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Solvability of a Maximum Quadratic Integral Equation of Arbitrary Orders

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Abstract

We investigate a new quadratic integral equation of arbitrary orders with maximum and prove an existence result for it. We will use a fixed point theorem due to Darbo as well as the monotonicity measure of noncompactness due to Banaś and Olszowy to prove that our equation has at least one solution in $C[0,1]$ which is monotonic on $[0,1]$.

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1 Introduction

In several papers, among them [1, 11], the authors studied differential and integral equations with maximum. In [6–9] Darwish *et al.* studied fractional integral equations with supremum. Also, in [4, 5], Caballero *et al.* studied the Volterra quadratic integral equations with supremum. They showed that these equations have monotonic solutions in the space $C[0, 1]$. Darwish [7] generalized and extended the Caballero *et al.* [4] results to the case of quadratic fractional integral equations with supremum.

In this paper we will study the fractional quadratic integral equation with maximum

$$y(t) = f(t) + \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}}ds,\ t \in J = [0,1],\ 0 < \beta < 1, \tag{1.1}$$

where $f, \varphi : J \to \mathbb{R}$, $T : C(J) \to C(J)$, $\sigma : J \to J$ and $\kappa : J \times J \to \mathbb{R}_+$.

By using the monotonicity measure of noncompactness due to Banaś and Olszowy [3] as well as the Darbo fixed point theorem, we prove the existence of monotonic solutions to (1.1) in $C[0, 1]$.

Now, we assume that $(E, \|\cdot\|)$ is a real Banach space. We denote by $B(x, r)$ the closed ball centred at x with radius r and $B_r \equiv B(\theta, r)$, where θ is a zero element of E. We let $X \subset E$. The closure and convex closure of X are denoted by $\overline{X}$ and $\mathrm{Conv}X$, respectively. The symbols $X + Y$ and λY are using for the usual algebraic operators on sets and $\mathfrak{M}_E$ and $\mathfrak{N}_E$ stand for the families defined by $\mathfrak{M}_E = \{A \subset E : A \neq \emptyset,\ A \text{ is bounded}\}$ and $\mathfrak{N}_E = \{B \subset \mathfrak{M}_E : B \text{ is relatively compact}\}$, respectively.

Definition 1.1 (See [2]). A function $\mu : \mathfrak{M}_E \to [0, +\infty)$ is called a measure of noncompactness in E if the following conditions:

1° $\emptyset \neq \{X \in \mathfrak{M}_E : \mu(X) = 0\} = \ker\mu \subset \mathfrak{N}_E$,

2° if $X \subset Y$, then $\mu(X) \leq \mu(Y)$,

3° $\mu(X) = \mu(\overline{X}) = \mu(\mathrm{Conv}X)$,

4° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, $0 \leq \lambda \leq 1$ and

5° if (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ with $X_n \supset X_{n+1} (n = 1,\ 2,\ 3,\ ...)$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$,

hold.

We will establish our result in the Banach space $C(J)$ of all defined, real and continuous functions on $J \equiv [0, 1]$ with standard norm $\|y\| = \max\{|y(\tau)| : \tau \in J\}$. Next, we define the measure of noncompactness related to monotonicity in $C(J)$; see [2, 3]. Let $\emptyset \neq Y \subset C(J)$ be a bounded set. For $y \in Y$ and $\varepsilon \geq 0$, the modulus of continuity of the function y , denoted by $\omega(y, \varepsilon)$, is defined by

$$\omega(y, \varepsilon) = \sup\{|y(t) - y(s)| : t, s \in J,\ |t - s| \leq \varepsilon\}.$$

Moreover, we let

$$\omega(Y,\varepsilon) = \sup\{\omega(y,\varepsilon) : y \in Y\}$$

and

$$\omega_0(Y) = \lim_{\varepsilon\to 0} \omega(Y,\varepsilon).$$

Define

$$d(y) = \sup_{t,s\in J,\ s\le t} (|y(t) - y(s)| - [y(t) - y(s)])$$

and

$$d(Y) = \sup_{y\in Y} d(y).$$

Notice that all functions in Y are nondecreasing on J if and only if $d(Y) = 0$.

Now, we define the map μ on $\mathfrak{M}_{C(J)}$ as

$$\mu(Y) = d(Y) + \omega_0(Y).$$

Clearly, μ satisfies all conditions in Definition 3, and therefore, it is a measure of noncompactness in $C(J)$ [3].

Definition 1.2. Let $\mathcal{P} : M \to E$ be a continuous mapping, where $\emptyset \ne M \subset E$. Suppose that $\mathcal{P}$ maps bounded sets onto bounded sets. Let Y be any bounded subset of M with $\mu(\mathcal{P}Y) \le \alpha\mu(Y)$, $\alpha \ge 0$, then $\mathcal{P}$ is called verify the Darbo condition with respect to a measure of noncompactness μ.

In the case $\alpha < 1$, the operator $\mathcal{P}$ is said to be a contraction with respect to μ.

Theorem 1.3 (See [10])**.** *Let $\emptyset \ne \Omega \subset E$ be a closed, bounded and convex set. If $\mathcal{P} : \Omega \to \Omega$ is a continuous contraction mapping with respect to μ, then $\mathcal{P}$ has a fixed point in Ω.*

We will need the following two lemmas in order to prove our results [4].

Lemma 1.4. *Let $r : J \to J$ be a continuous function and $y \in C(J)$. If, for $t \in J$,*

$$(Fy)(t) = \max_{[0,\sigma(t)]} |y(\tau)|,$$

then $Fy \in C(J)$.

Lemma 1.5. *Let (y_n) be a sequence in $C(J)$ and $y \in C(J)$. If (y_n) converges to $y \in C(J)$, then (Fy_n) converges uniformly to Fy uniformly on J.*

2 Main Theorem

Let us consider the following assumptions:

(a_1) $f \in C(J)$. Moreover, f is nondecreasing and nonnegative on J.

(a_2) The operator $T : C(J) \to C(J)$ is continuous and satisfies the Darbo condition with a constant c for the measure of noncompactness μ . Moreover, $Ty \geq 0$ if $y \geq 0$.

(a_3) There exist constants $a, b \geq 0$ such that $|(Ty)(t)| \leq a + b\|y\| \ \forall y \in C(J),\ t \in J$.

(a_4) The function $\varphi : J \to \mathbb{R}$ is $C^1(J)$ and nondecreasing.

(a_5) The function $\kappa : J \times J \to \mathbb{R}_+$ is continuous on $J \times J$ and nondecreasing $\forall t$ and s separately. Moreover, $\kappa^* = \sup\limits_{(t,s)\in J\times J} \kappa(t, s)$.

(a_6) The function $\sigma : J \to J$ is nondecreasing and continuous on J.

(a_7) $\exists\, r_0 > 0$ such that

$$\|f\| + \frac{\kappa^* r_0(a + br_0)}{\Gamma(\beta + 1)}(\varphi(1) - \varphi(0))^\beta \leq r_0 \tag{2.1}$$

and $\dfrac{ck^* r_0}{\Gamma(\beta + 1)} < (\varphi(1) - \varphi(0))^{-\beta}$.

Now, we define two operators $\mathcal{K}$ and $\mathcal{F}$ on $C(J)$ as follows

$$(\mathcal{K}y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t, s) \max_{[0,\sigma(s)]} |y(\tau)|}{(\varphi(t) - \varphi(s))^{1-\beta}} ds \tag{2.2}$$

and

$$(\mathcal{F}y)(t) = f(t) + (Ty)(t) \cdot (\mathcal{K}y)(t), \tag{2.3}$$

respectively. Solving (1.1) is equivalent to find a fixed point of the operator $\mathcal{F}$.

Under the above assumptions, we will prove the following theorem.

Theorem 2.1. *Assume the assumptions* $(a_1) - (a_7)$ *are satisfied. Then* (1.1) *has at least one solution* $y \in C(J)$ *which is nondecreasing on* J.

Proof. First, we claim that the operator $\mathcal{F}$ transforms $C(J)$ into itself. For this, it is sufficient to show that if $y \in C(J)$, then $\mathcal{K}y \in C(J)$. Let $y \in C(J)$ and $t_1,\ t_2 \in J$ $(t_1 \leq t_2)$ such that $|t_2 - t_1| \leq \varepsilon$ for fixed $\varepsilon > 0$, then we have

$$|(\mathcal{K}y)(t_2) - (\mathcal{K}y)(t_1)|$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\beta)}\left|\int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\
&\quad\left.-\int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}}ds\right|\\
&\le \frac{1}{\Gamma(\beta)}\left|\int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\
&\quad\left.-\int_0^{t_2} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right|\\
&\quad+\frac{1}{\Gamma(\beta)}\left|\int_0^{t_2} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\
&\quad\left.-\int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right|\\
&\quad+\frac{1}{\Gamma(\beta)}\left|\int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\
&\quad\left.-\int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}}ds\right|\\
&\le \frac{1}{\Gamma(\beta)}\int_0^{t_2} \frac{\varphi'(s)|\kappa(t_2,s)-\kappa(t_1,s)|\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2} \frac{\varphi'(s)|\kappa(t_1,s)|\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds+\frac{1}{\Gamma(\beta)}\int_0^{t_1}|\kappa(t_1,s)|\\
&\quad\times\varphi'(s)\;|(\varphi(t_2)-\varphi(s))^{\beta-1}-(\varphi(t_1)-\varphi(s))^{\beta-1}|\max_{[0,\sigma(s)]}|y(\tau)|\;ds\\
&\le \frac{\|y\|}{\Gamma(\beta)}\;\omega_\kappa(\varepsilon,.)\int_0^{t_2}\frac{\varphi'(s)}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds+\frac{\kappa^*\|y\|}{\Gamma(\beta)}\\
&\quad\times\left\{\int_0^{t_1}\varphi'(s)[(\varphi(t_1)-\varphi(s))^{\beta-1}-(\varphi(t_2)-\varphi(s))^{\beta-1}]ds\right.\\
&\quad\left.+\int_{t_1}^{t_2}\frac{\varphi'(s)}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right\}\\
&= \frac{\|y\|}{\Gamma(\beta+1)}\omega_\kappa(\varepsilon,.)(\varphi(t_2)-\varphi(0))^\beta+\frac{\kappa^*\|y\|}{\Gamma(\beta+1)}\\
&\quad\times\left[(\varphi(t_1)-\varphi(0))^\beta-(\varphi(t_2)-\varphi(0))^\beta+2(\varphi(t_2)-\varphi(t_1))^\beta\right]\\
&\le \frac{\|y\|}{\Gamma(\beta+1)}\omega_\kappa(\varepsilon,.)(\varphi(t_2)-\varphi(0))^\beta+\frac{2\kappa^*\|y\|}{\Gamma(\beta+1)}(\varphi(t_2)-\varphi(t_1))^\beta\\
&\le \frac{\|y\|}{\Gamma(\beta+1)}\;\omega_\kappa(\varepsilon,.)(\varphi(1)-\varphi(0))^\beta+\frac{2\kappa^*\|y\|}{\Gamma(\beta+1)}[\omega(\varphi,\varepsilon)]^\beta,
\end{aligned}
\tag{2.4}
$$

where we used

$$
\omega_\kappa(\varepsilon,.)=\sup_{t,\;\tau\in J,\;|t-\tau|\le\varepsilon}|\kappa(t,s)-\kappa(\tau,s)|
$$

and the fact that $\varphi(t_1) - \varphi(0) \leq \varphi(t_2) - \varphi(0)$. Notice that, since the function κ is uniformly continuous on $J \times J$ and the function φ is continuous on J, then when $\varepsilon \to 0$, we have that $\omega_\kappa(\varepsilon, .) \to 0$ and $\omega(\varphi, \varepsilon) \to 0$.

Therefore, $\mathcal{K}y \in C(J)$ and consequently, $\mathcal{F}y \in C(J)$.

Now, for $t \in J$, we have

$$\begin{aligned} |(\mathcal{F}y)(t)| &\leq \left| f(t) + \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}} ds \right| \\ &\leq \|f\| + \frac{a + b\|y\|}{\Gamma(\beta)} \int_0^t \kappa(t,s) \frac{\varphi'(s)}{(\varphi(t)-\varphi(s))^{1-\beta}} \max_{[0,\sigma(s)]} |y(\tau)| ds \\ &\leq \|f\| + \frac{(a + b\|y\|)\kappa^* \|y\|}{\Gamma(\beta+1)} (\varphi(t) - \varphi(0))^\beta. \end{aligned}$$

Hence

$$\|\mathcal{F}y\| \leq \|f\| + \frac{(a + b\|y\|)\kappa^* \|y\|}{\Gamma(\beta+1)} (\varphi(1) - \varphi(0))^\beta.$$

By assumption (a_7), if $\|y\| \leq r_0$, we get

$$\begin{aligned} \|\mathcal{F}y\| &\leq \|f\| + \frac{(a + br_0)\kappa^* r_0}{\Gamma(\beta+1)} (\varphi(1) - \varphi(0))^\beta \\ &\leq r_0. \end{aligned}$$

Therefore, $\mathcal{F}$ maps B_{r_0} into itself.

Next, we consider the operator $\mathcal{F}$ on the set $B_{r_0}^+ = \{y \in B_{r_0} : y(t) \geq 0, \ \forall t \in J\}$. It is clear that $B_{r_0}^+ \neq \emptyset$ is closed, convex and bounded. By these facts and our assumptions, we obtain $\mathcal{F}$ maps $B_{r_0}^+$ into itself.

In what follows, we will show that $\mathcal{F}$ is continuous on $B_{r_0}^+$. For this, let (y_n) be a sequence in $B_{r_0}^+$ such that $y_n \to y$ and we will show that $\mathcal{F}y_n \to \mathcal{F}y$. We have, for $t \in J$,

$$\begin{aligned} &|(\mathcal{F}y_n)(t) - (\mathcal{F}y)(t)| \\ &= \left| \frac{(Ty_n)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y_n(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}} ds \right. \\ &\quad \left. - \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}} ds \right| \\ &\leq \left| \frac{(Ty_n)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y_n(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}} ds \right. \\ &\quad \left. - \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y_n(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}} ds \right| \\ &\quad + \left| \frac{(Ty)(t)}{\Gamma(\beta)} \int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y_n(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}} ds \right. \end{aligned}$$

$$-\frac{(Ty)(t)}{\Gamma(\beta)}\int_0^t \frac{\varphi'(s)\kappa(t,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}}ds\Bigg|$$
$$\leq \frac{|(Ty_n)(t)-(Ty)(t)|}{\Gamma(\beta)}\int_0^t \frac{\varphi'(s)|\kappa(t,s)|\max_{[0,\sigma(s)]}|y_n(\tau)|}{(\varphi(t)-\varphi(s))^{1-\beta}}ds$$
$$+\frac{|(Ty)(t)|}{\Gamma(\beta)}\int_0^t \frac{\varphi'(s)|\kappa(t,s)|}{(\varphi(t)-\varphi(s))^{1-\beta}}\left|\max_{[0,\sigma(s)]}|y_n(\tau)|-\max_{[0,\sigma(s)]}|y(\tau)|\right|ds.$$

By applying Lemma 1.5, we get

$$\begin{aligned}\|\mathcal{F}y_n-\mathcal{F}y\| \;\leq\; & \frac{\kappa^* r_0(\varphi(1)-\varphi(0))^\beta\|Ty_n-Ty\|}{\Gamma(\beta+1)}\\ & +\frac{\kappa^*(a+br_0)(\varphi(1)-\varphi(0))^\beta\|y_n-y\|}{\Gamma(\beta+1)}. \qquad (2.5)\end{aligned}$$

By the continuity of T, $\exists n_1\in\mathbb{N}$ such that

$$\|Ty_n-Ty\|\leq \frac{\varepsilon\Gamma(\beta+1)}{2\kappa^* r_0(\varphi(1)-\varphi(0))^\beta},\ \ \forall n\geq n_1.$$

Also, $\exists n_2\in\mathbb{N}$ such that

$$\|y_n-y\|\leq \frac{\varepsilon\Gamma(\beta+1)}{2\kappa^*(a+br_0)(\varphi(1)-\varphi(0))^\beta},\ \ \forall n\geq n_2.$$

Now, take $n\geq \max\{n_1,n_2\}$, then (2.5) gives us that

$$\|\mathcal{F}y_n-\mathcal{F}y\|\leq\varepsilon.$$

This shows that $\mathcal{F}$ is continuous in $B_{r_0}^+$.

Next, let $Y\subset B_{r_0}^+$ be a nonempty set. Let us choose $y\in Y$ and $t_1,t_2\in J$ with $|t_2-t_1|\leq\varepsilon$ for fixed $\varepsilon>0$. Since no generality will loss, we will assume that $t_2\geq t_1$. Then, by using our assumptions and (2.4), we obtain

$$\begin{aligned}&|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)|\\ &\leq |f(t_2)-f(t_1)|+|(Ty)(t_2)(\mathcal{K}y)(t_2)-(Ty)(t_2)(\mathcal{K}y)(t_1)|\\ &\quad+|(Ty)(t_2)(\mathcal{K}y)(t_1)-(Ty)(t_1)(\mathcal{K}y)(t_1)|\\ &\leq \omega(f,\varepsilon)+|(Ty)(t_2)|\,|(\mathcal{K}y)(t_2)-(\mathcal{K}y)(t_1)|+|(Ty)(t_2)-(Ty)(t_1)|\,|(\mathcal{K}y)(t_1)|\\ &\leq \omega(f,\varepsilon)+\frac{(a+b\|y\|)\|y\|}{\Gamma(\beta+1)}\left[\omega_\kappa(\varepsilon,.)(\varphi(1)-\varphi(0))^\beta+2\kappa^*(\omega(\varphi,\varepsilon))^\beta\right]\\ &\quad+\frac{\omega(Ty,\varepsilon)\|y\|\kappa^*(\varphi(t_1)-\varphi(0))^\beta}{\Gamma(\beta+1)}\\ &\leq \omega(f,\varepsilon)+\frac{r_0(a+br_0)}{\Gamma(\beta+1)}\left[\omega_\kappa(\varepsilon,.)(\varphi(1)-\varphi(0))^\beta+2\kappa^*(\omega(\varphi,\varepsilon))^\beta\right]\end{aligned}$$

$$+\frac{\kappa^* r_0(\varphi(1)-\varphi(0))^\beta}{\Gamma(\beta+1)}\omega(Ty,\varepsilon).$$

Hence,

$$\begin{aligned}\omega(\mathcal{F}y,\varepsilon) &\leq \omega(f,\varepsilon)+\frac{r_0(a+br_0)}{\Gamma(\beta+1)}\left[\omega_\kappa(\varepsilon,.)(\varphi(1)-\varphi(0))^\beta+2\kappa^*(\omega(\varphi,\varepsilon))^\beta\right]\\ &\quad +\frac{\kappa^* r_0(\varphi(1)-\varphi(0))^\beta}{\Gamma(\beta+1)}\omega(Ty,\varepsilon).\end{aligned}$$

Consequently,

$$\begin{aligned}\omega(\mathcal{F}Y,\varepsilon) &\leq \omega(f,\varepsilon)+\frac{r_0(a+br_0)}{\Gamma(\beta+1)}\left[\omega_\kappa(\varepsilon,.)(\varphi(1)-\varphi(0))^\beta+2\kappa^*(\omega(\varphi,\varepsilon))^\beta\right]\\ &\quad +\frac{\kappa^* r_0(\varphi(1)-\varphi(0))^\beta}{\Gamma(\beta+1)}\omega(TY,\varepsilon).\end{aligned}$$

The uniform continuity of the function κ on $J\times J$ and the continuity of the functions f and φ on J, implies the last inequality becomes

$$\omega_0(\mathcal{F}Y)\leq\frac{\kappa^* r_0(\varphi(1)-\varphi(0))^\beta}{\Gamma(\beta+1)}\omega_0(TY). \tag{2.6}$$

In the next step, fix arbitrary $y\in Y$ and $t_1,t_2\in J$ with $t_2>t_1$. Then, by our assumptions, we have

$$\begin{aligned}&|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)|-[(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)]\\ &=\left|f(t_2)+\frac{(Ty)(t_2)}{\Gamma(\beta)}\int_0^{t_2}\frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\ &\quad\left.-f(t_1)-\frac{(Ty)(t_1)}{\Gamma(\beta)}\int_0^{t_1}\frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}}ds\right|\\ &\quad-\left[f(t_2)+\frac{(Ty)(t_2)}{\Gamma(\beta)}\int_0^{t_2}\frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\ &\quad\left.-f(t_1)-\frac{(Ty)(t_1)}{\Gamma(\beta)}\int_0^{t_1}\frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}}ds\right]\\ &\leq\{|f(t_2)-f(t_1)|-[f(t_2)-f(t_1)]\}\\ &\quad+\left|\frac{(Ty)(t_2)}{\Gamma(\beta)}\int_0^{t_2}\frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\\ &\quad\left.-\frac{(Ty)(t_1)}{\Gamma(\beta)}\int_0^{t_2}\frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right|\\ &\quad+\left|\frac{(Ty)(t_1)}{\Gamma(\beta)}\int_0^{t_2}\frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds\right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}} ds \Bigg| \\
&\quad - \Bigg\{ \Bigg[\frac{(Ty)(t_2)}{\Gamma(\beta)} \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&\quad - \frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \Bigg] \\
&\quad + \Bigg[\frac{(Ty)(t_1)}{\Gamma(\beta)} \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&\quad - \frac{(Tx)(t_1)}{\Gamma(\beta)} \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}} ds \Bigg] \Bigg\} \\
&\leq \frac{\{|(Ty)(t_2)-(Ty)(t_1)| - [(Ty)(t_2)-(Ty)(t_1)]\}}{\Gamma(\beta)} \\
&\quad \times \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&\quad + \frac{(Ty)(t_1)}{\Gamma(\beta)} \Bigg\{ \Bigg| \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&\quad - \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}} ds \Bigg| \\
&\quad - \Bigg[\int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&\quad - \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}} ds \Bigg] \Bigg\}. \qquad (2.7)
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds - \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}} ds \\
&= \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds - \int_0^{t_2} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&+ \int_0^{t_2} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds - \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&+ \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds - \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}} ds \\
&= \int_0^{t_2} \frac{\varphi'(s)(\kappa(t_2,s)-\kappa(t_1,s))\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&+ \int_{t_1}^{t_2} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}} ds \\
&+ \int_0^{t_1} \kappa(t_1,s)[(\varphi(t_2)-\varphi(s))^{\beta-1} - (\varphi(t_1)-\varphi(s))^{\beta-1}] \max_{[0,\sigma(s)]} |y(\tau)| ds.
\end{aligned}
$$

Since $\kappa(t_2,s) \geq \kappa(t_1,s)$ ($\kappa(t,s)$ is nondecreasing with respect to t), we have

$$\int_0^{t_2} \frac{\varphi'(s)(\kappa(t_2,s)-\kappa(t_1,s))\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds \geq 0 \tag{2.8}$$

and, since $\dfrac{1}{(\varphi(t_2)-\varphi(s))^{1-\beta}} \geq \dfrac{1}{(\varphi(t_1)-\varphi(s))^{1-\beta}}$ for $s \in [0,t_1)$ then

$$\begin{aligned}
&\int_0^{t_1} \varphi'(s)\kappa(t_1,s)[(\varphi(t_2)-\varphi(s))^{\beta-1}-(\varphi(t_1)-\varphi(s))^{\beta-1}]\max_{[0,\sigma(s)]}|y(\tau)|ds \\
&\quad + \int_{t_1}^{t_2} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds \\
&\geq \int_0^{t_1} \varphi'(s)\kappa(t_1,t_1)[(\varphi(t_2)-\varphi(s))^{\beta-1}-(\varphi(t_1)-\varphi(s))^{\beta-1}]\max_{[0,\sigma(t_1)]}|y(\tau)|ds \\
&\quad + \int_{t_1}^{t_2} \frac{\varphi'(s)\kappa(t_1,t_1)\max_{[0,\sigma(t_1)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds \\
&= \kappa(t_1,t_1)\max_{[0,\sigma(t_1)]}|y(\tau)|\left[\int_0^{t_2}\frac{\varphi'(s)ds}{(\varphi(t_2)-\varphi(s))^{1-\beta}} - \int_0^{t_1}\frac{\varphi'(s)ds}{(\varphi(t_1)-\varphi(s))^{1-\beta}}\right] \\
&= \kappa(t_1,t_1)\frac{(\varphi(t_2)-\varphi(0))^{\beta}-(\varphi(t_1)-\varphi(0))^{\beta}}{\beta}\max_{[0,\sigma(t_1)]}|y(\tau)| \\
&\geq 0.
\end{aligned} \tag{2.9}$$

Finally, (2.8) and (2.9) imply that

$$\int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds - \int_0^{t_1} \frac{\varphi'(s)\kappa(t_1,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_1)-\varphi(s))^{1-\beta}}ds \geq 0.$$

The above inequality and (2.7) leads us to

$$\begin{aligned}
&|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)| - [(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)] \\
&\leq \frac{|(Ty)(t_2)-(Ty)(t_1)| - [(Ty)(t_2)-(Ty)(t_1)]}{\Gamma(\beta)} \\
&\quad \times \int_0^{t_2} \frac{\varphi'(s)\kappa(t_2,s)\max_{[0,\sigma(s)]}|y(\tau)|}{(\varphi(t_2)-\varphi(s))^{1-\beta}}ds \\
&\leq \frac{\kappa^* r_0(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)}d(Ty).
\end{aligned}$$

Thus,

$$d(\mathcal{F}y) \leq \frac{\kappa^* r_0(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)}d(Ty)$$

and therefore,

$$d(\mathcal{F}Y) \leq \frac{\kappa^* r_0(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)}d(TY). \tag{2.10}$$

Finally, (2.6) and (2.10) give us that

$$\omega_0(\mathcal{F}Y) + d(\mathcal{F}Y) \leq \frac{\kappa^* r_0(\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta+1)}(\omega_0(\mathcal{F}Y) + d(TY))$$

or

$$\begin{aligned} \mu(\mathcal{F}Y) &\leq \frac{r_0\kappa^*(\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta+1)}\mu(TY) \\ &\leq \frac{\kappa^* c r_0(\varphi(1) - \varphi(0))^\beta}{\Gamma(\beta+1)}\mu(Y). \end{aligned}$$

Since $\dfrac{\kappa^* r_0 c}{\Gamma(\beta+1)} < (\varphi(1) - \varphi(0))^{-\beta}$, $\mathcal{F}$ is a contraction operator with respect to μ.

Finally, by Theorem 1.3, $\mathcal{F}$ has at least one fixed point, or equivalently, (1.1) has at least one nondecreasing solution in B_{r_0}. This finishes our proof. □

Next, we present the following numerical example in order to illustrate our results.

Example 2.2. Let us consider the following integral equation with maximum

$$y(t) = \arctan t + \frac{y(t)}{5\Gamma(1/2)} \int_0^t \frac{\sqrt{t^2+s^2}\max_{[0,\ln(s+1)]}|y(\tau)|}{2\sqrt{s+1}\sqrt{\sqrt{t+1}-\sqrt{s+1}}} ds, \ t \in J. \tag{2.11}$$

Notice that (2.11) is a particular case of (1.1), where $f(t) = \arctan t$, $(Ty)(t) = y(t)/5$, $\beta = 1/2$, $\varphi(s) = \sqrt{s+1}$, $\kappa(t,s) = \sqrt{t^2+s^2}$ and $\sigma(t) = \ln(t+1)$.

It is not difficult to see that assumptions (a_1), (a_2), (a_3), (a_4) , (a_5) and (a_6) are verified with $\|f\| = \pi/4$, $c = 1/5$, $a = 0$, $b = 1/5$ and $\kappa^* = \sqrt{2}$.

Now, the inequality (2.1) in assumption (a_7) takes the expression

$$\frac{\pi}{4} + \frac{\sqrt{2}\sqrt{\sqrt{2}-1}}{5\Gamma(3/2)} r_0^2 \leq r_0$$

which is satisfied by $r_0 = 1$. Moreover,

$$\frac{c\kappa^* r_0}{\Gamma(\beta+1)} = \frac{\sqrt{2}}{5\Gamma(3/2)} \cong 0.32 < (\varphi(1) - \varphi(0))^{-\beta} = \frac{1}{\sqrt{\sqrt{2}-1}} \cong 1.56.$$

Therefore, by Theorem 2.1, (2.11) has at least one continuous and nondecreasing solution which is located in the ball B_1.

Acknowledgements

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References

[1] D. Bainov and D. Mishev, *Oscillation Theory for Neutral Differenial Equations with Delay*, Hilger, Bristol, 1991.

[2] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, 60, Marcel Dekker, New York, 1980.

[3] J. Banaś and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math.* **41** (2001), 13–23.

[4] J. Caballero, B. López and K. Sadarangani, On monotonic solutions of an integral equation of Volterra type with supremum, *J. Math. Anal. Appl.* **305** (2005), 304–315.

[5] J. Caballero, B. López, K. Sadarangani, Existence of nondecreasing and continuous solutions for a nonlinear integral equation with supremum in the kernel, *Z. Anal. Anwend.* **26** (2007) 195–205.

[6] J. Caballero, M. A. Darwish and K. Sadarangani, Solvability of a fractional hybrid initial value problem with supremum by using measures of noncompactness in Banach algebras, *Appl. Math. Comput.* **224** (2013), 553–563.

[7] M. A. Darwish, On monotonic solutions of a singular quadratic integral equation with supremum, *Dynam. Systems Appl.* **17** (2008), 539–549.

[8] M. A. Darwish and K. Sadarangani, Nondecreasing solutions of a quadratic Abel equation with supremum in the kernel, *Appl. Math. Comput.* **219** (2013), 7830–7836.

[9] M. A. Darwish and K. Sadarangani, On a quadratic integral equation with supremum involving Erdélyi-Kober fractional order, *Math. Nachr.* **288** (2015), 566–576.

[10] J. Dugundji and A. Granas, *Fixed Point Theory*, Monografie Mathematyczne, PWN, Warsaw, 1982.

[11] S. G. Hristova and D. D. Bainov, Monontone–iterative techniques for V. Lakshmikantham for a boundary value problem for systems of impulsive differential equations with "supremum", *J. Math. Anal. Appl.* **172** (1993), 339–352.