ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2015. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1166 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{k=1}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k} - 2} = \frac{15-\sqrt{5}}{10}.$$

B-1167 Proposed by Atara Shriki and Opher Liba, Oranim College of Education, Israel.

Find the area of the polygon with vertices

$$(F_1, F_2), (F_3, F_4), (F_5, F_6), \ldots, (F_{2n-1}, F_{2n}) \quad \text{for } n \geq 3.$$

B-1168 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.**

Let n be a nonnegative integer. Find the minimum value of

$$\frac{F_n}{2F_n+3L_n(1+F_n)}+\frac{F_nL_n}{2F_nL_n+3(F_n+L_n)}+\frac{L_n}{2L_n+3F_n(1+L_n)}.$$

B-1169 **Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

Let p be a positive integer. Prove that for any integer $n>1$

$$\frac{F_{n+2}}{2}\leq\sqrt[p]{\frac{F_{n+1}^{p+1}-F_n^{p+1}}{(p+1)F_{n-1}}}\leq\sqrt[p]{\frac{F_n^p+F_{n+1}^p}{2}}.$$

B-1170 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Let $a>0$, $b>0$, and $c\geq 0$. Find

(i)

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!aF_{n+1}^c}}-\frac{n^2}{\sqrt[n]{n!bF_n^c}}\right),$$

(ii)

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!aL_{n+1}^c}}-\frac{n^2}{\sqrt[n]{n!bL_n^c}}\right).$$

SOLUTIONS

It Can Be Sharpened

B-1146 **Proposed by Titu Zvonaru, Comăneşti, Romania. (Vol. 52.2, May 2014)**

Prove that $F_{n+2}^2\geq 5F_{n-1}^2$ for all integers $n\geq 1$.

Solution by Brian D. Beasley, Department of Mathematics, Presbyterian College, Clinton, SC.

We claim that for any positive real number c with $c<\alpha^3$, $F_{n+2}^2\geq c^2F_{n-1}^2$ for sufficiently large n. In particular, $F_{n+2}^2\geq 9F_{n-1}^2$ for $n\geq 1$.

Fix a real number c with $0<c<\alpha^3=2+\sqrt{5}$. Since $F_{n+2}\geq cF_{n-1}$ trivially for $n=1$, we move on to consider $n>1$. Then $F_{n+2}\geq cF_{n-1}$ if and only if

$$\alpha^{n-1}(\alpha^3-c)\geq\beta^{n-1}(\beta^3-c).$$

If n is odd, then this inequality holds trivially, as its left side is positive while its right side is negative. If n is even, then this inequality holds if and only if $d^{n-1} \geq e$, where $d = \alpha/(-\beta) = \alpha + 1 > 1$ and $e = (c - \beta^3)/(\alpha^3 - c) > 0$. Thus the inequality holds if and only if

$$n \geq \frac{\log(e)}{\log(d)} + 1,$$

so we conclude that $F_{n+2} \geq cF_{n-1}$ for sufficiently large n. In particular, for $c = 3$, we have $F_{n+2} \geq cF_{n-1}$ for $n \geq 2$.

Addendum. We note that we may let c be arbitrarily close to α^3. For example, if $c = 4.2$, then $F_{n+2} \geq cF_{n-1}$ for $n \geq 6$; if $c = 4.23$, then $F_{n+2} \geq cF_{n-1}$ for $n \geq 8$; etc.

Also solved by Carlos Alirio Rico Acevedo, Gurdial Arora, Charles K. Cook, Kenneth B. Davenport, Pedro Fernando Fernández Espinoza, Dmitry Fleischman, Ralph P. Grimaldi, Russell Jay Hendel, (Tia Herring, Gary Knight, Shane Latchman, Ludwig Murillo, Zibusiso Ndimande, and Eyob Tarekegn; students) (jointly), Wei-Kai Lai, Kathleen E. Lewis, Sydney Marks and Leah Seader (jointly), Reiner Martin, Grey Martinsen and Pantelimon Stănică (jointly), Ángel Plaza, Jason L. Smith, and the proposer.

Close This Sum

B-1147 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 52.2, May 2014)

Find a closed form expression for

$$\sum_{\substack{0\leq a,b,c\leq n\\ a+b+c=n}} \frac{F_aF_bF_c}{a!b!c!}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

The sum is in the form of a convolution, its generating function is

$$G(x) = \sum_{n\geq 0}\left(\sum_{\substack{0\leq a,b,c,\leq n\\ a+b+c=n}} \frac{F_aF_bF_c}{a!b!c!}\right)x^n = \left(\sum_{a\geq 0}\frac{F_a}{a!}x^a\right)\left(\sum_{b\geq 0}\frac{F_b}{b!}x^b\right)\left(\sum_{c\geq 0}\frac{F_c}{c!}x^c\right).$$

Hence,

$$G(x) = \left(\sum_{k\geq 0}\frac{\alpha^k - \beta^k}{\alpha - \beta}\cdot\frac{x^k}{k!}\right)^3 = \left(\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}\right)^3.$$

Since $2\alpha + \beta = \alpha + 1 = \alpha^2$, and $\alpha + 2\beta = \beta + 1 = \beta^2$, we find

$$\begin{aligned}(\alpha - \beta)^3 G(x) &= e^{3\alpha x} - e^{(2\alpha+\beta)x} + 3e^{(\alpha+2\beta)x} - e^{3\beta x}\\ &= e^{3\alpha x} - e^{3\beta x} - 3(e^{\alpha^2 x} - e^{\beta^2 x}).\end{aligned}$$

After comparing the coefficients of x^n from both sides of this equation, we determine that

$$\sum_{\substack{0\le a,b,c,\le n\\ a+b+c=n}} \frac{F_aF_bF_c}{a!b!c!} = \frac{(3\alpha)^n-(3\beta)^n-3(\alpha^{2n}-\beta^{2n})}{(\alpha-\beta)^3 n!} = \frac{3^nF_n-3F_{2n}}{5n!}.$$

Also solved by Dmitry Fleischman, (Sydney Marks, Leah Seader, and Michelle Manin; students) (jointly), Reiner Martin, Ángel Plaza and Sergio Falcón (jointly), and the proposer.

The Exact Value of an Infinite Series

B-1148 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52.2, May 2014)

Prove that

$$\sum_{k=1}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k}+1} = \frac{1}{\sqrt{5}}.$$

Solution by students Tia Herring, Gary Knight, Shane Latchman, Ludwig Murillo, Zibusiso Ndimande and Eyob Tarekegn, jointly, Benedict College, Columbia, SC.

We will first prove by induction on n that

$$\sum_{k=1}^{n} \frac{F_{2^{k-1}}}{L_{2^k}+1} = \frac{F_{2^n}}{L_{2^n}+1}.$$

If $n=1$, then the equality becomes $\frac{F_1}{L_2+1} = \frac{F_2}{L_2+1}$, which is true. Assume that $\sum_{k=1}^{n} \frac{F_{2^{k-1}}}{L_{2^k}+1} = \frac{F_{2^n}}{L_{2^n}+1}$. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{F_{2^{k-1}}}{L_{2^k}+1} &= \sum_{k=1}^{n} \frac{F_{2^{k-1}}}{L_{2^k}+1} + \frac{F_{2^n}}{L_{2^{n-1}+1}} \\
&= \frac{F_{2^n}}{L_{2^n}+1} + \frac{F_{2^n}}{L_{2^{n-1}}+1} \\
&= \frac{L_{2^n}-1}{L_{2^n}-1}\cdot\frac{F_{2^n}}{L_{2^n}+1} + \frac{F_{2^n}}{L_{2^{n+1}}+1} \\
&= \frac{(L_{2^n}-1)F_{2^n}}{(L_{2^n})^2-1} + \frac{F_{2^n}}{L_{2^{n+1}}+1} \\
&= \frac{L_{2^n}F_{2^n}-F_{2^n}}{L_{2^{n+1}}+2(-1)^{2^n}-1} + \frac{F_{2^n}}{L_{2^{n-1}}+1} \\
&= \frac{L_{2^n}F_{2^n}}{L_{2^{n-1}}+1} = \frac{F_{2^{n+1}}}{L_{2^{n-1}}+1}
\end{aligned}$$

which proves that the inequality holds for $n+1$. In the previous string of equalities we used the following identities $(L_n)^2 = L_{2n}+2(-1)^n$ and $L_nF_n = F_{2n}$, [1, pp. 97, 80]. Now, since $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\alpha$, we have that $\lim_{n\to\infty}\frac{L_n}{F_n}=\lim_{n\to\infty}\frac{F_{n+1}+F_{n-1}}{F_n}=\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}+\frac{1}{\frac{F_n}{F_{n-1}}}\right)=\alpha+\frac{1}{\alpha}=\sqrt{5}$ and therefore,

$$\sum_{k=1}^{\infty}\frac{F_{2^{k-1}}}{L_{2^k}+1}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{F_{2^{k-1}}}{L_{2^k}+1}=\lim_{n\to\infty}\frac{F_{2^n}}{L_{2^n}+1}=\lim_{n\to\infty}\frac{1}{\frac{L_{2^n}}{F_{2^n}}}+\frac{1}{F_{2^n}}=\frac{1}{\sqrt{5}}.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Inc., 2001.

Also solved by Russell Jay Hendel, Reiner Martin, Ángel Plaza, and the proposer.

Inequalities with k-Fibonacci and k-Lucas Sequences

B-1149 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 52.2, May 2014)

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recurrently by $u_{n+1}=ku_n+u_{n-1}$ for $n\geq 1$, with respective initial conditions $F_{k,0}=0$; $F_{k,1}=1$ and $L_{k,0}=2$; $L_{k,1}=k$. Prove that

$$\sum_{i=1}^{n-1}(F_{k,i}-\sqrt{F_{k,i}F_{k,i+1}}+F_{k,i+1})^2+(F_{k,n}-\sqrt{F_{k,n}F_{k,1}}+F_{k,1})^2\geq\frac{F_{k,n}F_{k,n+1}}{k},\qquad(1)$$

$$\sum_{i=1}^{n-1}(L_{k,i}-\sqrt{L_{k,i}L_{k,i+1}}+L_{k,i+1})^2+(L_{k,n}-\sqrt{L_{k,n}L_{k,1}}+L_{k,1})^2\geq\frac{L_{k,n}L_{k,n+1}}{k}-2,\qquad(2)$$

for any positive integer n.

Solution by the proposers.

Both inequalities are consequence that for any positive real numbers x, y,

$$\frac{x+y}{2}\geq\frac{\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}}{2},$$

and hence, $x+y\geq\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}$.

Then the left-hand side of (1), LHS is

$$LHS\geq\frac{F_{k,1}^2+F_{k,2}^2}{2}+\frac{F_{k,2}^2+F_{k,3}^2}{2}+\cdots+\frac{F_{k,n}^2+F_{k,1}^2}{2}=\sum_{i=1}^{n}F_{k,i}^2=\frac{F_{k,n}F_{k,n+1}}{k},$$

where the last identity may be proved easily by induction. For $n = 1$ trivially, $1 = \frac{1k}{k} = 1$. Assume that the identity holds for $n - 1$. Then

$$\begin{aligned}\sum_{i=1}^{n} F_{k,i}^2 &= \sum_{i=1}^{n-1} F_{k,i}^2 + F_{k,n}^2 \\ &= \frac{F_{k,n-1}F_{k,n}}{k} + F_{k,n}^2 \\ &= \frac{F_{k,n-1}F_{k,n} + kF_{k,n}^2}{k} \\ &= \frac{F_{k,n}(F_{k,n-1} + kF_{k,n})}{k} \\ &= \frac{F_{k,n}F_{k,n+1}}{k}.\end{aligned}$$

Inequality (2) is proved in the same way by now using $\displaystyle\sum_{i=1}^{n} L_{k,i}^2 = \frac{L_{k,n}L_{k,n+1}}{k} - 2$.

Also solved by Dmitry Fleischman.

Maximum of a Lucas Fibonacci Ratio

B-1150 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52.2, May 2014)

Let n be a positive integer. For any positive integers $r_1, r_2, \ldots, r_n$, find the maximum value of

$$\frac{L_{r_1+r_2+\cdots+r_n}}{F_{r_1}F_{r_2}\cdots F_{r_n}}$$

as a function of n.

Solution by Reiner Martin, 65812 Bod Soden-Nevenbdin, Germany.

We will show that the maximum value is L_{2n}, which is attained by $r_1 = r_2 = \cdots = r_n = 2$.

Note that if some r_i is equal to 1, then we can replace it by 2 to get a larger value for the fraction, as $F_1 = F_2$ and as the Lucas numbers are increasing for positive indices. Thus, we can assume that $r_i \geq 2$ for all $i = 1, 2, \ldots, n$.

We proceed by showing that any such fraction does not decrease when we replace any r_i by 2. As the expression in question is symmetric in $r_i, r_2, \ldots, r_n$ it is enough to show that

$$\frac{L_{r_1+r_2+\cdots+r_n}}{F_{r_1}F_{r_2}\cdots F_{r_n}} \leq \frac{L_{2+r_2+\cdots+r_n}}{F_2F_{r_2}\cdots F_{r_n}}$$

for all $r_i, r_2, \ldots, r_n$. Writing $r = r_2 + \cdots + r_n$, this is equivalent to

$$L_{r_1+r}F_2 \leq L_{2+r}F_{r_1}.$$

But equation (18) in S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Halsted Press (1989) implies that

$$L_{2+r}F_{r_1} - L_{r_1+r}F_2 = L_rF_{r_1-2} - L_{r_1+r-2}F_0 = L_rF_{r_1-2} \geq 0.$$

Also solved by Dmitry Fleischman, Russell Jay Hendel, and the proposer.