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Flounders unlimited

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The aim of this paper is to provide an introduction to higher mathematical concepts and techniques as applied to real-world problems. An interesting case study is the 200-mile limit for fisheries presented in Wan (1989, *Mathematical Models and their Analysis* (New York: Harper and Row)). It is commented upon, re-examined and solved in a different manner inspired by Turing's (1952, *Philosophical Transactions of the Royal Society London*, **B237**, 37–72) paper as a suitable classroom example for undergraduate students taking a first course on mathematical modelling. Emphasis is made not only on the mathematics—always kept as elementary as possible—but also on interpretations, explanations and translation into plain language of the mathematical expressions and results.

1. Introduction

Historically, the width of territorial waters of maritime countries has grown from a modest 3-mile strip up to today's 200-mile limit, defining the so-called exclusive economic zone. The evolution of this width is a measure of how far the coastal surveillance can be pushed into the open sea, a fact that seems directly related to, and depending on, the reliability of war and diplomatic technologies. The 1996 conflict between the European Union and Canada on the Atlantic Halibut (a large flounder-like fish whose scientific name is *Hippoglossus hippoglossus*) fishery in the North Atlantic Ocean is an interesting example that can be exploited in the classroom as an exercise for undergraduate students taking a course on mathematical modelling.

In chapter 11 of the book by Wan (1989: 241–266) there is an interesting and readable account on how to relate the width of the exclusive zone with some biological and economic parameters. The chapter—and the rest of the book as well—is worth reading, though the mathematics is rather involved and heuristic ideas are somehow obscure. It is felt that there are easier ways to achieve the same result, so the aim of this paper is to develop a simpler derivation and to offer sensible interpretations of the various parameters involved. Comments on Wan's solution are also made.

2. A model

To begin with, let C be a maritime country with a rectilinear shoreline, and suppose that the mean marine biomass observed (i.e. fish) in the water column below

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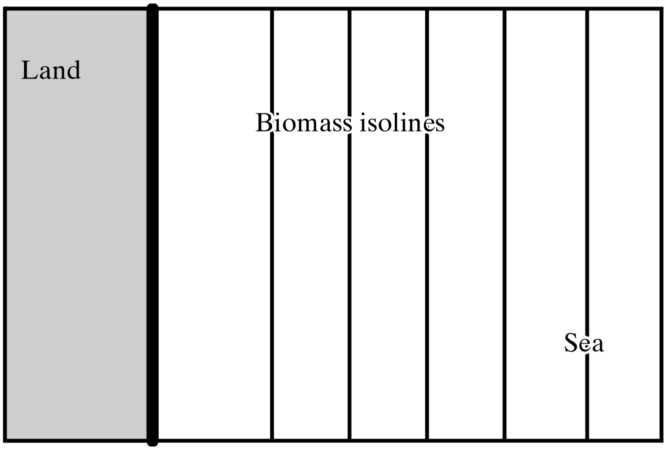

Figure 1. Geometry of the model.

any point in the sea surface in front of C depends only on the perpendicular distance between the point and the coastline; therefore, the biomass isolines are straight lines parallel to the coast. Spatially speaking, the problem is reduced to a one-dimensional one on an interval; see figure 1. Now, suppose that the fishery industry of C, usually working in coastal waters, finds itself in trouble due to the scarcity of fish and to competition with other countries' fishing fleets. In order to cope with the difficulties, the think tank in C decides to increase the width of the exclusive strip up to some width L. Is there any physically sensible idea behind this decision—ignoring war and/or diplomatic efforts?

A reasonably realistic and simple equation (see, e.g. Cushing 1983) accounting for biomass growth, fishing effort and spatial distribution can be written as a mildly non-linear one:

$$\frac{\partial u}{\partial t} = ru\left(1 - \frac{u}{K}\right) - Eu + D\frac{\partial^2 u}{\partial x^2}.$$

This is a *reaction–diffusion* equation, where $u = u(x, t)$ is the biomass found at x ε $(0, L)$ at time $t > 0$, and the right-hand side member shows a logistic growth $ru(1 - (u/K))$ with linear growth rate r and carrying capacity K, a linear fishing effort term $-Eu$ and a diffusion term $D(\partial^2 u/\partial x^2)$. The logistic and fishing effort terms together constitute the reaction part of the equation. Note that E is usually interpreted as a measure of the boat tonnage in the fishery business, and that D carries information about fish mobility along the interval $(0, L)$. It is worth noting that Wan dwells on the actual estimation of D and that this is not an easy task; see the book by Crank (1990).

The model is completed by adding boundary conditions at both ends of the interval. At $x = 0$, a zero-flux condition $\partial u/\partial x = 0$ is imposed: this simply means that—as a rule—fish do not swim on land. For $x = L$ a rather strong condition is imposed: $u(L, t) = 0$, meaning that any fish trying to leave $(0, L)$ through L is immediately captured by the foreign fleets waiting at the borderline. Moreover, an initial fish distribution $u(x, 0) = \varphi(x)$ must be supplied. Nevertheless, the initial condition does not play a role in this analysis.

In this presentation of the problem, emphasis will be made on the diffusion coefficient D as the most important governing parameter. The roots of the idea are found in the classical paper of Turing (1952) on morphogenesis of half a century ago.

2.1. *Dimensionless is not a mess*

Any sensible modelling stresses the importance of dimensionless equations and parameters; see, for example, the comments in the book by Fowler (1997). Dimensionless quantities allow for interesting relationships and interpretations; usually the number of parameters is reduced substantially and the ranges of most variables are kept to manageable limits.

The model equation and the spatial interval $(0, L)$ immediately offer a good choice of units in order to achieve a nice dimensionles model: there is a typical though unknown length scale L, so the spatial coordinate can be written as $x = x^*L$ and the spatial interval reduces to $(0, 1)$. Furthermore, both r and E have the dimension T^{-1}, so time can be scaled by either $1/r$ or $1/E$. Here the choice $t = t^*/r$ is made. The natural scaling for the logistic equation is $u = u^*K$, thereby expressing biomass as a fraction of the carrying capacity K. Plugging all these expressions into the model equation yields:

$$\frac{\partial u^*}{\partial t^*} = u^*(1 - u^*) - \frac{E}{r}u^* + \frac{D}{rL^2}\frac{\partial^2 u^*}{\partial x^{*2}},$$

where only two non-dimensional parameters are left: $\alpha = E/r$, the ratio of the fishing and growth rates, and $\delta = D/rL^2$, which compares the mobility coefficient D with the spatial extent L and the relaxation time $1/r$. Therefore, after dropping the asterisks, the dimensionless model reads:

$$\begin{aligned} \frac{\partial u}{\partial t} &= u(1-u) - \alpha u + \delta\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x} &= 0 \text{ if } x = 0 \\ u &= 0 \text{ if } x = 1 \end{aligned}$$

$$u(x, 0) = \bar{\varphi}(x) \text{ in } (0, 1).$$

3. Linear analysis

The standard way of presenting qualitative analyses of differential equations in elementary courses is linearization around the steady states. In this way, the problem is reduced to linear algebraic considerations.

3.1. *No diffusion*

The simplest population models deal with spatially homogeneous biomass distributions, and the analysis of spatially inhomogeneous ones also starts from this simplifying idea (Turing 1952). Therefore, first of all the steady states of the model *without diffusion* are computed and their stability is determined. From

$$\frac{\partial u}{\partial t} = F(u) = u(1 - u) - \alpha u = 0$$

it follows that either $u = 0$ or $u = 1 - \alpha$. The first one will be named the trivial zero state, and it is immediately seen that $\alpha < 1$ must hold for the second one to have a physical meaning because u is a positive quantity. But this inequality is equivalent to $E < r$, meaning that for a positive steady state to exist, the fishing effort must be smaller than the growth rate, in agreement with common-sense experience: too large an effort can lead to fish depletion.

Stability of the steady states is determined by the sign of the derivative $\mathrm{d}F/\mathrm{d}u$ evaluated at the corresponding points. Thus

$$\left[\frac{\mathrm{d}F}{\mathrm{d}u}\right]_{x=0} = 1 - \alpha > 0,$$

so the trivial state is an unstable one. This means that in the absence of spatial distribution effects—represented by the diffusive term—any small perturbation of the zero steady state, i.e. any small fish population, will grow in the future. This is good news, indeed, and the think tank in C should not spoil it. On the other hand,

$$\left[\frac{\mathrm{d}F}{\mathrm{d}u}\right]_{x=1-\alpha} = \alpha - 1 < 0$$

shows that the positive steady state $1 - \alpha$ is a stable one: for constant r and E with $\alpha < 1$, the dimensionless biomass will eventually settle at the value $1 - \alpha$. Whether this has some interest depends on economic considerations outside the scope of this paper.

3.2. *Enter diffusion*

It is a known fact that the 'conflict' between reaction and diffusion can change the linear stability of the diffusionless steady states. Therefore, the diffusion coefficient δ (or its dimensional counterpart D) can be considered as a *bifurcation parameter* (see the last section of this paper). For instance, in systems with more than

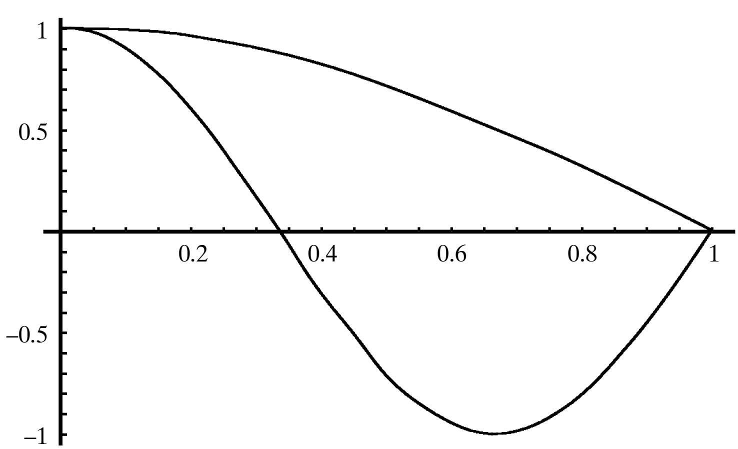

Figure 2. The functions $\cos(\pi x/2)$ and $\cos(3\pi x/2)$ in the interval [0,1].

one state variable, morphogenesis is the result of a diffusionless stable steady state switching into an unstable one under the influence of the diffusive terms. See, for example, the classical Turing (1952) article or the book by Murray (1989).

In the fishery scenario, δ can be thought of as a measure of how much fish biomass is leaving the territorial waters, and the following question is a natural one: Under which conditions will the addition of the diffusive term $\delta(\partial^2 u/\partial x^2)$ *not change* the steady state $u = 0$ from unstable into stable? In other words, diffusion must not prevent any initial biomass, however small, from growing.

To answer the question the linearized model equation at $u = 0$ is considered:

$$\frac{\partial u}{\partial t} = (1 - \alpha)u + \delta \frac{\partial^2 u}{\partial x^2}$$

with the same boundary conditions of the original non-linear model.

In order to solve the linear model the *ansatz* $u(x, t) = ae^{\sigma t} \cos qx$ is made, where a is the amplitude, σ is the parameter controlling growth (if $\sigma > 0$) or decay with time, and q is a wavenumber to be computed after the boundary conditions.

For $x = 0$, the condition

$$\left[\frac{\partial u}{\partial t}\right]_{x=0} = -qae^{\sigma t} \sin 0 = 0$$

is met by any q. On the other hand, at $x = 1$, $u(1, t) = ae^{\sigma t} \cos q = 0$ implies $\cos q = 0$, or $q = n(\pi/2)$ for odd n. Nevertheless, as u represents a biomass it cannot take negative values, so the only admissible value for n is $n = 1$ (see figure 2). Wan's approach to solving the equation is a different one. First, the reaction term is eliminated with the standard change $u = v \exp(ax + bt)$. It seems, from the modeller's viewpoint, that the importance of the 'diffusion versus reaction' mechanism is diminished. Second, the resulting diffusion equation and boundary conditions for v are solved by separation of variables. The Fourier expansion of the solution is then truncated at the first term, but the explanation is unclear.

Now, setting $u(x, t) = ae^{\sigma t} \cos(\pi x/2)$ in the linearized model equation one obtains

$$\sigma a e^{\sigma t} \cos \frac{\pi x}{2} = (1 - \alpha) a e^{\sigma t} \cos \frac{\pi x}{2} - \delta \left(\frac{\pi}{2}\right)^2 a e^{\sigma t} \cos \frac{\pi x}{2}$$

or, after cancelling the expression $ae^{\sigma t} \cos(\pi x/2)$:

$$\sigma = 1 - \alpha - \delta \left(\frac{\pi}{2}\right)^2 .$$

For the steady state $u = 0$ to be unstable, $\sigma > 0$ must hold, and this is equivalent to

$$\delta < 4\frac{1 - \alpha}{\pi^2}$$

which, after recovering the dimensional quantities, yields

$$\frac{D}{L^2 r} < 4\frac{1 - (E/r)}{\pi^2} \Leftrightarrow L > \frac{\pi}{2}\sqrt{\frac{D}{r - E}}.$$

Therefore, the width of the exclusive economic zone must grow with D. An easy interpretation is the following. After depletion of the usual species in a narrow zone, a switch towards fishing more mobile prey amounts to considering a larger value for D, and the width L suggested by the think tank in C must be enlarged accordingly.

Nevertheless, it seems that the most powerful reason for such a suggestion is more firmly based on the remark that L must also grow if E dangerously approaches r. Actually, a combination of both reasons should be the right answer.

4. Final comments and views

The analysis of this mathematical model is a good starting point for the introduction of higher mathematical concepts.

For instance, consider the idea of bifurcation pointed out in section 3.2. In a dynamical system with one state variable x and a parameter λ, let $x^*(\lambda)$ be a steady state. A value $\lambda = \lambda_{bif}$ is a bifurcation value for $x^*(\lambda)$ if the linear stability of $x^*(\lambda)$ changes as λ crosses through λ_{bif}. In the model, $\lambda = \delta$ and $\delta_{bif} = 4[(1-\alpha)/\pi^2]$ is the bifurcation value for the steady state $u(\delta) = 0$. The same stability analysis, when performed for the other steady state $u = 1 - \alpha$, shows that it is always stable independent of the value of δ. The discussion is summed up in the bifurcation table 1.

Steady state	No diffusion	With diffusion
$u = 0$	Unstable	Unstable if $\delta < \delta_{bif}$ Unstable if $\delta > \delta_{bif}$
$u = 1 - \alpha$	Stable	Stable

Table 1. Summing up the discussion.

In Wan's solution the concept of bifurcation is present under another form. It is said that the unstable solution bifurcates from the steady state. It is not clear what this should mean in a first reading and it needs some ingenuity to work it out.

The computation of q in section 3.2 is probably the easiest example of the general problem of eigenvalues and eigenfunctions of an operator. The model equation can be written in operator form as

$$\frac{\partial u}{\partial t} = A[u]$$

with $A := 1 - \alpha + \delta(\partial^2/\partial x^2)$ acting on the functional space $\xi[0, 1]$ whose elements are functions defined on $[0, 1]$ and satisfying the boundary conditions. Differentiability of $f \varepsilon \xi[0, 1]$ can be understood either in a *strictu sensu* to yield 'classical' solutions, or in weaker forms originating the so-called 'generalized' or distribution solutions. It is plain to see that the notion of a differential operator must include not only the operative differential expression, but also its application domain. Usually, introductory courses do not stress this fact, leading to quite important misunderstandings in both theory and applications.

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