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On the complex k -Fibonacci numbers

Sergio Falcon^{1*}

Abstract: We first study the relationship between the k -Fibonacci numbers and the elements of a subset of \mathbb{Q}^2 . Later, and since generally studies that are made on the Fibonacci sequences consider that these numbers are integers, in this article, we study the possibility that the index of the k -Fibonacci number is fractional; concretely, $\frac{2n+1}{2}$. In this way, the k -Fibonacci numbers that we obtain are complex. And in our desire to find integer sequences, we consider the sequences obtained from the moduli of these numbers. In this process, we obtain several integer sequences, some of which are indexed in The Online Enciplopedy of Integer Sequences (OEIS).

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1. Introduction

Classical Fibonacci numbers have been very used in different sciences such biology, demography, or economy (Hoggat, 1969; Koshy, 2001). Recently, they have been applied even in high-energy physics (El Naschie, 2001, 2006). But, there exist generalizations of these numbers given by such researchers

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Despite getting only about 12 years, dedicated to the University on time complete, at this time, he has published more than 70 articles in mathematics covering various aspects such as disclosure mathematics mathematical didactics and, above all, mathematical research of top level. Some of his research articles have been published in the best mathematics journals around the world.

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PUBLIC INTEREST STATEMENT

So far, the Fibonacci numbers have been generalized in different forms one being that made by ourselves and we call k -Fibonacci numbers.

But all these generalizations have been in the real numbers field.

What I propose in this study is to extend the area of Fibonacci numbers to the complex numbers field.

as Horadam (1961) and recently by Bolat and Kse (2010), Ramírez (2015), Salas (2011) and the current author Falcon and Plaza (2007a, 2007b, 2009a). In this paper, this last generalization is presented, so called the k -Fibonacci numbers.

1.1. On the k -Fibonacci numbers

For any positive real number k , the k -Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by $F_{k,n+1} = k F_{k,n} + F_{k,n-1}$ for $n \geq 1$ with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

For $k = 1$, the classical Fibonacci sequence is obtained and for $k = 2$, the Pell sequence appears.

The well-known Binet formula (Falcon & Plaza, 2007a; Horadam, 1961; Spinadel, 2002) allows us to relate the k -Fibonacci numbers to the characteristic roots σ_1 and σ_2 associated to the recurrence relation $r^2 = k r + 1$ so that $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$.

If σ denotes the positive characteristic root, $\sigma = \frac{k + \sqrt{k^2 + 4}}{2}$, the general term may be written as $F_{k,n} = \frac{\sigma^n - (-\sigma)^{-n}}{\sigma + \sigma^{-1}}$, and it is verified that the limit of the quotient of two terms of the sequence $\{F_{k,n}\}_{n \in \mathbb{N}}$ is $\lim_{n \rightarrow \infty} \frac{F_{k,n+r}}{F_{k,n}} = \sigma^r$.

In particular, if $k = 1$, then σ is the Golden Ratio, $\phi = \frac{1 + \sqrt{5}}{2}$; if $k = 2$, σ_2 is the Silver Ratio and for $k = 3$, we obtain the Bronze Ratio (Spinadel, 2002).

Among other properties that we can see in Falcon and Plaza (2007a, 2007b, 2009a), we will need the Simson Identity: $F_{k,n-1} F_{k,n+1} - F_{k,n}^2 = (-1)^n$.

2. The k -Fibonacci numbers and the set $\mathcal{A} = \{(a, b), a, b \in \mathbb{Q}\}$

Let us consider the set $\mathcal{A} = \{(a, b), a, b \in \mathbb{Q}\} \subset \mathbb{C}$. In \mathcal{A} , we define the operations $(a, b) + (c, d) = (a + c, b + d)$ and for a fixed number $k \in \mathbb{N} - \{0\}$,

$$(a, b) \cdot (c, d) = (a d + b c + k a c, a c + b d). \tag{1}$$

Then, \mathcal{A} is an abelian field, with the identity element being $(0, 1)$, and

$$(a, b)^{-1} = \frac{1}{k a b + b^2 - a^2} (-a, k a + b) \tag{2}$$

the inverse of the element $(a, b) \neq (0, 0)$.

From the definition of sum, it follows that $n(a, b) = (na, nb)$, for $n \in \mathbb{Q}$ and $(a, b)^2 = (a, b) \cdot (a, b)$.

2.1. The k -Fibonacci numbers and the pairs $(1, 0)^n$

Now, we consider the subset $\mathcal{F} \subset \mathcal{A}$, defined as $\mathcal{F} = \{(1, 0)^n, n = 1, 2, \dots\}$. The elements of \mathcal{F} are related to the k -Fibonacci numbers in the following form.

LEMMA 1 The elements of \mathcal{F} are of the form

$$(1, 0)^n = (F_{k,n}, F_{k,n-1}) \tag{3}$$

Proof We proceed by induction on n .

For $n = 1$, it is $(1, 0) = (F_{k,1}, F_{k,0})$.

Assume that $(1, 0)^n = (F_{k,n}, F_{k,n-1})$ holds. Then, $(1, 0)^{n+1} = (F_{k,n}, F_{k,n-1})(1, 0) = (F_{k,n-1} + k F_{k,n}, F_{k,n}) = (F_{k,n+1}, F_{k,n})$.

It follows that, from Equations (1) and (2), we can deduce that $(1, 0)^{-n} = (-1)^n(-F_{k,n-1}, F_{k,n})$. This formula allows us to define the k -Fibonacci numbers of negative index (as is known),

$$F_{k,-n} = (-1)^{n-1}F_{k,n} \tag{4}$$

On the other hand, $(1, 0)^n(1, 0)^{-n} = (1, 0)^0$ and taking into account the Simson Identity,

$$\begin{aligned} (1, 0)^n(1, 0)^{-n} &= (F_{k,n}, F_{k,n-1})(F_{k,-n}, F_{k,-n-1}) \\ &= (F_{k,n}F_{k,-n-1} + F_{k,n-1}F_{k,-n} + kF_{k,n}F_{k,-n}, F_{k,n}F_{k,-n} + F_{k,n-1}F_{k,-n-1}) \\ &= (-1)^n(F_{k,n}F_{k,n+1} - F_{k,n-1}F_{k,n} - kF_{k,n}^2, -F_{k,n}^2 + F_{k,n-1}F_{k,n+1}) \\ &= (-1)^n(F_{k,n}F_{k,n+1} - F_{k,n}(F_{k,n+1}), (-1)^n) = (0, 1). \end{aligned}$$

Consequently, we can define $(1, 0)^0 = (0, 1)$, with $(0, 1)$ being the multiplicative identity in \mathcal{F} .

From Formula (2), it is $(1, 1)^{-1} = \frac{1}{k}(-1, k + 1)$, and consequently, $(F_{k,n}, F_{k,n})^{-1} = (F_{k,n}(1, 1))^{-1} = \frac{1}{F_{k,n}} \frac{1}{k}(-1, k + 1)$.

The previous definition and the results obtained allow us to find some properties of the k -Fibonacci numbers, previously proven in papers (Falcon & Plaza, 2007a, 2007b, 2009a), in the next subsection. □

2.1.1. Convolution of the k -Fibonacci numbers

It is obvious that

$$(1, 0)^m(1, 0)^n = (1, 0)^{m+n} = (F_{k,m+n}, F_{k,m+n-1}) \tag{5}$$

On the other hand, and taking into account the Simson Identity,

$$\begin{aligned} (1, 0)^m(1, 0)^n &= (F_{k,m}, F_{k,m-1})(F_{k,n}, F_{k,n-1}) \\ &= (F_{k,m}F_{k,n-1} + F_{k,m-1}F_{k,n} + kF_{k,m}F_{k,n}, F_{k,m}F_{k,n} + F_{k,m-1}F_{k,n-1}) \\ &= (F_{k,m}(kF_{k,n} + F_{k,n-1}) + F_{k,m-1}F_{k,n}, F_{k,m}F_{k,n} + F_{k,m-1}F_{k,n-1}) \\ &= (F_{k,m}F_{k,n+1} + F_{k,m-1}F_{k,n}, F_{k,m}F_{k,n} + F_{k,m-1}F_{k,n-1}). \end{aligned}$$

Equating first elements of this pair with (5), we deduce the convolution formula (Falcon & Plaza, 2007b; Vajda, 1989): $F_{k,m+n} = F_{k,m}F_{k,n+1} + F_{k,m-1}F_{k,n}$.

And as particular cases, we will mention the following:

(1) If $m = n$, then $F_{k,2n} = \frac{1}{k}(F_{k,n+1}^2 - F_{k,n-1}^2)$

(2) If $m = n + 1$, then $F_{k,2n+1} = F_{k,n+1}^2 + F_{k,n}^2$

3. k -Fibonacci numbers of the half index

In this section, we will study the k -Fibonacci numbers of the half index.

We will call $F_{k, \frac{2n+1}{2}}$ a k -Fibonacci number of the half index.

Taking into account $(1, 0)^{1/2}(1, 0)^{1/2} = (1, 0)$ and Equation (1), it is

$(F_{k,1/2}, F_{k,-1/2})(F_{k,1/2}, F_{k,-1/2}) = (1, 0)$. Hence, applying definition:

$$\begin{aligned} (1, 0) &= (2F_{k,1/2}F_{k,-1/2} + kF_{k,1/2}^2, F_{k,1/2}^2 + F_{k,-1/2}^2) \\ &= (F_{k,1/2}(kF_{k,1/2} + F_{k,-1/2}), F_{k,1/2}^2 + F_{k,-1/2}^2) \\ &= (F_{k,1/2}(F_{k,3/2} + F_{k,-1/2}), F_{k,1/2}^2 + F_{k,-1/2}^2). \end{aligned}$$

Then, we obtain the system of quadratic equations

$$\begin{aligned} F_{k,1/2}(F_{k,3/2} + F_{k,-1/2}) &= 1 \\ F_{k,1/2}^2 + F_{k,-1/2}^2 &= 0 \end{aligned} \tag{6}$$

From the second equation, we obtain $F_{k,-1/2} = \pm i F_{k,1/2}$. If we suppose the real part of the complex number $F_{k,1/2}$ is positive, then we must take $F_{k,-1/2} = -i F_{k,1/2}$. Replacing in (6), the following equation holds:

$$F_{k,1/2}(F_{k,3/2} - i F_{k,1/2}) = 1 \tag{7}$$

Hence, we can accept for the k -Fibonacci numbers of index $-\frac{2n+1}{2}$ the following definition: $F_{k,-\frac{2n+1}{2}} = (-1)^{n+1} i F_{k,\frac{2n+1}{2}}$. This formula is very similar to Formula (4) for the k -Fibonacci numbers of negative integer indices.

3.1. Binnet identity

The Binnet identity for the k -Fibonacci numbers of integer indices (Falcon & Plaza, 2007b) continues being valid for the case of that $n = \frac{2r+1}{2}$ because its characteristic equation is the same in both cases, $r^2 - kr - 1 = 0$. This shows that we could have defined the k -Fibonacci numbers from this formula and then to then found the different sequences for $k = 1, 2, \dots$

It is noteworthy that many of the general formulas found for the k -Fibonacci numbers continue checking for the case of $n = (2r + 1)/2$, except perhaps that sometimes it is necessary to multiply by the factor $i = \sqrt{-1}$. We have proved in Falcon and Plaza (2007a, 2007b) and in this same paper, the formulas $F_{k,2n+1} = F_{k,n+1}^2 + F_{k,n}^2$ and $F_{k,2n} = \frac{1}{k}(F_{k,n+1}^2 - F_{k,n-1}^2)$. Next, we will prove that both formulas are also valid for any number if we take into account the number of the half index. From the preceding formulas,

$$\begin{aligned} (1, 0)^{\frac{n+1}{2}} (1, 0)^{\frac{n+1}{2}} &= (1, 0)^{n+1} \\ \rightarrow (F_{k,\frac{n+1}{2}}, F_{k,\frac{n-1}{2}}) (F_{k,\frac{n+1}{2}}, F_{k,\frac{n-1}{2}}) &= (F_{k,n+1}, F_{k,n}) \\ \rightarrow (2F_{k,\frac{n+1}{2}} F_{k,\frac{n-1}{2}} + k F_{k,\frac{n+1}{2}}^2, F_{k,\frac{n+1}{2}}^2 + F_{k,\frac{n-1}{2}}^2) &= (F_{k,n+1}, F_{k,n}) \end{aligned}$$

From the second terms of both pairs, $F_{k,n} = F_{k,\frac{n+1}{2}}^2 + F_{k,\frac{n-1}{2}}^2$.

And from the first terms,

$$\begin{aligned} F_{k,n+1} &= 2F_{k,\frac{n+1}{2}} F_{k,\frac{n-1}{2}} + k F_{k,\frac{n+1}{2}}^2 \\ &= F_{k,\frac{n+1}{2}} (k F_{k,\frac{n+1}{2}} + F_{k,\frac{n-1}{2}} + F_{k,\frac{n-1}{2}}) \\ &= \frac{1}{k} (F_{k,\frac{n+3}{2}} - F_{k,\frac{n-1}{2}}) (F_{k,\frac{n+3}{2}} + F_{k,\frac{n-1}{2}}) \\ &\rightarrow F_{k,n+1} = \frac{1}{k} (F_{k,\frac{n+3}{2}}^2 - F_{k,\frac{n-1}{2}}^2). \end{aligned}$$

Finally, if we substitute n by $2n$ in these formulae, we find both initial formulas.

Also the convolution formula remains valid, and its proof is similar to the preceding, from $(1, 0)^{n+m} = (1, 0)^{\frac{2n-1}{2}} (1, 0)^{\frac{2m+1}{2}}$ and we would obtain:

$$F_{k,n+m} = F_{k,\frac{2n-1}{2}} F_{k,\frac{2m+1}{2}} + F_{k,\frac{2n-1}{2}} F_{k,\frac{2m-1}{2}}$$

However, the Catalan formula for the k -Fibonacci numbers dictates that if n is an integer number (Falcon & Plaza, 2007a, 2007b, 2009a), then $F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n-r-1}F_{k,r}^2$ changes if the number is of the half index. In this case, the Catalan formula takes the form $F_{k,\frac{2n+1}{2}-r}F_{k,\frac{2n+1}{2}+r} - F_{k,\frac{2n+1}{2}}^2 = (-1)^{n-r-1}iF_{k,r}^2$. It is enough to apply the Binnet Identity, taking into account that $\sigma_1 \cdot \sigma_2 = -1$. Consequently, the Simson Identity $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$ is transformed into $F_{k,\frac{2n-1}{2}}F_{k,\frac{2n+3}{2}} - F_{k,\frac{2n+1}{2}}^2 = (-1)^ni$.

3.2. Some notes about the k -Fibonacci numbers of the half index

- (1) Both Real and imaginary parts of $F_{k,\frac{2n+1}{2}}$ never are integers. Consequently, $F_{k,\frac{2n+1}{2}}$ never is a Gaussian Integer (Weisstein, 2009).
- (2) For a fixed number k , it is verified that $F_{k,n} < \left|F_{k,\frac{2n+1}{2}}\right| < F_{k,n+1}$ except for the classical Fibonacci number $\left|F_{\frac{3}{2}}\right| < F_1$.
- (3) Taking into account that $|\sigma_2|$ decreases when n increases, the absolute value of $F_{k,\frac{2n+1}{2}}$ tends to the Real part of this number:

$$\lim_{n \rightarrow \infty} \left|F_{k,\frac{2n+1}{2}}\right| = \lim_{n \rightarrow \infty} \left(\operatorname{Re}(F_{k,\frac{2n+1}{2}})\right). \tag{8}$$

3.3. Another formula for the k -Fibonacci numbers of the half index

If, in the Binnet Identity $F_{k,\frac{2n+1}{2}} = \frac{\sigma_1^{\frac{2n+1}{2}} - \sigma_2^{\frac{2n+1}{2}}}{\sigma_1 - \sigma_2}$, we multiply both numerator and denominator of the fraction by $\sigma_1^{\frac{2n+1}{2}} + \sigma_2^{\frac{2n+1}{2}}$ and then we do the division, on obtaining the following formula for the calculus of the k -Fibonacci number of the half index:

$$F_{k,\frac{2n+1}{2}} = \frac{1}{\sigma_1^{\frac{2n+1}{2}} + \sigma_2^{\frac{2n+1}{2}}} \sum_{j=0}^{2n} (-1)^j \sigma_1^{2(n-j)}.$$

4. On the sequences of k -Fibonacci numbers of half index

Let us consider the k -Fibonacci sequence of complex numbers $\left\{F_{k,\frac{2n+1}{2}}\right\}_{n \in \mathbb{N}}$.

The Binnet Identity can be indicated as $F_{k,\frac{2n+1}{2}} = \frac{\sigma^{\frac{2n+1}{2}} - (-1)^{\frac{2n+1}{2}} \sigma^{-\frac{2n+1}{2}}}{\sqrt{k^2 + 4}}$

hence

$$F_{k,\frac{2n+1}{2}} = \frac{1}{\sqrt{k^2 + 4}} \left(\sigma^n \sqrt{\sigma} + (-1)^n \frac{1}{\sigma^n \sqrt{\sigma}} i \right) \tag{9}$$

Consequently,

$$\operatorname{Re}\left(F_{k,\frac{2n+1}{2}}\right) = \sigma^n \sqrt{\frac{\sigma}{k^2 + 4}} \tag{10}$$

$$\operatorname{Im}\left(F_{k,\frac{2n+1}{2}}\right) = (-1)^n \frac{1}{\sigma^n \sqrt{\sigma(k^2 + 4)}} \tag{11}$$

Hence, the real part of the first term of this sequence is $\operatorname{Re}\left(F_{k,\frac{1}{2}}\right) = \sqrt{\frac{\sigma}{k^2 + 4}}$ and the real parts of the successive terms is obtained multiplying the real part of the previous term by σ . Similarly, the imaginary part of the first term of this sequence is $\operatorname{Im}\left(F_{k,\frac{1}{2}}\right) = \frac{1}{\sqrt{(k^2+4)\sigma}}$ and the imaginary parts of the successive terms are obtained by multiplying the imaginary part of the previous term by $-\sigma^{-1} = \sigma_2$.

Consequently, this k -Fibonacci sequence takes the form

$$\frac{1}{\sqrt{(k^2 + 4)\sigma}} \left\{ (\sigma^{n+1} + (-1)^n \frac{1}{\sigma^n} i) \right\}.$$

From Equations (8) and (10), we deduce that the sequence $\left\{ |F_{k, \frac{2n+1}{2}}| \right\}$ diverges.

From Equation (9), we obtain the following interesting results.

4.1. Theorem

For all $n \in \mathbb{N}$, following equalities hold:

$$\begin{aligned} (1) \quad & \frac{\operatorname{Re}\left(F_{k, \frac{2n+1}{2}+r}\right)}{\operatorname{Re}\left(F_{k, \frac{2n+1}{2}}\right)} = \sigma^r \\ (2) \quad & \frac{\operatorname{Im}\left(F_{k, \frac{2n+1}{2}+r}\right)}{\operatorname{Im}\left(F_{k, \frac{2n+1}{2}}\right)} = (-1)^r \frac{1}{\sigma^r} \\ (3) \quad & \lim_{n \rightarrow \infty} \left| \frac{F_{k, \frac{2n+1}{2}+r}}{F_{k, \frac{2n+1}{2}}} \right| = \sigma^r \end{aligned}$$

The first two formulae are obvious. As for the third, we must bear in mind that the imaginary part tends to zero when the index tends to infinite, so its contribution to the modulus of the complex number decreases when n increases. In consequence,

$$\lim_{n \rightarrow \infty} \left| \frac{F_{k, \frac{2n+1}{2}+r}}{F_{k, \frac{2n+1}{2}}} \right| = \lim_{n \rightarrow \infty} \frac{\operatorname{Re}\left(F_{k, \frac{2n+1}{2}+r}\right)}{\operatorname{Re}\left(F_{k, \frac{2n+1}{2}}\right)} = \frac{\operatorname{Re}\left(F_{k, \frac{2n+1}{2}+r}\right)}{\operatorname{Re}\left(F_{k, \frac{2n+1}{2}}\right)} = \sigma^r.$$

The next theorem relates the k -Fibonacci numbers of the half index to the k -Fibonacci numbers of the integer index.

4.2. Theorem

For all integers k and for all $n \in \mathbb{N}$:

$$F_{k, \frac{2n+1}{2}} \left(F_{k, \frac{2n+3}{2}+r} - i F_{k, \frac{2n+1}{2}+r} \right) = F_{k, n+1} (F_{k, n+1+r} - i F_{k, n+r}) + i (-1)^n F_{k, r} \tag{12}$$

Proof Applying the Binnet identity to both sides of this equation, taking into account $(\sigma_1 \sigma_2)^{\frac{1}{2}} = -i$, and after removing $k^2 + 4$ from both denominators, it becomes

$$\begin{aligned} (LHS) &= (\sigma_1^{n+\frac{1}{2}} - \sigma_2^{n+\frac{1}{2}}) \left(\sigma_1^{n+\frac{1}{2}+1+r} - \sigma_2^{n+\frac{1}{2}+1+r} - i \sigma_1^{n+\frac{1}{2}+r} + i \sigma_2^{n+\frac{1}{2}+r} \right) \\ &= \sigma_1^{2n+2+r} + (-1)^n i \sigma_2^{r+1} - i \sigma_1^{2n+1+r} + (-1)^n \sigma_2^r \\ &\quad + i (-1)^n \sigma_1^{r+1} + \sigma_2^{2n+r+2} + (-1)^n \sigma_1^r - i \sigma_2^{2n+r+1} \\ (RHS) &= (\sigma_1^{n+1} - \sigma_2^{n+1}) (\sigma_1^{n+1+r} - \sigma_2^{n+1+r} - i \sigma_1^{n+r} + i \sigma_2^{n+r}) \\ &\quad + (-1)^n i (\sigma_1^r - \sigma_2^r) (\sigma_1 - \sigma_2) \\ &= \sigma_1^{2n+2+r} - (-1)^{n+1} \sigma_2^r - i \sigma_1^{2n+1+r} + i (-1)^{n+1} \sigma_2^{r-1} \\ &\quad - (-1)^{n+1} \sigma_1^r + \sigma_2^{2n+2+r} + i (-1)^{n+1} \sigma_1^{r-1} - i \sigma_2^{2n+1+r} \\ &\quad + (-1)^n i \sigma_1^{r+1} + (-1)^n i \sigma_1^{r-1} + (-1)^n i \sigma_2^{r-1} + (-1)^n i \sigma_2^{r+1} = (LHS). \end{aligned}$$

In particular, for $n = r = 0$, Equation (7) is obtained. □

4.3. On the sequences of k -Fibonacci numbers of the half index

Taking into account a k -Fibonacci number of the half index is a complex number which both real-part and imaginary part are never integers, the sequences of these numbers do not have greater interest. Of course, in these sequences, the initial relation is verified, that is $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$.

The sequences related to the modulus of these complex numbers are more interesting.

Let $|F_{k,r}|$ be the modulus of the k -Fibonacci number $F_{k,r}$, when $r = \frac{2n+1}{2}$.

The floor function of $|F_{k,n}|$ is the integral part of this number: $O_{k,n} = \text{Floor}[|F_{k,n}|]$. We can also say that they are obtained by the rounding down of $|F_{k,n}|$.

The round function of $|F_{k,n}|$ is the closest integer to this number: $R_{k,n} = \text{Round}[|F_{k,n}|]$.

The ceiling function of $|F_{k,n}|$ is the function whose value is the smallest integer, not less than $|F_{k,n}|$: $C_{k,n} = \text{Ceiling}[|F_{k,n}|]$ (http://en.wikipedia.org/wiki/Catalan_number). We can also say that they are obtained by the rounding up of $|F_{k,n}|$.

For $k = 1, 2, 3$, we will obtain the following sequences, none of which is indexed in Sloane (2006), from now on OEIS:

- (1) For $k = 1$:
 - (a) $O_1 = \{0, 0, 1, 2, 3, 6, 10, 16, \dots\}$
 - (b) $R_1 = \{1, 1, 2, 4, 6, 10, 17, 27, \dots\}$
 - (c) $C_1 = \{1, 1, 2, 3, 4, 7, 11, 17, 27, \dots\}$
- (2) For $k = 2$:
 - (a) $O_2 = \{0, 1, 3, 7, 18, 45, 108, 262, \dots\}$
 - (b) $R_2 = \{1, 1, 3, 8, 19, 45, 109, 263, \dots\}$
 - (c) $C_2 = \{1, 2, 4, 8, 19, 46, 109, 263, \dots\}$
- (3) For $k = 3$:
 - (a) $O_3 = \{0, 1, 5, 18, 59, 198, 654, \dots\}$
 - (b) $R_3 = \{1, 2, 5, 18, 60, 198, 654, \dots\}$
 - (c) $C_3 = \{1, 2, 6, 19, 60, 199, 655, \dots\}$

5. Integer sequences from $(1, 1)^n$

Let us remember that $(0, 1)$ is the unity element of $\mathcal{F} = \{(a, b)\}$, so $(0, 1)^n = (0, 1)$. Then, taking into account Equation (3), $(1, 0)^n = (F_{k,n}, F_{k,n-1})$, if a and b are non-null simultaneously, then

$$\begin{aligned} (a, b)^n &= (a(1, 0) + b(0, 1))^n = \sum_{j=0}^n \binom{n}{j} (a(1, 0))^{n-j} b^j \\ &= \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j (1, 0)^{n-j} = \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j (F_{k,n-j}, F_{k,n-j-1}). \end{aligned}$$

5.1. Expression of a k -Fibonacci number whose index is a multiple of another index

As $((1, 0)^n)^m = (1, 0)^{n \cdot m} = (F_{k,n \cdot m}, F_{k,n \cdot m - 1})$ and

$$((1, 0)^n)^m = (F_{k,n}, F_{k,n-1})^m = \sum_{j=0}^m \binom{m}{j} F_{k,n}^{m-j} F_{k,n-1}^j (F_{k,m-j}, F_{k,m-j-1}), \text{ we obtain}$$

$$F_{k,m,n} = \sum_{j=0}^m \binom{m}{j} F_{k,n}^{m-j} F_{k,n-1}^j F_{k,m-j} \tag{13}$$

For $m = 2$: $F_{k,2n} = \sum_{j=0}^2 \binom{2}{j} F_{k,n}^{2-j} F_{k,n-1}^j F_{k,2-j} = k F_{k,n}^2 + 2 F_{k,n} F_{k,n-1}$ (Falcon & Plaza, 2007a, 2007b).

For $m = 3$: $F_{k,3n} = \sum_{j=0}^3 \binom{3}{j} F_{k,n}^{3-j} F_{k,n-1}^j F_{k,3-j} = (k^2 + 4) F_{k,n}^3 - 3k F_{k,n-1} F_{k,n}^2 + 3k^2 F_{k,n-1}^2 F_{k,n}$ (Falcon & Plaza, 2007a, 2007b).

Equation (13) can be written as $F_{k,m,n} = F_{k,n} \sum_{j=0}^{m-1} \binom{m}{j} F_{k,n}^{m-j-1} F_{k,n-1}^j F_{k,m-j}$ and that denotes that $F_{k,m,n}$ is a multiple of $F_{k,n}$.

In short:

(1) If $a \neq 0$ and $b \neq 0$: $(a, b)^n = \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j (F_{k,n-j}, F_{k,n-j-1})$

(2) If $b = 0$: $(a, 0)^n = a^n (1, 0)^n = a^n (F_{k,n}, F_{k,n-1})$

(3) If $a = 0$: $(0, b)^n = b^n (0, 1)^n = b^n (0, 1)$

If $a = b = 1$, then $(1, 1)^n = \sum_{j=0}^n \binom{n}{j} (F_{k,n-j}, F_{k,n-j-1})$: the first terms of the sequence of powers $\{(1, 1)^n\}$ are the binomial transforms of the k -Fibonacci sequence (Falcon & Plaza, 2009b).

5.2. Integer sequences of coefficients from $(1, 1)^n$

In the sequel, we give the expressions of the first terms of this sequence $\{(1, 1)^n\}$, for $n = 0, 1, 2, \dots$

$(1, 1)^0 = (0, 1)$

$(1, 1)^1 = (1, 1)$

$(1, 1)^2 = (1, 1)(1, 1) = (k + 2, 2)$

$(1, 1)^3 = (k + 2, 2)(1, 1) = (k^2 + 3k + 4, k + 4)$

$(1, 1)^4 = (k^2 + 3k + 4, k + 4)(1, 1) = (k^3 + 4k^2 + 8k + 8, k^2 + 4k + 8)$

...

With the coefficients of the first terms of the pairs of the Second-Hand Side, we form Table 1.

The number a_{ij} is the coefficient of k^{i-j} in the first elements of the pairs of $(1, 1)^i$. The first diagonal is the sequence of powers of 2, $\{2^n\}$, $n = 0, 1, 2, \dots$

Table 1. Coefficients in $\sum_{j=0}^n \binom{n}{j} F_{k,n-j}$								
	1	2	3	4	5	6	7	8
1	1							
2	1	2						
3	1	3	4					
4	1	4	8	8				
5	1	5	13	20	16			
6	1	6	19	38	48	32		
7	1	7	26	63	104	112	64	
8	1	8	34	96	192	272	256	128

Any other coefficient of the row r and column c , can be calculated as $a_{r,c} = a_{r-1,c} + \sum_{j=1}^c a_{r-j,c-j}$. For instance, $192 = 104 + 63 + 19 + 5 + 1$.

Moreover: the sequence of the sums of the coefficients of each row is the bisection of the classical Fibonacci sequence $\{1, 3, 8, 21, 55, 144, 377, 987, \dots\} = A001906$ and its alternate sums is this same sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\} = A000045$. Hence, we can write $\sum_{j=0}^n \binom{n}{j} F_{n-j} = F_{2n}$ and $\sum_{j=0}^n (-1)^{n-j-1} \binom{n}{j} F_{n-j} = F_n$.

Only the first four column sequences are listed in OEIS as A000012, A000027, A034856, A006416.

Each diagonal sequence is the convolution of the preceding diagonal sequence and $A011782 = \{1, 1, 2, 4, 8, 16, 32, \dots\}$ and are listed in OEIS:

A000079, A001792, A049611, A049612, A055589, A055852, A055853, A055854, and A055855.

Finally, we indicate that the generating function of the diagonal sequence $D_n = \{1, n, \dots\}$ is
$$d(x) = \frac{(1-x)^{n-4}}{(1-2x)^{n-2}}.$$

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