following identities hold for all $n \geq 0$.

$$P_{2n-1} + P_{2n+1} = \sum_{r=0}^{n} \frac{2n}{2n-r}\binom{2n-r}{r}2^{2n-2r} \quad \text{and}$$

$$P_{2n} + P_{2n+2} = \sum_{r=0}^{n} \frac{2n+1}{2n+1-r}\binom{2n+1-r}{r}2^{2n+1-2r}.$$

Proof. We start with a Lemma.

Lemma 3. For all $n \geq 0$,

$$P_{n+1} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r}2^{n-2r}. \tag{6}$$

Proof. We will argue by mathematical induction. For $n = 0$ the result trivially holds. Assume that (6) holds and we should prove that

$$P_{n+2} = \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-r}{r} 2^{n+1-2r}.$$

In fact, if n is even, then

$$\sum_{r=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-r}{r} 2^{n+1-2r} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} 2^{n+1-2r} + \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r-1} 2^{n+1-2r}.$$

Setting $j = r - 1$ in the last sum, we get

$$\sum_{r=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1-r}{r} 2^{n+1-2r} = 2 \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} 2^{n-2r}$$

$$+ \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-j}{j} 2^{n-1-2j} = 2P_{n+1} + P_n.$$

Thus, by mathematical induction, formula (6) holds for all even integers $n \geq 0$. Similarly, it can be shown that (6) holds when n is odd.

From (6) and taking into account Pascal's identity

$$\binom{n-k}{k-1} + \binom{n+1-k}{k} = \frac{n+1}{n+1-k} \binom{n+1-k}{k},$$

we have

$$P_{2n-1} + P_{2n+1} = \sum_{r=0}^{n-1} \binom{2n-2-r}{r} 2^{2n-2-2r} + \sum_{r=0}^{n} \binom{2n-r}{r} 2^{2n-2r}.$$

Setting $r = k - 1$ in the first of the preceding sums, we obtain

$$P_{2n-1} + P_{2n+1} = \sum_{k=1}^{n} \binom{2n-1-k}{k-1} 2^{2n-2k} + \sum_{r=0}^{n} \binom{2n-r}{r} 2^{2n-2r}$$

$$= \binom{2n}{0} 2^{2n} + \sum_{k=1}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} 2^{2n-2k}.$$

Similarly,

$$P_{2n} + P_{2n+2} = \sum_{r=0}^{n-1} \binom{2n-1-r}{r} 2^{2n-1-2r} + \sum_{r=0}^{n} \binom{2n+1-r}{r} 2^{2n+1-2r}$$

$$= \binom{2n+1}{0} 2^{2n+1} + \sum_{r=1}^{n} \left[\binom{2n-r}{r-1} + \binom{2n+1-r}{r} \right] 2^{2n+1-2r}$$

$$= \sum_{r=0}^{n} \frac{2n+1}{2n+1-k} \binom{2n+1-r}{r} 2^{2n+1-2r}.$$

This completes the proof.

3. Divisibility Properties. Next, we consider sums of an even and an odd number of terms of odd order in the sequence of Pell numbers and we obtain the following results.

Theorem 4. For all $n \geq 0$,

$$P_{2n+1} \left| \sum_{k=0}^{2n} P_{2k+1}, \quad \text{and} \quad P_{2n} \right| \sum_{k=1}^{2n} P_{2k-1}.$$

Proof. Taking into account (1) and the fact that $\alpha^2 - 1 = 2\alpha$ and $\beta^2 - 1 = 2\beta$, we have

$$\begin{aligned}
&P_1 + P_3 + P_5 + \cdots + P_{4n+1}\\
&= \frac{\alpha^1 - \beta^1}{\alpha - \beta} + \frac{\alpha^3 - \beta^3}{\alpha - \beta} + \cdots + \frac{\alpha^{4n+1} - \beta^{4n+1}}{\alpha - \beta}\\
&= \frac{1}{\alpha - \beta}\left(\frac{\alpha^{4n+3} - \alpha}{\alpha^2 - 1} - \frac{\beta^{4n+3} - \beta}{\beta^2 - 1}\right)\\
&= \frac{1}{\alpha - \beta}\left(\frac{\alpha^{4n+2} - 1}{2} - \frac{\beta^{4n+2} - 1}{2}\right)\\
&= \frac{1}{2}\frac{\alpha^{4n+2} - \beta^{4n+2}}{\alpha - \beta} = \frac{1}{2}\left(\alpha^{2n+1} + \beta^{2n+1}\right)\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}\right)\\
&= \frac{1}{2}\left(2\sum_{r=0}^{n}\binom{2n+1}{2r}2^r\right)P_{2n+1}.
\end{aligned}$$

The second part again follows from (1). Indeed, since

$$\alpha^{2n} + \beta^{2n} = 2\sum_{r=0}^{n}\binom{2n}{2r}2^r,$$

we have that

$$\begin{aligned}
&P_1 + P_3 + P_5 + \cdots + P_{4n-1} = \frac{\alpha^1 - \beta^1}{\alpha - \beta} + \frac{\alpha^3 - \beta^3}{\alpha - \beta} + \cdots + \frac{\alpha^{4n-1} - \beta^{4n-1}}{\alpha - \beta}\\
&= \frac{1}{\alpha - \beta}\left(\frac{\alpha^{4n+1} - \alpha}{\alpha^2 - 1} - \frac{\beta^{4n+1} - \beta}{\beta^2 - 1}\right) = \frac{1}{\alpha - \beta}\left(\frac{\alpha^{4n} - 1}{2} - \frac{\beta^{4n} - 1}{2}\right)\\
&= \frac{1}{2}\frac{\alpha^{4n} - \beta^{4n}}{\alpha - \beta} = \frac{\alpha^{2n} + \beta^{2n}}{2} \cdot \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\\
&= \frac{\alpha^{2n} + \beta^{2n}}{2} \cdot P_{2n}
\end{aligned}$$

and the proof is complete.

Reference

1. S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section,* Ellis Horwood Limited, Chichester, 1989.

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