ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
HARRIS KWONG

Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2019. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1236 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that, for any integers $m \ge 0$ and $n > 1$,

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}^{m+1}}{F_k^{2m}} > \frac{2^{n(m+1)}}{F_{n+1}^m F_{n+2}^m}, \qquad \text{and} \qquad \sum_{k=1}^{n+1} \frac{F_k^{m+1}}{\binom{n}{k-1}^m} > \frac{(F_{n+3}-1)^{m+1}}{2^{mn}}.$$

B-1237 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Evaluate

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{\alpha^k + \alpha}\right), \qquad \text{and} \qquad \prod_{k=1}^{\infty} \left(1 - \frac{1}{\alpha^k + \alpha}\right).$$

B-1238 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Let $a > 1$ and consider the sequence of real numbers defined recursively by $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = \left(a + \frac{1}{a}\right) x_n - x_{n-1}, \qquad n \geq 1.$$

Prove that $\sum_{n=0}^{\infty} \frac{1}{x_{2^n}}$ is a rational number if and only if a is a rational number.

B-1239 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all integers n, prove that

$$\left(\frac{1}{L_n} - \frac{1}{L_{n+1}}\right)^4 + \left(\frac{1}{L_{n+1}} + \frac{1}{L_{n+2}}\right)^4 + \left(\frac{1}{L_{n+2}} + \frac{1}{L_n}\right)^4 = 2\left(\frac{1}{L_n} + \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}}\right)^4.$$

B-1240 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Assume $x_k > 0$ for $k = 1, 2, \ldots, n$. Prove that, for any positive integers $m \geq 1$ and $n > 1$,

$$\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{\substack{i=1\\ \text{cyclic}}}^{n} \frac{x_i x_{i+1}}{F_m x_i + F_{m+1} x_{i+1}}\right) \geq \frac{n^2}{F_{m+2}},$$

$$\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{\substack{i=1\\ \text{cyclic}}}^{n} \frac{x_i x_{i+1}}{L_m x_i + L_{m+1} x_{i+1}}\right) \geq \frac{n^2}{L_{m+2}}.$$

SOLUTIONS

Editor's Notes. In the solution to Elementary Problem B-1208 that appeared in the May issue, the first round of row reductions should be carried out according to $k = n+1, n, \ldots, 3$. The two rounds of row reductions can be combined into one. For $k = n+1, n, \ldots, 3$, subtract the sum of row $k-1$ and row $k-2$ from row k. Next, subtracting the first row from the second yields the last augmented matrix shown in the solution.

Another Application of the AM-GM Inequality

B-1216 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 55.4, November 2017)

Prove that, for any positive real number m, and any positive integer n,

$$F_n^m F_{n+1}^m \sum_{k=1}^n \frac{L_k^{m+1}}{F_k^{2m}} \geq n^{m+1} \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The proposed inequality follows from the AM-GM inequality and the identity $F_n F_{n+1} = \sum_{k=1}^n F_k^2$:

$$\begin{aligned}
F_n^m F_{n+1}^m \sum_{k=1}^m \frac{L_k^{m+1}}{F_k^{2m}} &\geq F_n^m F_{n+1}^n \cdot n \sqrt[n]{\prod_{k=1}^n \frac{L_k^{m+1}}{F_k^{2m}}} \\
&= \left(\frac{F_n F_{n+1}}{\sqrt[n]{\prod_{k=1}^n F_k^2}} \right)^m \cdot n \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}} \\
&= \left(\frac{\sum_{k=1}^n F_k^2}{\sqrt[n]{\prod_{k=1}^n F_k^2}} \right)^m \cdot n \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}} \\
&\geq \left(\frac{n \sqrt[n]{\prod_{k=1}^n F_k^2}}{\sqrt[n]{\prod_{k=1}^n F_k^2}} \right)^m \cdot n \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}} \\
&= n^{m+1} \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}}.
\end{aligned}$$

Solution 2 by Wei-Kai Lai and John Risher (student) (jointly), University of South Carolina Salkehatchie, Walterboro, SC.

According to Radon's Inequality, we know that

$$\sum_{k=1}^n \frac{L_k^{m+1}}{F_k^{2m}} \geq \frac{\left(\sum_{k=1}^n L_k\right)^{m+1}}{\left(\sum_{k=1}^n F_k^2\right)^m}.$$

To prove the claimed inequality, we therefore only need to prove that

$$F_n^m F_{n+1}^m \frac{\left(\sum_{k=1}^n L_k\right)^{m+1}}{\left(\sum_{k=1}^n F_k^2\right)^m} \geq n^{m+1} \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}}.$$

Since $\sum_{k=1}^n F_k^2 = F_n F_{n+1}$, the above inequality is equivalent to

$$\left(\sum_{k=1}^n L_k \right)^{m+1} \geq n^{m+1} \left(\prod_{k=1}^n L_k \right)^{\frac{m+1}{n}},$$

which is apparently true due to the AM-GM inequality.

Also solved by Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Hideyuki Ohtsuka, and the proposers.

Help From Exponential Generating Function

B-1217 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 55.4, November 2017)

Let $M_{k_i} = 2^{(i-1)k_i} L_{k_i}$. For integers $r \geq 1$ and $n \geq 0$, find a closed form expression for the sum

$$S_n = \sum_{\substack{0 \leq k, k_1, \ldots, k_r \leq n \\ k + k_1 + \cdots + k_r = n}} \frac{F_k M_{k_1} M_{k_2} \cdots M_{k_r}}{k!\, k_1!\, k_2! \, \cdots \, k_r!}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

The exponential generating function for the Fibonacci numbers is

$$G_F(x) = \sum_{k=0}^{\infty} \frac{F_k}{k!} x^k = \frac{1}{\sqrt{5}} \left(e^{\alpha x} - e^{\beta x}\right),$$

whereas the exponential generating function for the Lucas numbers is

$$G_L(x) = \sum_{k=0}^{\infty} \frac{L_k}{k!} x^k = e^{\alpha x} + e^{\beta x}.$$

It follows that the exponential generating function for M_{k_i} is

$$G_i(x) = \sum_{k_i=0}^{\infty} \frac{M_{k_i}}{k_i!} x^{k_i} = \sum_{k_i=0}^{\infty} \frac{L_{k_i}}{k_i!} \left(2^{i-1}x\right)^{k_i} = G_L(2^{i-1}x) = e^{2^{i-1}\alpha x} + e^{2^{i-1}\beta x}.$$

Due to convolution, we can now recognize S_n as the coefficient of x^n in the product

$$\begin{aligned}
& G_F(x)G_1(x)G_2(x)\cdots G_r(x) \\
= \; & \tfrac{1}{\sqrt{5}} \left(e^{\alpha x} - e^{\beta x}\right)\left(e^{\alpha x} + e^{\beta x}\right)\left(e^{2\alpha x} + e^{2\beta x}\right) \cdots \left(e^{2^{r-1}\alpha x} + e^{2^{r-1}\beta x}\right) \\
= \; & \tfrac{1}{\sqrt{5}} \left(e^{2\alpha x} - e^{2\beta x}\right)\left(e^{2\alpha x} + e^{2\beta x}\right) \cdots \left(e^{2^{r-1}\alpha x} + e^{2^{r-1}\beta x}\right) \\
= \; & \tfrac{1}{\sqrt{5}} \left(e^{4\alpha x} - e^{4\beta x}\right) \cdots \left(e^{2^{r-1}\alpha x} + e^{2^{r-1}\beta x}\right) \\
\vdots \; & \quad \vdots \\
= \; & \tfrac{1}{\sqrt{5}} \left(e^{2^r \alpha x} - e^{2^r \beta x}\right).
\end{aligned}$$

Therefore,

$$S_n = \frac{1}{\sqrt{5}} \left[\frac{(2^r\alpha)^n}{n!} - \frac{(2^r\beta)^n}{n!}\right] = \frac{2^{rn} F_n}{n!}.$$

Also solved by Raphael Schumacher (student), and the proposer.

Simplifying a Complicated Expression

B-1218 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 55.4, November 2017)

Find a closed form expression for

$$(L_{n+1}-1)F_n(F_{2n+2}-F_{n+2})+(1-F_n-F_{n+2})F_{n+2}(F_{2n+2}-F_{n+3})+(F_{2n+2}-F_{n+2})(F_{2n+2}-F_{n+3}).$$

Solution 1 by Charles K. Cook, Sumter, SC.

The well-known identities $F_{2n} = F_n L_n$ and $L_n = F_{n-1} + F_{n+1}$ will be used as needed. Let A represent the first term, B, the second, and C, the third, of the given sum. Expanding and using the above identities yields

$$\begin{aligned} A &= F_n F_{n+1} L_{n+1}^2 - F_n(F_{n+1} + F_{n+2}) L_{n+1} + F_n F_{n+2}, \\ B &= -F_{n+1} F_{n+2} L_{n+1}^2 + F_{n+2}(F_{n+1} + F_{n+3}) L_{n+1} - F_{n+2} F_{n+3}, \\ C &= F_{n+1}^2 L_{n+1}^2 - F_{n+1}(F_{n+2} + F_{n+3}) L_{n+1} + F_{n+2} F_{n+3}. \end{aligned}$$

The coefficient for L_{n+1}^2 in the sum is

$$F_n F_{n+1} - F_{n+1} F_{n+2} + F_{n+1}^2 = F_{n+1}(F_n - F_{n+2} + F_{n+1}) = 0,$$

whereas the coefficient for L_{n+1} is

$$\begin{aligned} &-F_n(F_{n+1} + F_{n+2}) + F_{n+2}(F_{n+1} + F_{n+3}) - F_{n+1}(F_{n+2} + F_{n+3}) \\ &= -F_n(F_{n+1} + F_{n+2}) + (F_{n+2} - F_{n+1}) F_{n+3} \\ &= 0, \end{aligned}$$

and the remaining terms are

$$F_n F_{n+2} - F_{n+2} F_{n+3} + F_{n+2} F_{n+3} = F_n F_{n+2}.$$

Thus, adding A, B, and C, the required closed form for the given sum is $F_n F_{n+2}$.

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

We use the well-known identities $F_{2m} = F_m L_m$, and $F_{m-1} + F_{m+1} = L_m$. Let $t = L_{n+1} - 1$. Then, we have

$$\begin{gathered} F_{2n+2} - F_{n+2} = F_{n+1} L_{n+1} - F_{n+1} - F_n = t F_{n+1} - F_n; \\ F_{2n+2} - F_{n+3} = F_{n+1} L_{n+1} - F_{n+1} - F_{n+2} = t F_{n+1} - F_{n+2}; \\ 1 - F_n - F_{n+2} = 1 - L_{n+1} = -t. \end{gathered}$$

By the above identities, the expression of the problem is

$$\begin{aligned} &t F_n(t F_{n+1} - F_n) - t F_{n+2}(t F_{n+1} - F_{n+2}) + (t F_{n+1} - F_n)(t F_{n+1} - F_{n+2}) \\ &= t^2 F_{n+1}(F_n - F_{n+2} + F_{n+1}) + t\left[F_{n+2}(F_{n+2} - F_{n+1}) - F_n(F_n + F_{n+1})\right] + F_n F_{n+2} \\ &= t(F_{n+2} F_n - F_n F_{n+2}) + F_n F_{n+2} \\ &= F_n F_{n+2}. \end{aligned}$$

Solution 3 by the proposer.

We use the identity $F_{2n+2} = F_{n+1} L_{n+1} = F_{n+1}(F_n + F_{n+2})$ to write the given expression as

$$\begin{aligned} F_n F_{n+2} \Bigg[&\frac{(F_{2n+2} - F_{n+1})(F_{2n+2} - F_{n+2})}{F_{n+1} F_{n+2}} \\ &- \frac{(F_{2n+2} - F_{n+1})(F_{2n+2} - F_{n+3})}{F_n F_{n+1}} + \frac{(F_{2n+2} - F_{n+2})(F_{2n+2} - F_{n+3})}{F_n F_{n+2}} \Bigg]. \end{aligned}$$

Let

$$P(x) = \frac{(x - F_{n+1})(x - F_{n+2})}{F_{n+1} F_{n+2}} - \frac{(x - F_{n+1})(x - F_{n+3})}{F_n F_{n+1}} + \frac{(x - F_{n+2})(x - F_{n+3})}{F_n F_{n+2}}.$$

We have

$$P(F_{n+1}) = P(F_{n+2}) = P(F_{n+3}) = 1.$$

Therefore, $P(x) \equiv 1$. Thus, a closed form for the expression is

$$F_n F_{n+2} \cdot P(F_{2n+2}) = F_n F_{n+2}.$$

Also solved by Brian D. Beasley, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, G. C. Greubel, Kantaphon Kuhapatanakul, Wei-Kai Lai, Ehren Metcalfe, Verónica Molina Reales (student), Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, Elizabeth S. Spoehel (student), and the proposers.

An Inequality with a Cyclic Sum

B-1219 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 55.4, November 2017)

Prove that, for any integer $n \geq 2$,

$$\frac{F_n^4 + F_n^2 + 1}{F_n} + \sum_{k=1}^{n-1} \frac{F_k^4 + F_k^2 F_{k+1}^2 + F_{k+1}^4}{F_k F_{k+1}} > 3F_n F_{n+1}.$$

Editor's Note: The condition on n should be $n \geq 3$.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since $F_1 = 1$, and

$$F_n F_{n+1} = \sum_{k=1}^{n} F_k^2,$$

the proposed inequality may be written as

$$\sum_{\substack{k=1 \\ \text{cyclic}}}^{n} \frac{F_k^4 + F_k^2 F_{k+1}^2 + F_{k+1}^4}{F_k F_{k+1}} > 3 \sum_{k=1}^{n} F_k^2,$$

which is a special case of the following more general inequality.

Lemma. *Let $a_1, \ldots, a_m$ be a sequence of positive real numbers. Then,*

$$\sum_{\substack{k=1 \\ cyclic}}^{m} \frac{a_k^4 + a_k^2 a_{k+1}^2 + a_{k+1}^4}{a_k a_{k+1}} \geq 3 \sum_{k=1}^{m} a_k^2.$$

Proof. It is enough to prove that, if $a, b > 0$, then

$$\frac{a^4 + a^2 b^2 + b^4}{ab} \geq \frac{3}{2}(a^2 + b^2),$$

which is equialent to

$$2(a^4 + a^2 b^2 + b^4) \geq 3ab(a^2 + b^2).$$

To complete the proof, observe that

$$a^4 + b^4 \geq a^3 b + ab^3 = ab(a^2 + b^2),$$
$$a^4 + 2a^2 b^2 + b^4 = (a^2 + b^2)(a^2 + b^2) \geq 2ab(a^2 + b^2).$$

To obtain a strict inequality, we need $m \geq 2$, and some of the terms in the sequence $a_1, \ldots, a_m$ have to be different. $\square$

Notice that the inequality in the problem becomes an identity when $n = 2$.

Also solved by Brian D. Beasley, Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai and John Risher (student) (jointly), Hideyuki Ohtsuka, and the proposers.

Gelin-Cesàro Identity Yields a Telescoping Product

B-1220 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 55.4, November 2017)

Prove that

$$\prod_{n=3}^{\infty} \left(1 - \frac{1}{F_n^4}\right) = \frac{\alpha^5}{12}.$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

Using the Gelin-Cesàro Identity $F_n^4 - 1 = F_{n-2}F_{n-1}F_{n+1}F_{n+2}$, we have

$$1 - \frac{1}{F_n^4} = \frac{F_n^4 - 1}{F_n^4} = \frac{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}{F_n^4}.$$

It follows from the telescoping property that, for $m \geq 4$,

$$\prod_{n=3}^{m} \left(1 - \frac{1}{F_n^4}\right) = \prod_{n=3}^{m} \frac{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}{F_n^4} = \frac{F_1 F_2^2}{F_3^2 F_4} \cdot \frac{F_{m+1}^2 F_{m+2}}{F_{m-1}F_m^2} = \frac{F_{m+1}^2 F_{m+2}}{12 F_{m-1} F_m^2}.$$

Since $\lim_{m\to\infty} F_{m+j}/F_m = \alpha^j$, we find

$$\prod_{n=3}^{\infty} \left(1 - \frac{1}{F_n^4}\right) = \lim_{m\to\infty} \frac{F_{m+1}^2 F_{m+2}}{12 F_{m-1} F_m^2} = \lim_{m\to\infty} \frac{1}{12} \left(\frac{F_{m+1}}{F_m}\right)^2 \frac{F_{m+2}}{F_{m-1}} = \frac{\alpha^2 \cdot \alpha^3}{12} = \frac{\alpha^5}{12}.$$

Also solved by Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Kantaphon Kuhapatanakul, Ángel Plaza, Raphael Schumacher (student), and the proposer.