ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-769 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that the inequality

$$\frac{F_1^6}{(F_1^4+F_1^2F_2^2+F_2^4)(\sqrt{2}F_1+F_2)}+\frac{F_2^6}{(F_2^4+F_2^2F_3^2+F_3^4)(\sqrt{2}F_2+F_3)}+\cdots$$
$$+\frac{F_{n-1}^6}{(F_{n-1}^4+F_{n-1}^2F_n^2+F_n^4)(\sqrt{2}F_{n-1}+F_n)}+\frac{F_n^6}{(F_n^4+F_n^2F_1^2+F_1^4)(\sqrt{2}F_n+F_1)}$$
$$\geq\frac{\sqrt{2}-1}{3}(F_{n+2}-1)$$

holds for all positive integers n.

H-770 Proposed by H. Ohtsuka, Saitama, Japan.

For an integer $n\geq 0$, find a closed form expression for the sum

$$S(n):=\sum_{k=0}^{n}\frac{1}{(L_{2^{k+1}}+1)(L_{2^k}+c)(L_{2^{k+1}}+c)\cdots(L_{2^n}+c)},$$

where $c\neq -L_{2^k}$ for $0\leq k\leq n$.

H-771 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Let $m>0$ and $\Gamma:(0,\infty)\to(0,\infty)$ be the gamma function. Calculate

$$\lim_{n\to\infty}\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}}\Gamma\left(\frac{x}{n}\sqrt[n]{F_n}^{m}\right)dx.$$

H-772 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

If ABC is a nonisosceles triangle then prove that

$$\sum_{\substack{cyclic\\permutations}} \frac{a^8}{(bF_n^2 + cF_{n+1}^2)(a-b)^2(a-c)^2} > \frac{288r^3\sqrt{3}}{F_{2n+1}}.$$

Here, a, b, c, r are the lengths of the sides and the radius of the inscribed circle of the triangle ABC, respectively.

SOLUTIONS

On Sums of Squares of Fibonomial Coefficients

H-738 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 51, No. 2, May 2013)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=0}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2 = \frac{L_{4n-1}+1}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2,$

(ii) $\displaystyle\sum_{\substack{a+b=2n\\a,b>0}} L_a L_b \binom{2n-1}{a}_F \binom{2n-1}{b}_F = \frac{L_{4n-1}-3}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2.$

Solution by the proposer

(i) The following identities are known (see [2]):

(A) $F_{2n} = F_n L_n$,
(B) $L_n^2 = 5F_n^2 + 4(-1)^n$,
(C) $L_n L_m = L_{n+m} + (-1)^m L_{n-m}$,
(D) $5F_n F_m = L_{n+m} - (-1)^m L_{n-m}$.

We use two properties

$$\binom{n}{k}_F = \frac{F_n}{F_k}\binom{n-1}{k-1}_F \quad \text{and} \quad \binom{n}{k}_F = \binom{n}{n-k}_F. \tag{1}$$

By (1) above, we have

$$\sum_{k=0}^{m} F_k^2 \binom{m}{k}_F^2 = \sum_{k=1}^{m} F_m^2 \binom{m-1}{k-1}_F^2 = F_m^2 \sum_{k=0}^{m-1} \binom{m-1}{k}_F^2. \tag{2}$$

For odd m we have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}_F^2 = 0, \tag{3}$$

because

$$LHS = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{m-k}^2 = -\sum_{k=0}^{m} (-1)^k \binom{m}{k}_F^2 = -LHS.$$

It was shown in [1] that

$$\sum_{k=0}^{2n} \binom{2n}{k}_F^2 = \prod_{k=1}^{n} \frac{L_{2k}F_{2(2k-1)}}{F_{2k}}. \tag{4}$$

We have

$$\begin{aligned}
\sum_{k=0}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2 &= 5\sum_{k=0}^{2n-1} F_k^2 \binom{2n-1}{k}_F^2 + 4\sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k}_F^2 \qquad \text{(by } (B)\text{)}\\
&= 5\sum_{k=0}^{2n-1} F_k^2 \binom{2n-1}{k}_F \qquad \text{(by (3))}\\
&= 5F_{2n-1}^2 \sum_{k=0}^{2n-2} \binom{2n-2}{k}_F^2 \qquad \text{(by (2))}\\
&= \frac{5F_{2n-1}^2 F_{2n}}{L_{2n}F_{2(2n-1)}} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 \qquad \text{(by (4))}\\
&= \frac{5F_{2n}F_{2n-1}}{L_{2n}L_{2n-1}} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 \qquad \text{(by } (A)\text{)}\\
&= \frac{L_{4n-1}+1}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 \qquad \text{(by } (C) \text{ and } (D)\text{)}.
\end{aligned}$$

(ii) The following identity is known (see [3]):

$$2\binom{m}{k}_F = L_k \binom{m-1}{k}_F + L_{m-k} \binom{m-1}{k-1}_F.$$

Squaring both sides of the above identity, we have

$$4\binom{m}{k}_F^2 = L_k^2 \binom{m-1}{k}_F^2 + L_{m-k}^2 \binom{m}{k-1}_F^2 + 2L_kL_{m-k} \binom{m-1}{k}_F \binom{m-1}{k-1}_F.$$

Using this identity, we have

$$\begin{aligned}
4\sum_{k=1}^{m-1} \binom{m}{k}_F^2 &= \sum_{k=1}^{m-1} \left(L_k^2 \binom{m-1}{k}_F^2 + L_{m-k}^2 \binom{m-1}{m-k}_F^2 + 2L_kL_{m-k} \binom{m-1}{k}_F \binom{m-1}{k-1}_F \right)\\
&= 2\sum_{k=1}^{m-1} L_k^2 \binom{m-1}{k}_F^2 + 2\sum_{\substack{a+b=m\\ a,b>0}} L_aL_b \binom{m-1}{a}_F \binom{m-1}{b}_F.
\end{aligned}$$

Therefore, putting $m = 2n$, we have

$$\sum_{\substack{a+b=2n\\ a,b>0}} L_a L_b \binom{2n-1}{a}_F \binom{2n-1}{b}_F = 2\sum_{k=1}^{2n-1}\binom{2n}{k}_F^2 - \sum_{k=1}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2$$

$$= 2\left(\sum_{k=0}^{2n}\binom{2n}{k}_F^2 - 2\right) - \left(\sum_{k=0}^{2n-1} L_k^2\binom{2n-1}{k}_F^2 - 4\right)$$

$$= 2\sum_{k=0}^{2n}\binom{2n}{k}_F^2 - \frac{L_{4n-1}+1}{L_{4n-1}-1}\sum_{k=0}^{2n}\binom{2n}{k}^2 \qquad \text{(by } (i))$$

$$= \frac{L_{4n-1}-3}{L_{4n-1}-1}\sum_{k=0}^{2n}\binom{2n}{k}_F^2.$$

References

[1] E. Kılıç, I. Akkuş and H. Ohtsuka, *Some generalized Fibonomial sums related to the Gaussian q-binomial sums*, Bull. Math. Sci. Math. Roumanie Tome, **55.103** (2012), 51–61.

[2] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, DOVER, 2008.

[3] E. W. Weisstein, *Fibonomial Coefficient*, From MathWorld-A Wolfram Web Resource `http://mathworld.wolfram.com/FibonomialCoefficient.html`.

More Sums of Squares of Fibonomial Coefficients

H-739 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 51, No. 3, August 2013)

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;m}$ by

$$\binom{n}{k}_{F;m} = \prod_{j=1}^{k}\frac{F_{m(n-j+1)}}{F_{mj}} \quad (\text{for } n \geq k > 0) \quad \text{with} \quad \binom{n}{0}_{F;m} = 1.$$

Prove that

$$\sum_{k=0}^{n}\alpha^{2mn(n-2k)}\binom{n}{k}_{F;2m}^2 = \sum_{k=0}^{2n}(-1)^{(m+1)(n-k)}\binom{2n}{k}_{F;m}^2.$$

Solution by E. Kılıç and I. Akkuş, Turkey.

Our way is to mechanically compute the desired sums by the qZeilberger algorithm (qZeilberger's own version, which is a Mathematica program).

The Gaussian q-binomial coefficient $\binom{n}{m}_q$ is defined, for all real n and integers m with $m \geq 0$,

$$\binom{n}{k}_{q^m} := \frac{(q^m;q^m)_n}{(q^m;q^m)_k(q^m;q^m)_{n-k}}$$

and as zero otherwise, where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The Binet form is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q}$$

with $q = \beta/\alpha = -\alpha^{-2}$. The link between the Fibonomial and Gaussian q-binomial coefficients is

$$\binom{n}{k}_{F;m} = \alpha^{mk(n-k)}\binom{n}{k}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

The claimed identity is

$$\sum_{k=0}^{n} \alpha^{2mn(n-2k)}\binom{n}{k}_{F;2m}^2 = \sum_{k=0}^{2n}(-1)^{(m+1)(n-k)}\binom{2n}{k}_{F;m}^2.$$

If we convert the claimed identity into q-notation, then we rewrite its LHS and RHS in terms of q-binomials as

$$(-1)^{mn}q^{-mn^2}\sum_{k=0}^{n} q^{2mk^2}\binom{n}{k}_{q^{2m}}^2,$$

and

$$(-1)^{n(m+1)}\sum_{k=0}^{2n}(-1)^k q^{mk(k-2n)}\binom{2n}{k}_{q^m}^2,$$

respectively.

The algorithm gives us the recurrence relation for both LHS and RHS :

$$\texttt{SUM[n]=}\frac{(-1)^m q^{m(1-2n)}(1+q^{2mn})(1-q^{2m(2n-1)})}{1-q^{2nm}}\texttt{SUM[n-1]} .$$

Since LHS(0)=RHS(0)= 1, we get that they are equal.

Also solved by the proposer.

Counting Dominating Sets in Paths

H-740 Proposed by Saeid Alikhani, Yazd, Iran and Emeric Deutsch, Brooklyn, NY. (Vol. 51, No. 3, August 2013)

Given a simple graph G with vertex set V, a dominating set of G is any subset S of V such that every vertex in $V \backslash S$ is adjacent to at least one vertex in S. Find the number of dominating sets of the path P_n with n vertices.

Solution by Harris Kwong, Fredonia, NY.

Let d_n denote the number of dominating sets of P_n. Label the vertices of P_n as $v_1, v_2, \ldots, v_n$ such that the vertices v_i and v_{i+1} are adjacent for $1 \le i \le n-1$. It is clear that $d_2 = 3$, because P_2 has three dominating sets: $\{v_1\}$, $\{v_2\}$, and $\{v_1, v_2\}$.

Consider $n \ge 3$. A dominating set S may or may not contain v_n. If it does, then $S \setminus \{v_n\}$ is a dominating set of P_{n-1}. Conversely, any dominating set of P_{n-1} can be expanded to a dominating set of P_n by including v_n in it. Hence, there are d_{n-1} choices for S in this case.

If S does not contain v_n, it must contain v_{n-1} so as to dominate v_n. Then $S \setminus \{v_{n-1}\}$ is a dominating set of P_{n-2}. Conversely, any dominating set of P_{n-2} can be expanded to a dominating set of P_n that also contains v_{n-1} but not v_n by adding v_{n-1} to it. Thus, there are d_{n-2} choices for S in this case.

We have just proved that $d_n = d_{n-1} + d_{n-2}$ for $n \geq 3$. When $n = 3$, after we remove v_2 and v_3 in the second case, we are left with only one vertex v_1, which may or may not be contained in S. In this regard, we may define $d_1 = 2$. Along with $d_2 = 3$, we see that $d_n = F_{n+2}$.

Also solved by the proposers.

An Application of the AM-GM Inequality

H-741 Proposed by Charlie Cook, Sumter, South Carolina. (Vol. 51, No. 3, August 2013)

If $n \geq 2$ and $m \geq 1$, then

$$m(L_n - F_n)(L_nF_n)^{(m-1)/2} \leq L_n^m - F_n^m,$$

where L_n and F_n are the Lucas and Fibonacci numbers, respectively.

Solution by Ángel Plaza, Las Palmas, Spain.

The proposed inequality is a particular case of the following more general inequality: If $0 < y \leq x$ and $m \geq 1$, then $m(x-y)(xy)^{(m-1)/2} \leq x^m - y^m$.

Proof. Last inequality may be written as

$$m(xy)^{(m-1)/2} \leq x^{m-1} + \cdots + x^{m-1-j}y^j + \cdots + y^{m-1}$$

$$(xy)^{(m-1)/2} \leq \frac{x^{m-1} + \cdots + x^{m-1-j}y^j + \cdots + y^{m-1}}{m}.$$

which follows immediately by the AM-GM inequality.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Robinson Higuita, Harris Kwong, Hideyuki Ohtsuka, and the proposer.

Errata: In problem **H-765**, the right-hand side of (iii) should be "$(L_nL_{n+1} - 2)^2$" instead of "$(L_nL_{n+1} - 1)^2$".

The published solution to **H-737** works for primes $p \geq 5$, but not for $p = 3$. Indeed, when p is 3, F_p is not necessarily prime to L_{mp} and is not prime to L_p. Here is a fix from the same solver.

For $p = 3$, we have $\binom{3n-1}{3-1}_F = F_{3n-1}F_{3n-2} := a_n$, say. Then a_n is a linear combination of α^{6n}, β^{6n} and $(-1)^n$, where α and β are the zeros of $x^2 - x - 1$. Thus, (a_n) is a linear recurrence with characteristic polynomial $(x^2 - L_6x + (\alpha\beta)^6)(x+1)$, which modulo 16 is $x^3 - x^2 - x + 1$. Therefore, we may prove that $a_n \equiv (-1)^{n-1} \pmod{F_3^2L_3 = 16}$ by induction as $a_{n+3} \equiv a_{n+2} + a_{n+1} - a_n \pmod{16}$.