

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY  
HARRIS KWONG

Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at [kwong@fredonia.edu](mailto:kwong@fredonia.edu). If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2018. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of *The Fibonacci Quarterly* are all available on the web free of charge at [www.fq.math.ca/](http://www.fq.math.ca/).

### BASIC FORMULAS

The Fibonacci numbers  $F_n$  and the Lucas numbers  $L_n$  satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also,  $\alpha = (1 + \sqrt{5})/2$ ,  $\beta = (1 - \sqrt{5})/2$ ,  $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ , and  $L_n = \alpha^n + \beta^n$ .

### PROBLEMS PROPOSED IN THIS ISSUE

**B-1221** Proposed by José Luis Díaz-Barrero, Technical University of Catalonia (Barcelona Tech), Barcelona, Spain.

For any positive integer  $n$ , show that

$$\frac{1}{54F_{2n}} \begin{vmatrix} 4 & F_n & L_n \\ F_n & (F_{n+1} + L_n)^2 & F_{2n} \\ L_n & F_{2n} & F_{n+2}^2 \end{vmatrix}$$

is a perfect square, and find its value.

**B-1222** Proposed by Kenny B. Davenport, Dallas, PA.

Let  $H_n$  denote the  $n$ th harmonic number. Prove that

$$\sum_{n=2}^{\infty} \frac{H_{n-1}F_n}{n2^n} = \frac{\ln 16 \cdot \ln \alpha}{\sqrt{5}}, \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{H_{n-1}L_n}{n2^n} = (\ln 2)^2 + 4(\ln \alpha)^2.$$

**B-1223** Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers  $n$  and  $a$ , prove that

$$\sum_{k=1}^n F_k(F_{k+1}^a + F_{k+2}^a - F_{n+2}^a - 1) \leq 0.$$

**B-1224** Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any positive integer  $n$ , prove that

$$\sum_{k=1}^n \binom{n}{k} \frac{F_k}{k} = \sum_{k=1}^n \frac{F_{2k}}{k}, \quad \text{and} \quad \sum_{k=1}^n \binom{n}{k} \frac{L_k}{k} = \sum_{k=1}^n \frac{L_{2k} - 2}{k}.$$

**B-1225** Proposed by Jathan Austin, Salisbury University, Salisbury, MD.

Construct a sequence  $\{M_n\}_{n=1}^\infty$  of  $3 \times 3$  matrices with positive entries that satisfy the following conditions:

- (A)  $|M_n|$  is the product of nonzero Fibonacci numbers.
- (B) The determinant of any  $2 \times 2$  submatrix of  $M_n$  is a Fibonacci number or the product of nonzero Fibonacci numbers.
- (C)  $\lim_{n \rightarrow \infty} |M_{n+1}|/|M_n| = 1 + 2\alpha$ .

## SOLUTIONS

### Cauchy-Schwarz or Bergström Again!

**B-1201** Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.  
(Vol. 55.1, February 2017)

If  $a, b, c > 0$ , then prove that, for any positive integer  $n$ ,

$$\begin{aligned} \frac{a^3}{aF_n + bF_{n+1}} + \frac{b^3}{bF_n + aF_{n+1}} &\geq \frac{a^2 + b^2}{F_{n+2}}, \\ \frac{a^3}{aL_n + bL_{n+1}} + \frac{b^3}{bL_n + aL_{n+1}} &\geq \frac{a^2 + b^2}{L_{n+2}}, \\ \frac{a^3}{aF_n + bF_{n+1} + cF_{n+2}} + \frac{b^3}{bF_n + cF_{n+1} + aF_{n+2}} + \frac{c^3}{cF_n + aF_{n+1} + bF_{n+2}} &\geq \frac{a^2 + b^2 + c^2}{2F_{n+2}}, \\ \frac{a^3}{aL_n + bL_{n+1} + cL_{n+2}} + \frac{b^3}{bL_n + cL_{n+1} + aL_{n+2}} + \frac{c^3}{cL_n + aL_{n+1} + bL_{n+2}} &\geq \frac{a^2 + b^2 + c^2}{2L_{n+2}}. \end{aligned}$$

**Solution by Brian Bradie, Christopher Newport University, Newport News, VA.**

Let  $a, b, c, x, y, z$  be positive real numbers. By the Cauchy-Schwarz inequality,

$$\frac{a^3}{ax + by} + \frac{b^3}{bx + cz} = \frac{a^4}{a^2x + aby} + \frac{b^4}{b^2x + aby} \geq \frac{(a^2 + b^2)^2}{(a^2 + b^2)x + 2aby}.$$

Now, by the arithmetic mean - geometric mean inequality,  $2ab \leq a^2 + b^2$ , so

$$\frac{a^3}{ax + by} + \frac{b^3}{bx + cz} \geq \frac{(a^2 + b^2)^2}{(a^2 + b^2)x + (a^2 + b^2)y} = \frac{a^2 + b^2}{x + y}. \quad (1)$$

With  $x = F_n$  and  $y = F_{n+1}$ , (1) becomes

$$\frac{a^3}{aF_n + bF_{n+1}} + \frac{b^3}{bF_n + aF_{n+1}} \geq \frac{a^2 + b^2}{F_n + F_{n+1}} = \frac{a^2 + b^2}{F_{n+2}}.$$

With  $x = L_n$  and  $y = L_{n+1}$ , (1) becomes

$$\frac{a^3}{aL_n + bL_{n+1}} + \frac{b^3}{bL_n + aL_{n+1}} \geq \frac{a^2 + b^2}{L_n + L_{n+1}} = \frac{a^2 + b^2}{L_{n+2}}.$$

Next, by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \frac{a^3}{ax + by + cz} + \frac{b^3}{bx + cy + az} + \frac{c^3}{cx + ay + bz} \\ &= \frac{a^4}{a^2x + aby + caz} + \frac{b^4}{b^2x + bcy + abz} + \frac{c^4}{c^2x + cay + bcz} \\ &\geq \frac{(a^2 + b^2 + c^2)^2}{(a^2 + b^2 + c^2)x + (ab + bc + ca)(y + z)}. \end{aligned}$$

Now, the inequality  $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$  is equivalent to

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

so

$$\begin{aligned} & \frac{a^3}{ax + by + cz} + \frac{b^3}{bx + cy + az} + \frac{c^3}{cx + ay + bz} \\ &\geq \frac{(a^2 + b^2 + c^2)^2}{(a^2 + b^2 + c^2)x + (a^2 + b^2 + c^2)(y + z)} = \frac{a^2 + b^2 + c^2}{x + y + z}. \end{aligned} \quad (2)$$

With  $x = F_n$ ,  $y = F_{n+1}$ , and  $z = F_{n+2}$ , (2) becomes

$$\begin{aligned} & \frac{a^3}{aF_n + bF_{n+1} + cF_{n+2}} + \frac{b^3}{bF_n + cF_{n+1} + aF_{n+2}} + \frac{c^3}{cF_n + aF_{n+1} + bF_{n+2}} \\ &\geq \frac{a^2 + b^2 + c^2}{F_n + F_{n+1} + F_{n+2}} = \frac{a^2 + b^2 + c^2}{2F_{n+2}}. \end{aligned}$$

With  $x = L_n$ ,  $y = L_{n+1}$ , and  $z = L_{n+2}$ , (2) becomes

$$\begin{aligned} & \frac{a^3}{aL_n + bL_{n+1} + cL_{n+2}} + \frac{b^3}{bL_n + cL_{n+1} + aL_{n+2}} + \frac{c^3}{cL_n + aL_{n+1} + bL_{n+2}} \\ &\geq \frac{a^2 + b^2 + c^2}{L_n + L_{n+1} + L_{n+2}} = \frac{a^2 + b^2 + c^2}{2L_{n+2}}. \end{aligned}$$

*Editor's Note:* Ricardo used Bergström inequality to derive (1) and (2).

Also solved by **Dmitry Fleischman**, **Hideyuki Ohtsuka**, **Ángel Plaza**, **Henry Ricardo**, **Nicușor Zlota**, and the proposer.

Root and Ratio Tests

**B-1202** Proposed by D. M. Băţineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania; Neculai Stanciu, George Emil Palade School, Buzău, Romania; and Gabriel Tica, Mihai Viteazul National College, Băileşti, Dolj, Romania.  
(Vol. 55.1, February 2017)

Let  $(a_n)_{n \geq 1}$  be a positive real sequence such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 a_n} = a$ . Evaluate

$$\lim_{n \rightarrow \infty} \left( \sqrt[n+1]{\frac{a_{n+1} F_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_n F_n}{n!}} \right) \quad \text{and} \quad \lim_{n \rightarrow \infty} \left( \sqrt[n+1]{\frac{a_{n+1} L_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_n L_n}{n!}} \right).$$

**Solution by the proposers.**

We claim that both limits equal to  $a\alpha/e$ . Given an infinite sequence  $(b_n)_{n \geq 1}$ , it is known that if  $\lim_{n \rightarrow \infty} |b_{n+1}/b_n| = L$ , then  $\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|} = L$ . Apply this to  $c_n = a_n F_n / (n! n^n)$ . We find

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \rightarrow \infty} \frac{a_{n+1} F_{n+1}}{(n+1)! (n+1)^{n+1}} \cdot \frac{n! n^n}{a_n F_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 a_n} \cdot \frac{F_{n+1}}{F_n} \left( \frac{n}{n+1} \right)^{n+2} = \frac{a\alpha}{e}.$$

Thus,  $\lim_{n \rightarrow \infty} \sqrt[n]{c_n} = a\alpha/e$  as well. Define

$$u_n = \sqrt[n+1]{\frac{a_{n+1} F_{n+1}}{(n+1)!}} \cdot \sqrt[n]{\frac{n!}{a_n F_n}} = \frac{\sqrt[n+1]{c_{n+1}}}{\sqrt[n]{c_n}} \cdot \frac{n+1}{n},$$

such that

$$\sqrt[n+1]{\frac{a_{n+1} F_{n+1}}{(n+1)!}} - \sqrt[n]{\frac{a_n F_n}{n!}} = \sqrt[n]{\frac{a_n F_n}{n!}} (u_n - 1) = \sqrt[n]{c_n} \cdot n(u_n - 1).$$

It suffices to show that  $\lim_{n \rightarrow \infty} n(u_n - 1) = 1$ . Note that  $\lim_{n \rightarrow \infty} u_n = 1$ , and

$$\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \cdot \frac{1}{\sqrt[n+1]{c_{n+1}}} \left( \frac{n+1}{n} \right)^n = e.$$

Therefore,

$$\lim_{n \rightarrow \infty} n(u_n - 1) = \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n = 1 \cdot \ln e = 1.$$

The proof of the other limit is similar, and is omitted here.

*Editor's Note:* Plaza noted that the inequalities follow from a result obtained by the first two proposers in [1], and Ohtsuka used a result from [2] to derive the inequalities directly.

REFERENCES

- [1] D. M. Băținețu-Giurgiu and N. Stanciu, *New methods for calculations of some limits*, The Teaching of Mathematics, **16**(2) (2013), 82–88.  
 [2] Gh. Toader, *Lalescu sequences*, Publikacije Elektrotehničkog fakulteta Univerziteta u Beogradu, Serija Matematika i fizika, **9** (1998), 19–28.

Also solved by **I. V. Fedak, Dmitry Fleishcman, Hamza Mahmood (student), Soumitra Mandal, Hideyuki Ohtsuka, Ángel Plaza, and Raphael Schumacher (student)**.

**Fibonacci Numbers with Fibonacci Numbers as Subscripts**

**B-1203** Proposed by **Hideyuki Ohtsuka, Saitama, Japan.**  
 (Vol. 55.1, February 2017)

Prove that, for any positive integer  $n$ ,

- (i) 
$$\sum_{k=1}^n F_{F_{3k}} F_{F_{3k-1}} F_{F_{3k-2}} = \frac{1}{5} \sum_{k=1}^{3n} F_{2F_k};$$
  
 (ii) 
$$\sum_{k=1}^n L_{L_{3k}} L_{L_{3k-1}} L_{L_{3k-2}} = 2n + \sum_{k=1}^{3n} (-1)^{L_k} L_{2L_k}.$$

**Solution by Jaroslav Seibert, University of Pardubice, Czech Republic.**

Using the Binet’s formula for the Fibonacci numbers, we find

$$\begin{aligned} F_{F_{3k}} F_{F_{3k-1}} F_{F_{3k-2}} &= \left( \frac{\alpha^{F_{3k}} - \beta^{F_{3k}}}{\sqrt{5}} \right) \left( \frac{\alpha^{F_{3k-1}} - \beta^{F_{3k-1}}}{\sqrt{5}} \right) \left( \frac{\alpha^{F_{3k-2}} - \beta^{F_{3k-2}}}{\sqrt{5}} \right) \\ &= \frac{1}{5\sqrt{5}} \left[ \alpha^{2F_{3k}} - (\alpha\beta)^{F_{3k-2}} \alpha^{2F_{3k-1}} - (\alpha\beta)^{F_{3k-1}} \alpha^{2F_{3k-2}} - (\alpha\beta)^{F_{3k}} \right. \\ &\quad \left. + (\alpha\beta)^{F_{3k}} + (\alpha\beta)^{F_{3k-1}} \beta^{2F_{3k-2}} + (\alpha\beta)^{F_{3k-2}} \beta^{2F_{3k-1}} - \beta^{2F_{3k}} \right]. \end{aligned}$$

Since  $\alpha\beta = -1$ , and  $F_{3k-1}$  and  $F_{3k-2}$  are both odd for any integer  $k$ , we have  $(\alpha\beta)^{F_{3k-1}} = (\alpha\beta)^{F_{3k-2}} = -1$ . Thus,

$$\begin{aligned} F_{F_{3k}} F_{F_{3k-1}} F_{F_{3k-2}} &= \frac{1}{5} \left( \frac{\alpha^{2F_{3k}} - \beta^{2F_{3k}}}{\sqrt{5}} + \frac{\alpha^{2F_{3k-1}} - \beta^{2F_{3k-1}}}{\sqrt{5}} + \frac{\alpha^{2F_{3k-2}} - \beta^{2F_{3k-2}}}{\sqrt{5}} \right) \\ &= \frac{1}{5} (F_{2F_{3k}} + F_{2F_{3k-1}} + F_{2F_{3k-2}}). \end{aligned}$$

Finally,

$$\sum_{k=1}^n F_{F_{3k}} F_{F_{3k-1}} F_{F_{3k-2}} = \frac{1}{5} \sum_{k=1}^n (F_{2F_{3k}} + F_{2F_{3k-1}} + F_{2F_{3k-2}}) = \frac{1}{5} \sum_{k=1}^{3n} F_{2F_k},$$

which proves (i).

The proof of (ii) proceeds in a similar manner. Using the Binet's formula for the Lucas numbers, we find

$$\begin{aligned} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} &= (\alpha^{L_{3k}} + \beta^{L_{3k}})(\alpha^{L_{3k-1}} + \beta^{L_{3k-1}})(\alpha^{L_{3k-2}} + \beta^{L_{3k-2}}) \\ &= \alpha^{2L_{3k}} + (\alpha\beta)^{L_{3k-2}}\alpha^{2L_{3k-1}} + (\alpha\beta)^{L_{3k-1}}\alpha^{2L_{3k-2}} + (\alpha\beta)^{L_{3k}} \\ &\quad + (\alpha\beta)^{L_{3k}} + (\alpha\beta)^{L_{3k-1}}\beta^{2L_{3k-2}} + (\alpha\beta)^{L_{3k-2}}\beta^{2L_{3k-1}} + \beta^{2L_{3k}}. \end{aligned}$$

It is known that  $L_{3k-1}$  and  $L_{3k-2}$  are both odd, and  $L_{3k}$  is even for any integer  $k$ . Hence,  $(\alpha\beta)^{L_{3k-1}} = (\alpha\beta)^{L_{3k-2}} = -1$ , and  $(\alpha\beta)^{L_{3k}} = 1$ . Thus,

$$\begin{aligned} L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} &= (\alpha^{2L_{3k}} + \beta^{2L_{3k}}) - (\alpha^{2L_{3k-1}} + \beta^{2L_{3k-1}}) - (\alpha^{2L_{3k-2}} + \beta^{2L_{3k-2}}) + 2 \\ &= L_{2L_{3k}} - L_{2L_{3k-1}} - L_{2L_{3k-2}} + 2 \\ &= (-1)^{L_{3k}}L_{2L_{3k}} + (-1)^{L_{3k-1}}L_{2L_{3k-1}} + (-1)^{L_{3k-2}}L_{2L_{3k-2}} + 2, \end{aligned}$$

which proves that

$$\sum_{k=1}^n L_{L_{3k}}L_{L_{3k-1}}L_{L_{3k-2}} = 2n + \sum_{k=1}^{3n} (-1)^{L_k}L_{2L_k}.$$

*Editor's Note:* Plaza quoted the general formulas for the products  $F_{x_1}F_{x_2}F_{x_3}$  and  $L_{x_1}L_{x_2}L_{x_3}$  in [2], and Davenport applied the following symmetric identities from [1]:

$$\begin{aligned} 5F_xF_yF_z &= F_{x+y+z} - (-1)^xF_{-x+y+z} - (-1)^yF_{x-y+z} - (-1)^zF_{x+y-z}, \\ L_xL_yL_z &= L_{x+y+z} + (-1)^xL_{-x+y+z} + (-1)^yL_{x-y+z} + (-1)^zL_{x+y-z}. \end{aligned}$$

#### REFERENCES

- [1] P. S. Bruckman, *Solution to Problem B-890*, The Fibonacci Quarterly, **38.5** (2000), 469–470.
- [2] H. H. Ferns, *Products of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **7.1** (1969), 1–13.

Also solved by Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleishman, Ángel Plaza, Raphael Schumacher (student), and the proposer.

#### A Double Binomial Sum

**B-1204** Proposed by Steve Edwards, Kennesaw State University, Marietta, GA.  
(Vol. 55.1, February 2017)

For non-negative integers  $n$ , express

$$A_n = \sum_{j=0}^n \frac{1}{2^{2j}} \sum_{i=0}^{n+j} \binom{n+j-i}{n-j} \binom{n+j}{i} \quad \text{and} \quad B_n = \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \sum_{i=0}^{n+j} \binom{n+j-i}{n-j-1} \binom{n+j}{i}$$

in terms of Fibonacci numbers.

**Solution by Hideyuki Ohtsuka, Saitama, Japan.**

We use the well-known identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}.$$

We have

$$\begin{aligned}
 A_n &= \sum_{j=0}^n \frac{1}{2^{2j}} \sum_{i=0}^{n+j} \frac{(n+j-i)!}{(n-j)!(2j-i)!} \cdot \frac{(n+j)!}{i!(n+j-i)!} \\
 &= \sum_{j=0}^n \frac{1}{2^{2j}} \sum_{i=0}^{n+j} \frac{(n+j)!}{(n-j)!(2j)!} \cdot \frac{(2j)!}{i!(2j-i)!} \\
 &= \sum_{j=0}^n \frac{1}{2^{2j}} \binom{n+j}{n-j} \sum_{i=0}^{n+j} \binom{2j}{i} = \sum_{j=0}^n \frac{1}{2^{2j}} \binom{n+j}{n-j} \sum_{i=0}^{2j} \binom{2j}{i} \\
 &= \sum_{j=0}^n \frac{1}{2^{2j}} \binom{n+j}{n-j} \cdot 2^{2j} = \sum_{k=0}^n \binom{2n-k}{k} = F_{2n+1},
 \end{aligned}$$

and

$$\begin{aligned}
 B_n &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \sum_{i=0}^{n+j} \frac{(n+j-i)!}{(n-j-1)!(2j-i+1)!} \cdot \frac{(n+j)!}{i!(n+j-i)!} \\
 &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \sum_{i=0}^{n+j} \frac{(n+j)!}{(n-j-1)!(2j+1)!} \cdot \frac{(2j+1)!}{i!(2j-i+1)!} \\
 &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \binom{n+j}{n-j-1} \sum_{i=0}^{n+j} \binom{2j+1}{i} \\
 &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \binom{n+j}{n-j-1} \sum_{i=0}^{2j+1} \binom{2j+1}{i} \\
 &= \sum_{j=0}^{n-1} \frac{1}{2^{2j+1}} \binom{n+j}{n-j-1} \cdot 2^{2j+1} = \sum_{k=0}^{n-1} \binom{2n-1-k}{k} = F_{2n}.
 \end{aligned}$$

Also solved by Brian Bradie, I. V. Fadek, Dmitry Fleischman, Jaroslav Seibert, and the proposer.

**Power-Mean and Jensen's Inequalities**

**B-1205** Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania; and Neculai Stanciu, George Emil Palade School, Buzău, Romania.  
(Vol. 55.1, February 2017)

Prove that

$$n^{m-1} \sum_{k=1}^n F_k^{2m} \geq F_n^m F_{n+1}^m$$

for any positive integers  $n$  and  $m$ .

**Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

It is enough to apply the power-mean arithmetic mean inequality to the sequence  $(F_k^2)_{1 \leq k \leq n}$ , as follows:

$$\sqrt[m]{\frac{1}{n} \sum_{k=1}^n F_k^{2m}} \geq \frac{1}{n} \sum_{k=1}^n F_k^2 = \frac{F_n F_{n+1}}{n}.$$

It follows that

$$n^{m-1} \sum_{k=1}^n F_k^{2m} \geq F_n^m F_{n+1}^m.$$

**Solution 2 by Henry Ricardo, New York Math Circle, Purchase, NY.**

Noting that, for any positive integer  $m$ , the function  $f(x) = x^m$  is convex on the interval  $(0, \infty)$ , and that  $\sum_{k=1}^n F_k^2 = F_n F_{n+1}$ , we use Jensen's inequality to conclude that

$$\frac{1}{n} \sum_{k=1}^n f(F_k^2) \geq f\left(\frac{1}{n} \sum_{k=1}^n F_k^2\right),$$

or

$$n^{m-1} \sum_{k=1}^n F_k^{2m} \geq (F_n F_{n+1})^m = F_n^m F_{n+1}^m.$$

**Also solved by Maria Aristizabal (student), Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai, Soumitra Mandal, Hideyuki Ohtsuka, Nicușor Zlota, and the proposer.**