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Ruin Probability Functions and Severity of Ruin as a Statistical Decision Problem

Emilio Gómez-Déniz [1,*], José María Sarabia [2] and Enrique Calderín-Ojeda [3]

[1] Department of Quantitative Methods and Institute of Tourism and Sustainable Economic Development (TIDES), University of Las Palmas de Gran Canaria, E-35017 Las Palmas de Gran Canaria, Spain
[2] Department of Economics, University of Cantabria, E-39005 Santander, Spain; sarabiaj@unican.es
[3] Centre for Actuarial Studies, Department of Economics, The University of Melbourne, Melbourne, VIC 3010, Australia; enrique.calderin@unimelb.edu.au
* Correspondence: emilio.gomez-deniz@ulpgc.es

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Abstract: It is known that the classical ruin function under exponential claim-size distribution depends on two parameters, which are referred to as the mean claim size and the relative security loading. These parameters are assumed to be unknown and random, thus, a loss function that measures the loss sustained by a decision-maker who takes as valid a ruin function which is not correct can be considered. By using squared-error loss function and appropriate distribution function for these parameters, the issue of estimating the ruin function derives in a mixture procedure. Firstly, a bivariate distribution for mixing jointly the two parameters is considered, and second, different univariate distributions for mixing both parameters separately are examined. Consequently, a catalogue of ruin probability functions and severity of ruin, which are more flexible than the original one, are obtained. The methodology is also extended to the Pareto claim size distribution. Several numerical examples illustrate the performance of these functions.

Keywords: loss function; exponential distribution; pareto distribution; ruin function; severity of ruin; upper bound

MSC: 62P05; 91B30; 97M30

1. Introduction

In a classical continuous time-surplus process $\{U(t)\}_{t\geq 0}$ when the insurer's initial surplus is given by u, an explicit expression for the probability of ultimate ruin exists, as is already well-known, for a limited number of claim-size distributions such as the exponential and mixed exponential distributions. When the claim-size distribution is exponential, simple analytic results for the ruin probability in infinite time may be possible. Nevertheless, as Grandell (1990) has pointed out, there is really no reason to believe that the exponential distribution is a realistic description of the claim behavior.

Furthermore, the class of mixtures of exponential distributions is somewhat limited because its mode is necessarily located at 0 (see Gerber et al. (1987)). Therefore, some efforts have been carried out in the statistical literature to find other probabilistic models to revise this issue.

Although for most of the general claim amount distributions, e.g., heavy-tailed, the Laplace transform technique does not work, explicit expressions under other assumptions, such as Pareto distributions, have been obtained but they are too complicated and require large computation to calculate the values of the ultimate ruin probability. For example, Garcia (2005) derived complicated exact solutions under series representation and Seal (1980) and Wei and Yang (2004) under integral representations. Grandell and Segerdahl (1971) showed that for the gamma claim amount distribution under some restrictions on the parameters, the exact value of ruin probability can be computed via a formula which involves a complicated integral. In Ramsay (2003), an expression based on numerical integration was derived for the probability of ultimate ruin under the classical compound Poisson risk model, given an initial reserve of u in the case of Pareto individual claim amount distributions. Furthermore, Albrecher et al. (2011) have obtained closed-form expressions for ruin probability functions under some kind of dependence assumption also using the mixing representation. In this regard, as Asmussen and Albrecher (2010) pointed out, the ideal situation is to come up with closed-form solutions for the ruin probabilities; however, these are limited. More recently, Tamturk and Utev (2018) computed ruin probabilities via a quantum mechanics approach and Sarabia et al. (2018) obtained ruin mixtures function in an aggregation of dependent risk model using mixtures of exponential distributions and finally Gómez-Déniz et al. (2016) has obtained closed-form expressions for the probability and severity of ruin when the claim size is assumed to follow a Lindley distribution. More recent results in the calculation of the ruin probabilities can be found in Kyprianou (2014) where the survival probability is defined for a general spectrally negative Lévy process. To compute these probabilities, it is necessary to invert the Laplace transform (see Kuznetzov et al. (2012b)). In addition, when the Laplace exponent is a rational function, it admits an explicit expression written in terms of the sum of complex exponentials. A relevant example is the case where the jump size is phase-type distributed, which can in theory approximate any Lévy process by phase-type fitting (Egami and Yamazaki (2014)). The joint density of ruin time and overshoot can also be simply computed once the scale function, i.e., the survival probability, is known (see Yamazaki (2017)).

In this paper, new ruin probability functions and severity of ruin are simply derived by a mixture mechanism, which is based on the use of appropriate loss functions. This procedure has resulted very useful to obtain new and adequate probability functions to fit insurance claim data and to derive new credibility expressions.

The structure of the paper is as follows. In Section 2 we revise some basic elements of the classical ruin theory. The mixture mechanism proposed in this work is given in Section 3. The main results are provided in Section 4. Here, closed-form expressions for mixture ruin probabilities and mixture severity of ruin when the mixing distribution belongs to the exponential family of distributions are given. Additionally, an expression for the upper bound of mixture ruin probability function is illustrated. Next numerical applications are shown in Section 5 and some final comments are presented in the last Section.

2. Background

The surplus process of an insurance portfolio is defined as the wealth obtained by the premium payments minus the reimbursements made at the times of claims. When this process becomes negative (if ever), we say that ruin has occurred. Let $\{U(t)\}_{t\geq 0}$ be a classical continuous time surplus process, the surplus process at time t given the initial surplus u, the dynamic of $\{U(t)\}_{t\geq 0}$ is given by

$$U(t) = u + ct - S(t),$$

where $S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregate claims amount up to time t, $S(t) = 0$ if $N(t) = 0$ where $N(t)$ is a time homogenous Poisson process with intensity δ. Here, $u \geq 0$ is the insurer's initial risk surplus at $t = 0$ and, as usual, we consider $c = (1+\theta)\delta p_1$ is the insurer's rate of premium income per unit time with loading $\theta > 0$, where $p_1 = E(X_i)$, $i = 1, \ldots, N(t)$.[1]

The probability of ultimate ruin, $\psi(u)$, given an initial surplus of $u \geq 0$, is defined as

$$\psi(u) = \Pr\left[U(t) < 0 \text{ for some } t > 0 | U(0) = u\right],$$

[1] Although the safety loading factor could be positive, negative or even zero (see for instance Schmidli (1999)), throughout this work it will be assumed that it only takes positive values.

this quantity is small for u sufficiently large.

It is well-known (see Gerber (1979); Rolski et al. (1999) and Dickson (2005); among others) that the general solution to $\psi(u)$ satisfies the following Volterra integral equation,

$$\psi(u) = \frac{1}{1+\theta}\left[K(u) + \int_0^u \psi(u-x)H(x)\,dx\right],$$

where $H(x) = \frac{1-F(x)}{p_1}, K(u) = \int_u^\infty H(x)\,dx$.

When $F(x) = 1 - \exp(-\lambda x)$, i.e., the claim size follows an exponential distribution with parameter $\lambda > 0$, it is possible to obtain a closed-form expression for $\psi(u)$ (see Gerber (1979) and Dickson (2005)) in the exponential claim-size case. This results in

$$\psi(u) = \frac{1}{1+\theta}\exp\left(-\frac{\lambda\theta u}{1+\theta}\right), \quad u \geq 0,\ \lambda > 0,\ \theta > 0. \tag{1}$$

Please note that the net profit condition is guaranteed in (1) since it has been assumed that the premium income rate is calculated based on the expression $c = (1+\theta)\delta\lambda$, i.e., expected value premium principle. If we make in (1) the change of variable $v = \theta/(1+\theta)$, the explicit ruin probability function under the exponential claim size can be rewritten as

$$\psi(u|\Pi) = (1-v)\exp(-\lambda v u), \tag{2}$$

for $u \geq 0$, where $\Pi = (\lambda, v)$, with $\lambda > 0$ and $0 < v < 1$.

On the other hand, practitioners working in ruin theory are also interested in the amount of the insurer's deficit at the time of ruin given that ruin occurs, i.e., given an initial surplus u, the time T_u is defined by

$$T_u = \inf\{t : U(t) < 0\},$$

being $T_u = \infty$ if $U(t) \geq 0$ for all $t > 0$. Defining the severity of ruin as

$$G(u,y) = \Pr(T_u < \infty, U(T_u) \geq -y),$$

it can be seen (see Gerber et al. (1987); Dickson and Waters (1992) and Dickson (2005), among others) that under exponential claim size

$$G(u,y) = \psi(u|\lambda)(1-\exp(-\lambda y)). \tag{3}$$

This last result is related to the exact distribution of the overshoot in the first passage times of the jump diffusion process due to the memoryless property of the exponential distribution. See Kou and Wang (2003) and references therein for details.

The main idea of this paper is to assume that Π is unknown and the estimation of $\psi(u|\Pi)$ is achieved by minimizing the expected loss

$$E_{P(\Pi)}\left[L\left(\psi(u|\Pi), \tilde{\psi}(u)\right)\right] \tag{4}$$

with respect to $\tilde{\psi}(u)$. The function (4) measures the loss sustained by a decision-maker who takes $\psi(u|\Pi)$ instead of $\tilde{\psi}(u)$. Here, $L(\cdot,\cdot)$ is a loss function usually taken as squared-error loss function, i.e., $L(a,z) = (a-z)^2$ and $P(\Pi)$ is the distribution function of Π. It is simple to see that in this case the value of $\tilde{\psi}(u)$ which minimizes the expected loss above is given by,

$$\tilde{\psi}(u) = \int_\Pi \psi(u|\Pi)\,dP(\Pi). \tag{5}$$

Therefore, according to expression (5), the ruin function can be viewed as a mixture procedure, i.e., the two parameters that the ruin function depends on can be considered to be random variables following a distribution function $P(\Pi)$. Then, the final ruin function can be obtained by compounding (mixing).

As a reviewer has pointed out, if the loading factor takes negative values, which is not the case assumed here, the conditional probability of ruin is one; however in general, this is not true for the overall unconditional probability of ruin.

3. Main Results

In practice, the ruin probability function defined in (1) depends on the relative safety loading $\theta > 0$ and on the mean claim amount $\lambda > 0$. In the following, we are assuming: (i) that both parameters are unknown and random, and they follow a bivariate probability distribution; (ii) that the λ parameter is unknown and random and therefore it follows certain probability density function with domain in the parametric space of λ, say Λ; and (iii) that the parameter which is unknown is θ and then proceed to mix the ruin function with some appropriate probability density functions. Mixture distributions arise in many contexts in the statistical literature and they arise naturally where a statistical population contains two or more subpopulations. In this case, we might assume that the insurance firm is interested in computing the ruin probabilities depending on these two parameters. As these parameters are not exactly known, there exists some lack of fit in the ruin probabilities that can be explained by the existence of some factor of heterogeneity in the population of policyholders or portfolios of the firm, then resulting in different ruin probabilities from portfolio to portfolio. For the exponential claims size the mixture proposed here can be considered an extension of the ruin probability obtained when finite mixture of exponential distributions is used.

3.1. Both Parameters Unknown

Let us suppose that v and λ parameters in (2) are unknown and random. Let us also define $\psi(u|\Pi) = \psi_{\lambda,v}(u)$. We suppose as well that the practitioner is capable of assigning a probability density function for $\Pi = (\lambda, v)$ with $\lambda > 0$ and $0 < v < 1$ according to the following scheme:

$$\begin{aligned} \psi(u|\Pi) &= \psi_{\lambda,v}(u), \\ \Pi &\sim P_{\Lambda,\Upsilon}(\lambda, v), \end{aligned}$$

then, we will determine the mixture ruin probability function by the well-known compounding formula

$$\tilde{\psi}_{BV}(u) = E_{\Pi}\left[\psi_{\lambda}(u)\right] = \int_{\Lambda}\int_{\Upsilon} \psi_{\lambda,v}(u)\, dP_{\Lambda,\Upsilon}(\lambda, v). \tag{6}$$

The subscript BV in (6) indicates that the ruin function is obtained when a bivariate distribution is chosen as mixing model.

Observe that in any case both parameters should not be independent. Thus, the distribution $dP_{\Lambda,\Upsilon}(\lambda, v)$ must be flexible enough to allow for correlation (positive) between both parameters. In this sense, any bivariate distribution (for example obtained via copula modeling) with the adequate support might be chosen as mixing distribution.

3.2. Mean Claim-Size Parameter Unknown

Let us now consider the situation, usually assumed in actuarial statistics, where λ is random and unknown and take $\psi(u|\Pi) = \psi_{\lambda}(u)$. Then, the practitioner is able to assign a probability density function for the parameter λ in its domain $(0, \infty)$. Let us also assume that the parameter θ in (1) is known. Now, we have the following statistical representation,

$$\begin{array}{rcl} \psi(u|\Lambda = \lambda) & = & \psi_\lambda(u), \\ \Lambda & \sim & P_\Lambda(\lambda), \end{array}$$

then we will determine the mixture ruin probability function by the compounding formula

$$\tilde{\psi}(u) = E_\Lambda\left[\psi_\lambda(u)\right] = \int_0^\infty \psi_\lambda(u)\, dP_\Lambda(\lambda). \tag{7}$$

In risk theory, a correct estimation of the length of the tail for a ruin function is vital when judging its suitability as a proposed model. As we will show later in the numerical experiments, this methodology facilitates to obtain ultimate ruin probability functions with slower rates of decrease as u tends to infinite. This is compatible with the fact that the new ruin functions obtained have long(heavy)-tail probability distributions, i.e., with tails that decay more slowly than exponentially.

The cumulative distribution function $P_\Lambda(\lambda)$ can be viewed as a prior distribution in a Bayesian methodology. It is common in actuarial statistics the use of the Bayesian approach. In this regard, a prior distribution is assumed for the parameters in the field of credibility theory. Several works have dealt with this issue, including analysis about the Bayesian sensitivity of the model assumed.

Let us also consider that the practitioner decides to select a model from the single-parameter natural exponential family of distributions. This family is determined by

$$q(\lambda) = \exp\left(\omega\, \lambda - \kappa(\omega)\right), \quad \lambda \in \Lambda \tag{8}$$

where ω is denoted as the natural parameter satisfying that

$$\kappa(\omega) = \log \int e^{\omega\lambda} q(\lambda)\, d\lambda < \infty,$$

being $\kappa(\omega)$ a function of the natural parameter ω.

A large number of well-known distributions such as normal, Poisson, or Gamma, belong to this family. The moment generating function is given by

$$M_\Lambda(t) = \exp\left(\kappa(\omega + t) - \kappa(\omega)\right). \tag{9}$$

Let us now consider that Λ in the ruin probability function (1) follows the single-parameter natural exponential family of distribution (8). Then by using (7) and (1) and taking into account (9), we have that the mixture ruin probability function is given by

$$\tilde{\psi}(u) = \frac{1}{1+\theta} M_\Lambda\left(-\frac{\theta u}{1+\theta}\right). \tag{10}$$

For $u = 0$, we have that $\tilde{\psi}(0) = 1/(1+\theta)$. It is already known (see for example Dickson (2005, p. 133)) that for the classical risk process, Lundberg's inequality provides an upper bound for the ruin probability, i.e., $\psi(u|\lambda) \leq e^{-R(\lambda)u}$, where $R(\lambda)$ is a function depending on λ, known as the adjustment coefficient, i.e., the positive solution of the equation

$$\delta M_X(r|\lambda) - \delta - cr = 0, \tag{11}$$

being $M_X(r|\lambda) = \int_0^\infty e^{rx}\, dF(x|\lambda)$, i.e. the moment generating function of the claim-size distribution where $F(\cdot)$ is the cumulative distribution function of the exponential distribution. As the solution of

Equation (11) is $R(\lambda) = \lambda - \delta/c$, then from $\psi(u|\lambda) \leq e^{-R(\lambda)u}$, by multiplying both sides by $dP_\Lambda(\lambda)$ and integrating from 0 to infinity, we obtain the upper bound of $\tilde{\psi}(u)$, which is given by

$$\widehat{\psi}(u) = M_\Lambda\left(-\frac{\theta u}{1+\theta}\right). \tag{12}$$

Now, to obtain mixture of severity of ruin we adopt the following scheme:

$$\begin{aligned} G(u,y|\Lambda=\lambda) &= G_\lambda(u,y), \\ \Lambda &\sim P_\Lambda(\lambda), \end{aligned}$$

then the mixture severity ruin function is again determined by the compounding formula

$$\tilde{G}(u,y) = E_\Lambda\left[G_\lambda(u,y)\right] = \int_0^\infty G_\lambda(u,y)\, dP_\Lambda(\lambda). \tag{13}$$

Let us now consider that Λ in the probability of severity of ruin function (3) follows the single-parameter exponential family of distribution (8), then we have that the mixture severity of ruin function is given by

$$\tilde{G}(u,y) = \tilde{\psi}(u)\left[1 - \frac{M_\Lambda\left(-y - \frac{\theta u}{1+\theta}\right)}{M_\Lambda\left(-\frac{\theta u}{1+\theta}\right)}\right]. \tag{14}$$

Please note that $\lim_{y\to\infty} \tilde{G}(u,y) = \tilde{\psi}(u)$.

3.3. Safety Loading Parameter Unknown

Let us now denote as $\psi(u|\Pi) = \psi_v(u)$ the situation where the parameter v is considered unknown. In this case, we will use the following scheme:

$$\begin{aligned} \psi(u|\Upsilon=v) &= \psi_v(u), \\ \Upsilon &\sim P_\Upsilon(v), \end{aligned}$$

when the v parameter is unknown and random. In this case, we will again determine the mixture ruin probability function by the compounding formula

$$\tilde{\psi}(u) = E_\Upsilon\left[\psi_v(u)\right] = \int_0^1 \psi_v(u)\, dP_\Upsilon(v). \tag{15}$$

After using (1) we get

$$\tilde{\psi}(u) = M_\Upsilon(-\lambda u) - M_\Upsilon^*(-\lambda u), \tag{16}$$

where $M_\Upsilon(t)$ is the moment generating function of the random variable Υ and $M_\Upsilon^*(t) = E_\Upsilon(ve^{tv})$.

The mixture severity of ruin function is derived from

$$\begin{aligned} \tilde{G}(u,y) &= E_\Upsilon\left[G_v(u,y)\right] = \int_0^1 G_v(u,y)\, dP_\Upsilon(v) \\ &= \tilde{\psi}(u)(1 - \exp(-\lambda y)), \end{aligned}$$

where $\tilde{\psi}(u)$ is given in (16).

4. Specific Results

As we are interested in deriving a closed-form expression when (6) is used, we shall consider firstly the probability density function recently proposed by Gómez-Déniz et al. (2014). This model was employed in the context of credibility premiums when the total claim amount distribution was examined. See Appendix for details. Expressions for the mixture ruin probability function and mixture severity of ruin are shown in Table 1. These expressions are obtained after some algebra which we do not reproduce here. In these expressions ${}_2F_1(a,b;c,z)$ represents the hypergeometric function which has the integral representation

$$ {}_2F_1(a,b;c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt. $$

Now, by using some particular probability density functions that belong to the class (8) and they admit closed-form expression for the moment generating function, as mixing distributions, new ruin probability functions and severity of ruin functions also given in closed-form are obtained when the λ parameter is assumed to be random. In this sense, the gamma, exponential, Lindley and inverse Gaussian distributions were chosen for this issue. The Lindley distribution (Lindley (1958) and Ghitany et al. (2008), among others) is a one parameter continuous distribution, that is becoming more popular in the recent years due to its simplicity. Besides, it has a closed-form expression for its moment generating function. The corresponding probability density functions and moment generating functions are included in the Appendix.

On the other hand, when the v parameter is assumed to be unknown, the mixture ruin probability and mixture severity of ruin functions are obtained by considering the uniform distribution, the confluent hypergeometric distribution, the classical beta distribution and the arcsin distribution, respectively. The confluent hypergeometric distribution with support on $(0,1)$ is a generalization of the classical beta distribution proposed in Gordy (1998) and appearing in Nadarajah (2005). The arcsine distribution is a particular case of the Beta distribution obtained when $\alpha = \beta = 1/2$. When the practitioner has no knowledge at all about the distribution of Y, then a uniform structure function to obtain the mixture ruin probability function or alternatively the arcsin distribution might be considered.

As it can be observed, for all examined cases, simple closed-form formulae are derived. When the Lindley and inverse Gaussian distributions are considered, expressions for the severity of ruin depend on two functions, $\mathcal{G}_L(\beta,\theta,u,y)$ and $\mathcal{G}_{IG}(\beta,\alpha,\theta,u,y)$

$$ \begin{aligned} \mathcal{G}_L(\beta,\theta,u,y) &= \frac{(1+y+\beta(1+\theta)+\theta(1+u+y))}{1+\beta+\theta(1+\beta+u)} \\ &\times \left[\frac{\beta+\theta(\beta+u)}{(y+\beta(1+\theta)+\theta(y+u))}\right]^2 \end{aligned} $$

and

$$ \begin{aligned} \mathcal{G}_{IG}(\beta,\alpha,\theta,u,y) &= \exp\left\{\frac{1}{\alpha}\left[\sqrt{\frac{\beta(\beta(1+\theta)+2\alpha^2\theta u)}{(1+\theta)}} \right.\right. \\ &- \left.\left.\sqrt{\frac{\beta((\beta+2\alpha^2 y)(1+\theta)+2\alpha^2\theta u)}{(1+\theta)}}\right]\right\}, \end{aligned} $$

respectively.

Table 1. Specific examples of mixture ruin probability functions and mixture severity of ruin.

Distribution on Π	Mixture Ruin Function	Mixture Severity of Ruin
Bivariate	$\tilde{\psi}_{BV}(u) \equiv \frac{\beta}{\alpha+\beta-\gamma}\left(\frac{\sigma}{\sigma+u}\right)^{\gamma}$	$\tilde{G}_{BV}(u,y) = \tilde{\psi}_{BV}(u)\left[1-\left(\frac{\sigma+u}{y}\right)^{\gamma}\frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\alpha)}{\Gamma(\alpha-\gamma)\Gamma(\alpha+\beta+1)} \times {}_2F_1\left(\alpha,\gamma;\alpha+\beta-\gamma;-\frac{u+\sigma}{y}\right)\right],\ y>0$
Distribution on Λ	**Mixture Ruin Function**	**Mixture Severity of Ruin**
Gamma	$\tilde{\psi}_G(u) = \frac{(1+\theta)^{\alpha-1}}{[1+\theta(1+\beta u)]^{\alpha}}$	$G_G(u,y) = \tilde{\psi}_G(u)\left\{1-\left[\frac{1+\theta(1+\beta u)}{(1+\theta)(1+\beta y)+\theta\beta u}\right]^{\alpha}\right\}$
Exponential	$\tilde{\psi}_E(u) = \frac{1}{1+\theta(1+\beta u)}$	$G_E(u,y) = \tilde{\psi}_E(u)\left[1-\frac{1+\theta(1+\beta u)}{(1+\theta)(1+\beta y)+\theta\beta u}\right]$
Lindley	$\tilde{\psi}_L(u) = \frac{\beta^2(1+\beta+\theta(1+\beta+u))}{(1+\beta)(\beta+\theta(\beta+u))^2}$	$G_L(u,y) = \psi_L(u)\left\{1-\mathcal{G}_L(\beta,\alpha,\theta,u,y)\right\}$
Inverse Gaussian	$\tilde{\psi}_{IG}(u) = \frac{1}{1+\theta}\exp\left\{\frac{1}{\alpha}\left[\beta-\sqrt{\frac{\beta(\beta(1+\theta)+2\alpha^2\theta u)}{(1+\theta)}}\right]\right\}$	$G_{IG}(u,y) = \tilde{\psi}_{IG}(u)\left\{1-\mathcal{G}_{IG}(\beta,\alpha,\theta,u,y)\right\}$
Distribution on Υ	**Mixture Ruin Function**	**Mixture Severity of Ruin**
Uniform	$\tilde{\psi}_U(u) = \frac{\lambda u+e^{-\lambda u}-1}{\lambda^2 u^2},\quad u>0$	$G_U(u,y) = \tilde{\psi}_U(u)(1-\exp(-\lambda y))$
CH	$\tilde{\psi}_{CH}(u) = \frac{\beta}{\alpha+\beta}\frac{{}_1F_1(\alpha,\alpha+\beta+1,-\sigma-\lambda u)}{{}_1F_1(\alpha,\alpha+\beta,-\sigma)}$	$G_{CH}(u,y) = \tilde{\psi}_{CH}(u)(1-\exp(-\lambda y))$
Beta	$\tilde{\psi}_B(u) = \frac{\beta}{\alpha+\beta}\ {}_1F_1(\alpha;\alpha+\beta+1;-\lambda u)$	$G_B(u,y) = \tilde{\psi}_B(u)(1-\exp(-\lambda y))$
Power	$\tilde{\psi}_P(u) = \frac{1}{\alpha+1}\ {}_1F_1(\alpha;\alpha+2;-\lambda u)$	$G_B(u,y) = \tilde{\psi}_P(u)(1-\exp(-\lambda y))$
arcsin	$\tilde{\psi}_{AS}(u) = \frac{1}{2}\ {}_1F_1(1/2,2,-\lambda u)$	$G_{AS}(u,y) = \tilde{\psi}_{AS}(u)(1-\exp(-\lambda y))$

Pareto distribution has also been employed as a competitive alternative to the exponential distribution to model the claim-size data in classical risk theory (Ramsay and Usabel (1997); Dickson and Waters (1992)); Ramsay (2003) and Politis (2006); among others). In this case, a closed-form expression for the ruin probability function cannot be obtained, and a numerical procedure can be easily implemented to approximate ruin probabilities. In this sense, Ramsay and Usabel (1997) obtained numerically, via product integration, ruin probabilities by assuming that the claim size follows a Pareto distribution with integer parameter and with cumulative distribution function given by

$$F(x) = 1-\left(1+\frac{x}{m}\right)^{-(m+1)}, \quad x>0, \ \ m=1,2,\ldots$$

In Ramsay (2003) a ruin probability function is derived under this assumption, which is given by

$$\psi(u;m,a,b) = \frac{1}{\theta}\left(1+\frac{u}{m}\right)^{-m}\mathbb{E}\left[g_m(Z,\theta)\right],$$

with

$$g_m(x;\theta) = \frac{\theta^2}{\left[\theta+xe^{-x}Ei_m(x)\right]^2+\left[\pi e^{-x}x^m/(m-1)!\right]^2},$$

where $Ei_m(x)$ is a generalization of the exponential integral $Ei(x)$ given by

$$Ei_m(x) = \frac{x^{m-1}}{(m-1)!}\left[\gamma+\log x-\sum_{r=1}^{m-1}\frac{1}{r}\right]+\sum_{\substack{r=0\\ r\neq m-1}}^{\infty}\frac{x^r}{(r-m+1)r!},$$

being γ the Euler's constant ($\gamma \approx 0.5772156649$) and Z follows a gamma distribution with parameters m and $1 + u/m$. By writing $\theta = v/(1-v)$, $v \in (0,1)$ and using expression (15), we can derive new ruin probability functions via

$$\tilde{\psi}(u) = \left(1 + \frac{u}{m}\right)^{-m} \int_0^1 \frac{1-v}{v} E\left[g_m\left(Z, \frac{v}{1-v}\right)\right] p(v)\, dv. \quad (17)$$

Finally, by taking some of the aforementioned probability distributions as $p(v)$, we obtain new ruin probability functions (not given in closed form) that can be evaluated numerically.

5. Numerical Experiments

In the following, we will proceed to numerically illustrate the usefulness of the explicit ruin functions obtained in this manuscript. For that reason, different values of the premium loading factor θ, usually used in the literature, will be considered. Furthermore, probabilities of severity of ruin have been computed for a fixed value of $\theta = 0.25$. Then, to assess the performance of the mixture ruin probabilities introduced in this paper, the values obtained under the classical exponential ruin probability function have been calculated, they are shown in Table 2 for various values of the initial risk surplus u and the loading factor θ. It is observable that the probability decreases with the loading factor θ and the initial surplus level u. Then, the mixture ruin probabilities for the different models considered before are exhibited in Table 3. The choice of the parameter values has been obtained by setting equal the expectation of the mixing distribution to one (the value of parameter λ) when λ is considered to be random. On the other hand, when the parameter v is random and unknown, $\theta/(1+\theta)$ has been set equal to the mean of the mixing distribution. Moreover, for the bivariate mixing distribution, we have selected $\gamma = 0.03$, $\sigma = 3$, while the parameters α and β were chosen in a way such that the marginal means are 1 and $\theta/(1+\theta)$ respectively. As with the classical exponential case, these probabilities reduce when both u and θ increase.

Furthermore, expression (17) has now been used to compute mixture ruin probabilities (see lower part of Table 4); by comparing the obtained results with the ones achieved in Ramsay (2003) (see upper part of Table 4). Please note that only the beta distribution was used as mixing distribution. Finally, some values of the mixture severity of ruin function are exhibited in Table 5 for $\theta = 0.25$. In this case, we have firstly considered the classical exponential and then gamma, Lindley and inverse Gaussian as mixing distributions. The choice of the parameters is the same as the one used in Table 4.

Table 2. Ruin probabilities for exponential claims with $\lambda = 1$ and different values of the premium loading factor θ.

u	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$
1	0.830092	0.654985	0.477688	0.372251	0.303265
2	0.757957	0.536256	0.342278	0.242499	0.183940
3	0.692091	0.439049	0.245253	0.157973	0.111565
4	0.631949	0.359463	0.175731	0.102910	0.067667
5	0.577033	0.294304	0.125917	0.067039	0.041042
6	0.526889	0.240955	0.090223	0.043672	0.024893
7	0.481103	0.197278	0.064648	0.028449	0.015098
8	0.439296	0.161517	0.046322	0.018533	0.009157
9	0.401121	0.132239	0.033191	0.012073	0.005554
10	0.366264	0.108268	0.023782	0.007865	0.003368
50	0.009650	0.000036	3.850×10^{-8}	2.82×10^{-10}	6.94×10^{-12}
100	0.000102	1.640×10^{-9}	2.22×10^{-15}	1.39×10^{-19}	9.64×10^{-23}

Table 3. Mixture ruin probabilities for the different models considered.

$\tilde{\psi}_{BV}(u)$						$\tilde{\psi}_G(u)$, $\alpha = 2$, $= \beta = 1/2$				
u	$\alpha = 1.21$ $\beta = 11.79$	$\alpha = 1.09$ $\beta = 4.26$	$\alpha = 1.06$ $\beta = 2.06$	$\alpha = 1.05$ $\beta = 1.36$	$\alpha = 1.04$ $\beta = 1.01$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$
1	0.909091	0.800000	0.666667	0.571429	0.500000	0.831758	0.661157	0.489796	0.387543	0.320000
2	0.898100	0.790328	0.658606	0.564520	0.493955	0.763889	0.555556	0.375000	0.280000	0.222222
3	0.890382	0.783536	0.652947	0.559669	0.489710	0.704000	0.473373	0.296296	0.211720	0.163265
4	0.884442	0.778309	0.648590	0.555935	0.486443	0.650888	0.408163	0.240000	0.165680	0.125000
5	0.879617	0.774063	0.645053	0.552902	0.483789	0.603567	0.355556	0.198347	0.133175	0.098765
6	0.526889	0.240955	0.090223	0.043672	0.024893	0.561224	0.312500	0.166667	0.109375	0.080000
7	0.875559	0.770492	0.642076	0.550351	0.481557	0.523187	0.276817	0.142012	0.091428	0.066116
8	0.872058	0.767411	0.639509	0.548151	0.479632	0.488889	0.246914	0.122449	0.077562	0.055555
9	0.868982	0.764704	0.637254	0.546217	0.477940	0.457856	0.221607	0.106667	0.066627	0.047337
10	0.866240	0.762291	0.635243	0.544494	0.476432	0.429688	0.200000	0.093750	0.057851	0.040816
50	0.824438	0.725506	0.604588	0.518218	0.453441	0.084876	0.022222	0.007653	0.004164	0.002743
100	0.807942	0.710989	0.592491	0.507849	0.444368	0.029562	0.006611	0.002136	0.001136	0.000739
$\psi_L(u)$, $\beta = \sqrt{2}$						$\psi_{IG}(u)$, $\alpha = 1, \beta = 1$				
u	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$
1	0.832812	0.664911	0.496879	0.396288	0.329431	0.833247	0.666071	0.498347	0.397569	0.330430
2	0.767518	0.566663	0.392956	0.300174	0.242641	0.768697	0.568483	0.393376	0.299056	0.240461
3	0.711083	0.492388	0.323521	0.240273	0.190909	0.712858	0.493426	0.320614	0.234702	0.183940
4	0.661881	0.434490	0.274170	0.199681	0.156854	0.663937	0.433616	0.267050	0.189374	0.145262
5	0.618656	0.388225	0.237447	0.170497	0.132858	0.620637	0.384737	0.226020	0.155878	0.117345
6	0.580416	0.350490	0.209139	0.148573	0.115094	0.581987	0.344021	0.193683	0.130277	0.096433
7	0.546376	0.319180	0.186697	0.131536	0.101441	0.547240	0.309591	0.167639	0.110210	0.080333
8	0.515901	0.292817	0.168498	0.117936	0.090635	0.515814	0.280121	0.146303	0.094168	0.067667
9	0.488477	0.270341	0.153459	0.106840	0.081879	0.487240	0.254640	0.128578	0.081137	0.057531
10	0.463681	0.250967	0.140834	0.097622	0.074645	0.461140	0.232419	0.113683	0.070412	0.049302
50	0.147573	0.063149	0.032234	0.021630	0.016275	0.103113	0.022243	0.005169	0.002066	0.001075
100	0.078514	0.032393	0.016351	0.010934	0.008213	0.030961	0.003601	0.000484	0.000140	0.000058

Table 3. *Cont.*

$\psi_B(u)$					$\psi_{CH}(u)$					
u	$\alpha=1$ $\beta=9$	$\alpha=1$ $\beta=3$	$\alpha=1$ $\beta=1$	$\alpha=1$ $\beta=0.33$	$\alpha=1$ $\beta=1.1$	$\alpha=1$ $\beta=9$ $\sigma=0.1$	$\alpha=1$ $\beta=3$ $\sigma=0.1$	$\alpha=1$ $\beta=1$ $\sigma=0.1$	$\alpha=1$ $\beta=0.33$ $\sigma=0.1$	$\alpha=1$ $\beta=0.1$ $\sigma=0.1$
1	0.824511	0.621830	0.367879	0.167418	0.389129	0.826698	0.631616	0.395369	0.224775	0.651310
2	0.759969	0.527252	0.283834	0.120673	0.302500	0.762419	0.536637	0.306354	0.163002	0.456691
3	0.704272	0.455508	0.227754	0.091973	0.244135	0.706900	0.464402	0.246685	0.124855	0.341109
4	0.655801	0.399725	0.188645	0.073327	0.203097	0.658545	0.408106	0.204899	0.099936	0.268109
5	0.613293	0.355394	0.160270	0.060566	0.173121	0.616107	0.363277	0.174469	0.082798	0.219271
6	0.575751	0.319479	0.138958	0.051421	0.150483	0.578603	0.326893	0.151541	0.070465	0.184886
7	0.542384	0.289886	0.122468	0.044604	0.132887	0.545249	0.296867	0.133752	0.061238	0.159596
8	0.512553	0.265138	0.109380	0.039352	0.118873	0.515414	0.271722	0.119601	0.054108	0.140303
9	0.485740	0.244170	0.098767	0.035191	0.107474	0.488583	0.250391	0.108103	0.048446	0.125136
10	0.461522	0.226200	0.090000	0.031819	0.098035	0.464337	0.232091	0.098589	0.043847	0.112913
50	0.152137	0.056541	0.019600	0.006555	0.021517	0.153571	0.058331	0.021619	0.009102	0.022970
100	0.082505	0.029117	0.009900	0.003289	0.010879	0.083346	0.030067	0.010930	0.004571	0.011508

Table 4. Classical ruin probabilities for Pareto distribution (above) with $m = 1$ and different values of the premium loading factor θ. Probabilities based on expression (17) and Beta mixing distribution (below).

u	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.50$	$\theta = 0.75$	$\theta = 1.00$
10	0.627722	0.372683	0.206648	0.138243	0.102523
20	0.498175	0.245262	0.119275	0.075909	0.055050
30	0.411440	0.178339	0.081426	0.051056	0.036887
40	0.347896	0.137560	0.060856	0.038038	0.027509
50	0.299157	0.110519	0.048164	0.030142	0.021847
60	0.260646	0.091524	0.039650	0.024884	0.018080
70	0.229552	0.077594	0.033588	0.021150	0.015402
80	0.204018	0.067029	0.029075	0.018369	0.013404
90	0.182761	0.058793	0.025596	0.016222	0.011859
100	0.164859	0.052226	0.022838	0.014516	0.010630
u	$\alpha = 1$ $\beta = 9$	$\alpha = 1$ $\beta = 3$	$\alpha = 1$ $\beta = 1$	$\alpha = 1$ $\beta = 0.33$	$\alpha = 1$ $\beta = 1.1$
10	0.656535	0.413391	0.206860	0.084824	0.221515
20	0.560347	0.322256	0.150307	0.059235	0.161739
30	0.497584	0.270725	0.121416	0.046852	0.130993
40	0.451470	0.236169	0.103170	0.039268	0.111501
50	0.415428	0.210890	0.090365	0.034055	0.097784
60	0.386135	0.191370	0.080780	0.030212	0.087497
70	0.361666	0.175726	0.073284	0.027243	0.079438
80	0.340804	0.162838	0.067230	0.024868	0.072923
90	0.322732	0.151995	0.062221	0.022918	0.067527
100	0.306875	0.142719	0.057997	0.021285	0.062971

Table 5. Probabilities of severity of ruin with a premium loading factor, $\theta = 0.25$.

	y	Exponential	Mixing Distribution		
			Gamma	Lindley	Inverse Gaussian
$u = 0$	1	0.505696	0.444444	0.411775	0.415263
	5	0.794610	0.734694	0.680567	0.721115
	10	0.799964	0.777777	0.736850	0.777757
	∞	0.800000	0.800000	0.800000	0.800000
$u = 5$	1	0.186035	0.155555	0.137258	0.152318
	5	0.292321	0.305556	0.286781	0.325647
	10	0.294290	0.336621	0.330539	0.366766
	∞	0.294304	0.355555	0.388225	0.384737
$u = 10$	1	0.068438	0.072000	0.066816	0.078125
	5	0.107539	0.160494	0.162841	0.187195
	10	0.108263	0.183673	0.197878	0.217767
	∞	0.108268	0.200000	0.250967	0.232419
$u = 100$	1	1.040×10^{-9}	0.000562	0.001506	0.000515
	5	1.630×10^{-9}	0.002222	0.006353	0.001880
	10	1.640×10^{-9}	0.003486	0.010625	0.002720
	∞	1.640×10^{-9}	0.006611	0.032393	0.003601

6. Final Comments

A wide catalogue of new closed-form ruin probability functions and probabilities of severity of ruin which can be useful for the actuarial community have been obtained. These expressions are much simpler than those traditionally used in the literature except for the formula based on the exponential claim size. The reason for this is because majority of these expressions do not require special functions.

In addition to this, explicit formulation for the upper bound of the mixture ruin probability functions have been derived. Comparisons between the different models obtained have been illustrated with several numerical examples.

Despite these new expressions for the ruin probability functions and severity of ruin have been calculated by assuming a squared-error loss function, other loss functions (see for example, Heilmann (1989), for a wide catalogue of loss functions used in the actuarial literature) might be employed to analyze alternative formulation to the one presented here. In addition, as the assumption of the exponential-size jumps is quite optimistic, the case of completely monotone jump densities that generalizes the situation of mixture of exponentials deserved to be studied at some future time. In this regard, it might be feasible to consider some cases included in the meromorphic family see (Kuznetzov et al. (2012a)).

Finally, a similar development to the one carried out in this work could easily be implemented when other plausible alternatives to the exponential and Pareto distributions for the claims amount are considered. In this sense, the gamma distribution with a positive integer shape parameter, i.e., Erlang distribution could be used.

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Appendix A. Continuous Distributions

The probability density function together with the moment generating function of the models acting as mixing distributions in this work are included.

- Bivariate distribution proposed by Gómez-Déniz et al. (2014).

$$f(x,y) = \frac{\sigma^{\gamma}}{B(\alpha-\gamma,\beta)\Gamma(\gamma)} x^{\alpha-1}(1-x)^{\beta-1} y^{\gamma-1}\exp(-\sigma x y), \tag{A1}$$

for $0 < x < 1,\ y > 0, \alpha > 0,\ \beta > 0, \gamma > 0, \sigma > 0$ and $\alpha > \gamma$, where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt$$

is the gamma function and $B(z_1, z_2)$ is the beta function given by

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt.$$

The marginal distributions, which can be obtained by integrating (A1) with respect to y and x, respectively, are known univariate distributions. Thus, the marginal distribution of X is a beta distribution with parameters $\alpha-\gamma$ and β, i.e.

$$f_X(x) = \frac{1}{B(\alpha-\gamma,\beta)} x^{\alpha-\gamma-1}(1-x)^{\beta-1}. \tag{A2}$$

The marginal distribution of Y is given by

$$f_Y(y) = \frac{\sigma^\gamma \Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha+\beta)B(\alpha-\gamma,\gamma)} y^{\gamma-1}\,{}_1F_1(\alpha,\alpha+\beta,-\sigma y), \tag{A3}$$

where ${}_1F_1(\cdot,\cdot,\cdot)$ is the confluent hypergeometric function, also called Kummer's function, given by

$${}_1F_1(m,n,z) = \sum_{k=0}^{\infty} \frac{(m)_k z^k}{(n)_k k!},$$

and $(m)_j = \Gamma(m+j)/\Gamma(m)$, $j \geq 1$, $(m)_0 = 1$ is the Pochhammer symbol.

Using Kummer's first theorem we have that (A3) can be rewritten as

$$f_Y(y) = \frac{\sigma^\gamma \Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha+\beta)B(\alpha-\gamma,\gamma)} y^{\gamma-1} e^{-\sigma y}\,{}_1F_1(\beta,\alpha+\beta,\sigma y). \tag{A4}$$

The covariance of (A1) is given by

$$cov(X,Y) = \frac{\beta\gamma}{\sigma(\alpha+\beta-\gamma)(\gamma-\alpha+1)}, \tag{A5}$$

which admits correlation of any sign. Thus, we have

$$cov(X,Y) \begin{cases} > 0 & \text{if } 0 < \alpha-\gamma < 1, \\ < 0 & \text{if } \alpha-\gamma > 1. \end{cases}$$

- Lindley distribution.

$$\begin{aligned} p(y) &= \frac{\beta^2}{1+\beta}(1+y)\exp(-\beta y), \quad y>0,\ \beta>0, \\ M(t) &= \frac{\beta^2(1+\beta-t)}{(1+\beta)(\beta-t)^2}. \end{aligned}$$

- Inverse Gaussian distribution.

$$\begin{aligned} p(y) &= \sqrt{\frac{\beta}{2\pi y^3}} \exp\left[-\frac{\beta}{2\alpha^2 y}(y-\alpha)^2\right], \quad y>0,\ \alpha>0,\ \beta>0, \\ M(t) &= \exp\left[\frac{\beta}{\alpha}\left(1-\sqrt{1-\frac{2\alpha^2 t}{\beta}}\right)\right]. \end{aligned}$$

- The confluent hypergeometric distribution.

$$\begin{aligned} p(y) &= \frac{y^{\alpha-1}(1-y)^{\beta-1}e^{-\sigma y}}{B(\alpha,\beta)\,{}_1F_1(\alpha,\alpha+\beta,-\sigma)}, \quad 0<y<1,\ \alpha>0,\ \beta>0,\ \sigma\geq 0, \\ M(t) &= \frac{{}_1F_1(\alpha,\alpha+\beta,t-\sigma)}{{}_1F_1(\alpha,\alpha+\beta,-\sigma)}, \end{aligned}$$

where ${}_1F_1(a;c;x)$ represents the confluent hypergeometric function given by

$${}_1F_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 z^{a-1}(1-z)^{c-a-1}e^{xz}dz, \quad c>a>x.$$

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