ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-825 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

If $a, b, c > 0$ and n is a positive integer, prove that

$$2\left(\left(\frac{a}{F_n b + F_{n+1}c}\right)^3 + \left(\frac{b}{F_n c + F_{n+1}a}\right)^3 + \left(\frac{c}{F_n a + F_{n+1}b}\right)^3\right)$$
$$+ \quad 3\frac{abc}{(F_n a + F_{n+1}b)(F_n b + F_{n+1}c)(F_n c + F_{n+1}a)} \geq \frac{9}{F_{n+2}^3}.$$

H-826 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For an integer $n \geq 0$, prove that

$$\sum_{\substack{a+b=n \\ a,b\geq 0}} \frac{1}{L_{2^a 3^b} F_{2^a 3^{b+1}}} = \frac{F_{3^{n+1}-2^{n+1}}}{F_{3^{n+1}} F_{2^{n+1}}}.$$

H-827 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_{n+1}/(na_n) = a > 0$. Compute

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left(\sqrt[n+1]{a_{n+1}}^{F_m/F_{m+1}} - \left(\sqrt[n]{a_n}\right)^{F_m/F_{m+1}}\right) n^{F_{m-1}/F_m}\right)\right).$$

<u>H-828</u> Proposed by Kenneth Davenport, Dallas, PA

Find a closed form expression for

$$\sum_{k=0}^{n} kT_k^2,$$

where $(T_k)_{k\geq 0}$ is the sequence of Tribonacci numbers satisfying $T_0 = 0$, $T_1 = T_2 = 1$, and $T_{k+3} = T_{k+2} + T_{k+1} + T_k$ for all $k \geq 0$.

SOLUTIONS

<u>Closed form for the sum of a series</u>

<u>H-791</u> Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 54, No. 2, May 2016)

For an integer $n \geq 0$, find a closed form expression for the sum

$$\sum_{k=0}^{n} \frac{(-1)^{2^k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}}\cdots L_{3^n})^2}.$$

Solution by the proposer

We find the identity

$$\sum_{k=0}^{n} \frac{(-1)^{2^k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}}\cdots L_{3^n})^2} = -\frac{1}{F_{3^{n+1}}}. \tag{1}$$

The proof of (1) is by mathematical induction on n. For $n = 0$, we have $LS = RS = -1/2$. We use the identities:

(1) $L_aF_a = F_{2a}$ (see [1] (13));
(2) $F_{a+b} + (-1)^bF_{a-b} = F_aL_b$ (see [1] (15a)).

We assume (1) holds for n. For $n+1$, we have

$$\begin{aligned}
&\sum_{k=0}^{n+1} \frac{(-1)^{2^k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}}\cdots L_{3^{n+1}})^2} \\
&= \frac{1}{F_{3^{n+2}}L_{3^{n+1}}^2} + \frac{1}{L_{3^{n+1}}^2}\sum_{k=0}^{n} \frac{(-1)^{2^k}}{F_{3^{k+1}}(L_{3^k}L_{3^{k+1}}\cdots L_{3^n})^2} \\
&= \frac{1}{F_{3^{n+2}}L_{3^{n+1}}^2} - \frac{1}{L_{3^{n+1}}^2F_{3^{n+1}}} = -\frac{F_{3^{n+2}} - F_{3^{n+1}}}{F_{3^{n+2}}F_{3^{n+1}}L_{3^{n+1}}^2} \\
&= -\frac{F_{2\cdot 3^{n+1}}L_{3^{n+1}}}{F_{3^{n+2}}F_{2\cdot 3^{n+1}}L_{3^{n+1}}} \qquad \text{(by (1) and (2))} \\
&= -\frac{1}{F_{3^{n+2}}}.
\end{aligned}$$

Therefore, (1) holds for all $n \geq 0$.

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Partially solved by Dmitry Fleischman.

Some Tribonacci identities

H-792 Proposed by George A. Hisert, Berkeley, CA (Vol. 54, No. 3, August 2016)

Consider the 3-sequence $T_{i+1} = T_i + T_{i-1} + T_{i-2}$ for all integers i with $T_0 = 0$, $T_1 = T_2 = 1$. Let $S_i = T_i + T_{i-1}$. Prove that for all integers n positive or negative, we have $T_n^2 - T_{n+1}T_{n-1} = T_{-(n+1)}$ and $T_{n+1}T_{n-2} - T_nT_{n-1} = S_{-(n+1)}$.

Solution by Brian Bradie

Anantakitpaisal and Kuhapatanakul [1] provide the following proof for the identity $T_n^2 - T_{n-1}T_{n+1} = T_{-(n+1)}$. It is known that

$$\begin{bmatrix} T_{n+k} \\ T_{n+k-1} \\ T_{n+k-2} \end{bmatrix} = A^n \begin{bmatrix} T_k \\ T_{k-1} \\ T_{k-2} \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It follows that

$$\begin{aligned} T_n^2 - T_{n-1}T_{n+1} &= \begin{vmatrix} T_{n+1} & T_{n+2} & 1 \\ T_n & T_{n+1} & 0 \\ T_{n-1} & T_n & 0 \end{vmatrix} \\ &= \left| A^n \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \quad A^n \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix} \quad A^n A^{-n} \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \right| \\ &= |A^n| \begin{vmatrix} 1 & 1 & T_{-n+1} \\ 0 & 1 & T_{-n} \\ 0 & 0 & T_{-n-1} \end{vmatrix} \\ &= 1 \cdot T_{-(n+1)} = T_{-(n+1)}. \end{aligned}$$

Proceeding as above, we have

$$\begin{aligned} T_{n+1}T_{n-2} - T_nT_{n-1} &= \begin{vmatrix} T_{n+2} & T_n & 1 \\ T_{n+1} & T_{n-1} & 0 \\ T_n & T_{n-2} & 0 \end{vmatrix} \\ &= \left| A^n \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix} \quad A^n \begin{bmatrix} T_0 \\ T_{-1} \\ T_{-2} \end{bmatrix} \quad A^n A^{-n} \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \right| \\ &= |A^n| \begin{vmatrix} 1 & 0 & T_{-n+1} \\ 1 & 0 & T_{-n} \\ 0 & 1 & T_{-n-1} \end{vmatrix} \\ &= T_{-n+1} - T_{-n} = T_{-n-1} + T_{-n-2} = S_{-n-1} = S_{-(n+1)}. \end{aligned}$$

[1] P. Anantakitpaisal and K. Kuhapatanakul, *Reciprocal sums of the Tribonacci numbers*, Journal of Integer Sequences, **19** (2016), Article 16.2.1.

Also solved by Dmitry Fleischman, Hideyuki Ohtsuka, and the proposer.

The limit of an expression involving double factorials and Fibonacci numbers

H-793 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Bogdan Andrei Stanciu, Braşov, Romania (Vol. 54, No. 3, August 2016)

Let $e_n = (1+1/n)^n$. Compute

$$\lim_{n\to\infty}\left(e_{n+1}\sqrt[n+1]{(2n+1)!!F_{n+1}} - e_n\sqrt[n]{(2n-1)!!F_n}\right).$$

Compute the similar limit with all the F's replaced by L's.

Solution by Brian Bradie

For large n, we have the following asymptotic equalities

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &\sim e\left(1-\frac{1}{2n}\right),\\ (2n-1)!! &\sim 2^{n+1/2}n^n e^{-n},\\ \sqrt[n]{(2n-1)!!} &\sim \frac{2n}{e},\\ \sqrt[n]{F_n} &\sim \alpha.\end{aligned}$$

It follows that

$$e_n\sqrt[n]{(2n-1)!!F_n} \sim (2n-1)\alpha,$$

so

$$\lim_{n\to\infty}(e_{n+1}\sqrt[n+1]{(2n+1)!!F_{n+1}} - e_n\sqrt[n]{(2n-1)!!F_n}) = 2\alpha.$$

The same holds for F replaced by L.

Also solved by Kenneth Davenport, Dmitry Fleischman, Ángel Plaza, David Terr, and the proposers.

An upper bound for a sum of cubic roots

H-794 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania (Vol. 54, No. 3, August 2016)

Prove that

$$\sqrt[3]{\frac{F_n}{5F_{n+2}}} + \sqrt[3]{\frac{F_{n+1}}{5F_{n+2}+3F_{n+1}}} + \sqrt[3]{\frac{F_{n+2}}{5F_{n+2}+3F_n}} < \sqrt[3]{4} \quad \text{for all} \quad n \geq 0.$$

Solution by Ángel Plaza

Since $F_{n+2} = F_{n+1} + F_n$, it follows that $\frac{F_n}{5F_{n+2}} \leq \frac{1}{10}$, $\frac{F_{n+1}}{5F_{n+2}+3F_{n+1}} \leq \frac{1}{8}$, and $\frac{F_{n+2}}{5F_{n+2}+3F_n} \leq \frac{1}{5}$. Therefore, the left side of the proposed inequality, LS, is

$$LS < \sqrt[3]{\frac{1}{10}} + \sqrt[3]{\frac{1}{8}} + \sqrt[3]{\frac{1}{5}} = 1.5489 < \sqrt[3]{4} = 1.5874.$$

Also solved by Brian Bradie, Miguel Cidra, Kenneth Davenport, Dmitry Fleischman, Wei-Kai Lai, Hideyuki Ohtsuka, and the proposers.

Sums of arc-tangents

H-795 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 54, No. 3, August 2016)

Prove that

$$\sum_{k=1}^{2n} \tan^{-1}\left(\frac{2}{L_{2k-1}}\right) = 2\sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{F_{4k-2}}\right).$$

Solution by Ángel Plaza

By induction. For $n = 1$, we have

$$\tan^{-1}\left(\frac{2}{L_1}\right) + \tan^{-1}\left(\frac{2}{L_3}\right) = \tan^{-1} 2 + \tan^{-1}\frac{1}{2} = \frac{\pi}{2} = 2\tan^{-1} 1 = 2\tan^{-1}\left(\frac{1}{F_2}\right).$$

Let us now assume that the identity holds for n by the induction hypothesis. We have to prove that it also holds for $n+1$. Equivalently, we will prove

$$\tan^{-1}\left(\frac{2}{L_{4n+1}}\right) + \tan^{-1}\left(\frac{2}{L_{4n+3}}\right) = 2\tan^{-1}\left(\frac{1}{F_{4n+2}}\right).$$

Taking tan of the left side, we obtain

$$\begin{aligned} \tan LS &= \frac{\frac{2}{L_{4n+1}} + \frac{2}{L_{4n+3}}}{1 - \frac{2}{L_{4n+1}}\frac{2}{L_{4n+3}}} = \frac{2\left(L_{4n+1} + L_{4n+3}\right)}{L_{4n+1}L_{4n+3} - 4} \\ &= \frac{10F_{4n+2}}{L_{8n+4} - 1}, \end{aligned}$$

since $L_{4n+1}L_{4n+3} = L_{8n+4} + 3$.

On the other hand, taking tan of the right side, we have

$$\tan RS = \frac{\frac{2}{F_{4n+2}}}{1 - \left(\frac{1}{F_{4n+2}}\right)^2} = \frac{2F_{4n+2}}{F_{4n+2}^2 - 1}.$$

The conclusion follows since

$$5F_{4n+2}^2 - 5 = L_{8n+4} - 1.$$

Also solved by Miguel Cidra, Mithun Kumar Das, Kenneth Davenport, Dmitry Fleischman, David Terr, and the proposer.