

## ADVANCED PROBLEMS AND SOLUTIONS

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### PROBLEMS PROPOSED IN THIS ISSUE

**H-825** Proposed by **D. M. Băţineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania**

If  $a, b, c > 0$  and  $n$  is a positive integer, prove that

$$2 \left( \left( \frac{a}{F_n b + F_{n+1} c} \right)^3 + \left( \frac{b}{F_n c + F_{n+1} a} \right)^3 + \left( \frac{c}{F_n a + F_{n+1} b} \right)^3 \right) + 3 \frac{abc}{(F_n a + F_{n+1} b)(F_n b + F_{n+1} c)(F_n c + F_{n+1} a)} \geq \frac{9}{F_{n+2}^3}.$$

**H-826** Proposed by **Hideyuki Ohtsuka, Saitama, Japan**

For an integer  $n \geq 0$ , prove that

$$\sum_{\substack{a+b=n \\ a,b \geq 0}} \frac{1}{L_{2a} 3^b F_{2a+3b+1}} = \frac{F_{3n+1} - 2^{n+1}}{F_{3n+1} F_{2n+1}}.$$

**H-827** Proposed by **D. M. Băţineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania**

Let  $(a_n)_{n \geq 0}$  be a sequence of positive real numbers such that  $\lim_{n \rightarrow \infty} a_{n+1}/(na_n) = a > 0$ . Compute

$$\lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} \left( \left( n^{+1} \sqrt[n]{a_{n+1}}^{F_m/F_{m+1}} - (n \sqrt[n]{a_n})^{F_m/F_{m+1}} \right) n^{F_{m-1}/F_m} \right) \right).$$

**H-828 Proposed by Kenneth Davenport, Dallas, PA**

Find a closed form expression for

$$\sum_{k=0}^n kT_k^2,$$

where  $(T_k)_{k \geq 0}$  is the sequence of Tribonacci numbers satisfying  $T_0 = 0, T_1 = T_2 = 1,$  and  $T_{k+3} = T_{k+2} + T_{k+1} + T_k$  for all  $k \geq 0$ .

**SOLUTIONS**

**Closed form for the sum of a series**

**H-791 Proposed by Hideyuki Ohtsuka, Saitama, Japan  
(Vol. 54, No. 2, May 2016)**

For an integer  $n \geq 0$ , find a closed form expression for the sum

$$\sum_{k=0}^n \frac{(-1)^{2k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}} \cdots L_{3^n})^2}.$$

**Solution by the proposer**

We find the identity

$$\sum_{k=0}^n \frac{(-1)^{2k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}} \cdots L_{3^n})^2} = -\frac{1}{F_{3^{n+1}}}. \tag{1}$$

The proof of (1) is by mathematical induction on  $n$ . For  $n = 0$ , we have  $LS = RS = -1/2$ . We use the identities:

- (1)  $L_a F_a = F_{2a}$  (see [1] (13));
- (2)  $F_{a+b} + (-1)^b F_{a-b} = F_a L_b$  (see [1] (15a)).

We assume (1) holds for  $n$ . For  $n + 1$ , we have

$$\begin{aligned} & \sum_{k=0}^{n+1} \frac{(-1)^{2k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}} \cdots L_{3^{n+1}})^2} \\ &= \frac{1}{F_{3^{n+2}}L_{3^{n+1}}^2} + \frac{1}{L_{3^{n+1}}^2} \sum_{k=0}^n \frac{(-1)^{2k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}} \cdots L_{3^n})^2} \\ &= \frac{1}{F_{3^{n+2}}L_{3^{n+1}}^2} - \frac{1}{L_{3^{n+1}}^2 F_{3^{n+1}}} = -\frac{F_{3^{n+2}} - F_{3^{n+1}}}{F_{3^{n+2}}F_{3^{n+1}}L_{3^{n+1}}^2} \\ &= -\frac{F_{2 \cdot 3^{n+1}}L_{3^{n+1}}}{F_{3^{n+2}}F_{2 \cdot 3^{n+1}}L_{3^{n+1}}} \quad (\text{by (1) and (2)}) \\ &= -\frac{1}{F_{3^{n+2}}}. \end{aligned}$$

Therefore, (1) holds for all  $n \geq 0$ .

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

**Partially solved by Dmitry Fleischman.**

Some Tribonacci identities

**H-792 Proposed by George A. Hisert, Berkeley, CA**  
**(Vol. 54, No. 3, August 2016)**

Consider the 3-sequence  $T_{i+1} = T_i + T_{i-1} + T_{i-2}$  for all integers  $i$  with  $T_0 = 0$ ,  $T_1 = T_2 = 1$ . Let  $S_i = T_i + T_{i-1}$ . Prove that for all integers  $n$  positive or negative, we have  $T_n^2 - T_{n+1}T_{n-1} = T_{-(n+1)}$  and  $T_{n+1}T_{n-2} - T_nT_{n-1} = S_{-(n+1)}$ .

**Solution by Brian Bradie**

Anantakitpaisal and Kuhapatanakul [1] provide the following proof for the identity  $T_n^2 - T_{n-1}T_{n+1} = T_{-(n+1)}$ . It is known that

$$\begin{bmatrix} T_{n+k} \\ T_{n+k-1} \\ T_{n+k-2} \end{bmatrix} = A^n \begin{bmatrix} T_k \\ T_{k-1} \\ T_{k-2} \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It follows that

$$\begin{aligned} T_n^2 - T_{n-1}T_{n+1} &= \begin{vmatrix} T_{n+1} & T_{n+2} & 1 \\ T_n & T_{n+1} & 0 \\ T_{n-1} & T_n & 0 \end{vmatrix} \\ &= \left| A^n \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \quad A^n \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix} \quad A^n A^{-n} \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \right| \\ &= |A^n| \begin{vmatrix} 1 & 1 & T_{-n+1} \\ 0 & 1 & T_{-n} \\ 0 & 0 & T_{-n-1} \end{vmatrix} \\ &= 1 \cdot T_{-(n+1)} = T_{-(n+1)}. \end{aligned}$$

Proceeding as above, we have

$$\begin{aligned} T_{n+1}T_{n-2} - T_nT_{n-1} &= \begin{vmatrix} T_{n+2} & T_n & 1 \\ T_{n+1} & T_{n-1} & 0 \\ T_n & T_{n-2} & 0 \end{vmatrix} \\ &= \left| A^n \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix} \quad A^n \begin{bmatrix} T_0 \\ T_{-1} \\ T_{-2} \end{bmatrix} \quad A^n A^{-n} \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \right| \\ &= |A^n| \begin{vmatrix} 1 & 0 & T_{-n+1} \\ 1 & 0 & T_{-n} \\ 0 & 1 & T_{-n-1} \end{vmatrix} \\ &= T_{-n+1} - T_{-n} = T_{-n-1} + T_{-n-2} = S_{-n-1} = S_{-(n+1)}. \end{aligned}$$

[1] P. Anantakitpaisal and K. Kuhapatanakul, *Reciprocal sums of the Tribonacci numbers*, Journal of Integer Sequences, **19** (2016), Article 16.2.1.

**Also solved by Dmitry Fleischman, Hideyuki Ohtsuka, and the proposer.**

**The limit of an expression involving double factorials and Fibonacci numbers**

**H-793** Proposed by D. M. Bătinețu-Giurgiu, Bucharest, and Bogdan Andrei Stanciu, Brașov, Romania (Vol. 54, No. 3, August 2016)

Let  $e_n = (1 + 1/n)^n$ . Compute

$$\lim_{n \rightarrow \infty} \left( e_{n+1} \sqrt[n+1]{(2n+1)!!F_{n+1}} - e_n \sqrt[n]{(2n-1)!!F_n} \right).$$

Compute the similar limit with all the  $F$ 's replaced by  $L$ 's.

**Solution by Brian Bradie**

For large  $n$ , we have the following asymptotic equalities

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &\sim e \left(1 - \frac{1}{2n}\right), \\ (2n-1)!! &\sim 2^{n+1/2} n^n e^{-n}, \\ \sqrt[n]{(2n-1)!!} &\sim \frac{2n}{e}, \\ \sqrt[n]{F_n} &\sim \alpha. \end{aligned}$$

It follows that

$$e_n \sqrt[n]{(2n-1)!!F_n} \sim (2n-1)\alpha,$$

so

$$\lim_{n \rightarrow \infty} (e_{n+1} \sqrt[n+1]{(2n+1)!!F_{n+1}} - e_n \sqrt[n]{(2n-1)!!F_n}) = 2\alpha.$$

The same holds for  $F$  replaced by  $L$ .

Also solved by Kenneth Davenport, Dmitry Fleischman, Ángel Plaza, David Terr, and the proposers.

**An upper bound for a sum of cubic roots**

**H-794** Proposed by D. M. Bătinețu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania (Vol. 54, No. 3, August 2016)

Prove that

$$\sqrt[3]{\frac{F_n}{5F_{n+2}}} + \sqrt[3]{\frac{F_{n+1}}{5F_{n+2} + 3F_{n+1}}} + \sqrt[3]{\frac{F_{n+2}}{5F_{n+2} + 3F_n}} < \sqrt[3]{4} \quad \text{for all } n \geq 0.$$

**Solution by Ángel Plaza**

Since  $F_{n+2} = F_{n+1} + F_n$ , it follows that  $\frac{F_n}{5F_{n+2}} \leq \frac{1}{10}$ ,  $\frac{F_{n+1}}{5F_{n+2} + 3F_{n+1}} \leq \frac{1}{8}$ , and  $\frac{F_{n+2}}{5F_{n+2} + 3F_n} \leq \frac{1}{5}$ . Therefore, the left side of the proposed inequality,  $LS$ , is

$$LS < \sqrt[3]{\frac{1}{10}} + \sqrt[3]{\frac{1}{8}} + \sqrt[3]{\frac{1}{5}} = 1.5489 < \sqrt[3]{4} = 1.5874.$$

Also solved by Brian Bradie, Miguel Cidra, Kenneth Davenport, Dmitry Fleischman, Wei-Kai Lai, Hideyuki Ohtsuka, and the proposers.

Sums of arc-tangents

**H-795** Proposed by Hideyuki Ohtsuka, Saitama, Japan  
(Vol. 54, No. 3, August 2016)

Prove that

$$\sum_{k=1}^{2n} \tan^{-1} \left( \frac{2}{L_{2k-1}} \right) = 2 \sum_{k=1}^n \tan^{-1} \left( \frac{1}{F_{4k-2}} \right).$$

**Solution by Ángel Plaza**

By induction. For  $n = 1$ , we have

$$\tan^{-1} \left( \frac{2}{L_1} \right) + \tan^{-1} \left( \frac{2}{L_3} \right) = \tan^{-1} 2 + \tan^{-1} \frac{1}{2} = \frac{\pi}{2} = 2 \tan^{-1} 1 = 2 \tan^{-1} \left( \frac{1}{F_2} \right).$$

Let us now assume that the identity holds for  $n$  by the induction hypothesis. We have to prove that it also holds for  $n + 1$ . Equivalently, we will prove

$$\tan^{-1} \left( \frac{2}{L_{4n+1}} \right) + \tan^{-1} \left( \frac{2}{L_{4n+3}} \right) = 2 \tan^{-1} \left( \frac{1}{F_{4n+2}} \right).$$

Taking tan of the left side, we obtain

$$\begin{aligned} \tan LS &= \frac{\frac{2}{L_{4n+1}} + \frac{2}{L_{4n+3}}}{1 - \frac{2}{L_{4n+1}} \frac{2}{L_{4n+3}}} = \frac{2(L_{4n+1} + L_{4n+3})}{L_{4n+1}L_{4n+3} - 4} \\ &= \frac{10F_{4n+2}}{L_{8n+4} - 1}, \end{aligned}$$

since  $L_{4n+1}L_{4n+3} = L_{8n+4} + 3$ .

On the other hand, taking tan of the right side, we have

$$\tan RS = \frac{\frac{2}{F_{4n+2}}}{1 - \left( \frac{1}{F_{4n+2}} \right)^2} = \frac{2F_{4n+2}}{F_{4n+2}^2 - 1}.$$

The conclusion follows since

$$5F_{4n+2}^2 - 5 = L_{8n+4} - 1.$$

Also solved by Miguel Cidra, Mithun Kumar Das, Kenneth Davenport, Dmitry Fleischman, David Terr, and the proposer.