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NON-LOCAL PROBLEMS WITH INTEGRAL GLUING CONDITION FOR LOADED MIXED TYPE EQUATIONS INVOLVING THE CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this work, we study the existence and uniqueness of solutions to non-local boundary value problems with integral gluing condition. Mixed type equations (parabolic-hyperbolic) involving the Caputo fractional derivative have loaded parts in Riemann-Liouville integrals. Thus we use the method of integral energy to prove uniqueness, and the method of integral equations to prove existence.

1. Introduction and formulation of a problem

The models of fractional-order derivatives are more adequate than the previously used integer-order models, because fractional-order derivatives and integrals enable the description of the memory and hereditary properties of different substances [23]. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which such effects are neglected.

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, etc., (see [6, 7, 8, 15, 18, 19]). There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas, Srivastava, Trujillo [14], Miller and Ross [20], Podlubny [23], Samko, Kilbas , Marichev. [28] and the references therein.

Very recently, some basic theory for the initial boundary value problems of fractional differential equations involving a Riemann-Liouville differential operator of order $0 < \alpha \leq 1$ has been discussed by Lakshmikantham and Vatsala [16, 17]. In a series of papers (see [4, 5]) the authors considered some classes of initial value problems for functional differential equations involving Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha \leq 1$: For more details concerning geometric and physical interpretation of fractional derivatives of Riemann-Liouville and Caputo types see [24]. Note that works [3, 12, 13, 22] are devoted to the studying of boundary value problems (BVP) for parabolic-hyperbolic equations, involving fractional derivatives. BVPs for the mixed type equations involving the Caputo and

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the Riemann-Liouville fractional differential operators were investigated in works [10, 11].

For the first time it was given the most general definition of a loaded equations and various loaded equations are classified in detail by Nakhushev [21]. Note that with intensive research on problem of optimal control of the agro-economical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate BVPs for such class of partial differential equations .More results on the theory of BVP's for the loaded equations parabolic, parabolic-hyperbolic and elliptic-hyperbolic types were published in works [9, 1]. Integral boundary conditions have various applications in thermoelasticity, chemical engineering, population dynamics, etc. Gluing conditions of integral form were used in [2, 27].

We consider the equation:

$$\begin{gathered} u_{xx} - {}_C D_{0y}^{\alpha} u + p(x,y)\int_x^1 (t-x)^{\beta-1} u(t,0)dt = 0 \quad \text{for } y>0 \\ u_{xx} - u_{yy} - q(x+y)\int_{x+y}^1 (t-x-y)^{\gamma-1} u(t,0)dt = 0 \quad \text{for } y<0\,, \end{gathered} \tag{1.1}$$

with the operator

$${}_C D_{0y}^{\alpha} f = \frac{1}{\Gamma(1-\alpha)}\int_0^y (y-t)^{-\alpha} f'(t)dt, \tag{1.2}$$

where $0<\alpha,\ \beta,\gamma<1$, $p(x,y)$ and $q(x+y)$ are given functions. Let Ω be domain, bounded with segments: $A_1A_2=\{(x,y): x=1,\ 0<y<h\}$, $B_1B_2=\{(x,y): x=0,\ 0<y<h\}$, $B_2A_2=\{(x,y): y=h,\ 0<x<1\}$ at the $y>0$, and characteristics : $A_1C: x-y=1$; $B_1C: x+y=0$ of the equation (1.1) at $y<0$, where $A_1(1;0)$, $A_2(1;h)$, $B_1(0;0)$, $B_2(0;h)$, $C(1/2;-1/2)$.

Let us introduce: $\theta(x)=\frac{x+1}{2}+i\frac{x-1}{2}$, $i^2=-1$,

$$D_{xa}^{-\beta} f(x) = \frac{1}{\Gamma(\beta)}\int_x^a (t-x)^{\beta-1} f(t)dt, \quad 0<\beta<1. \tag{1.3}$$

$\Omega^+=\Omega\cap(y>0)$, $\Omega^-=\Omega\cap(y<0)$, $I_1=\{x: \frac12<x<1\}$, $I_2=\{y: 0<y<h\}$. In the domain Ω we study the following problem.

Problem I. Find a solution $u(x,y)$ of equation (1.1) from the class of functions:

$$W=\{u(x,y): u(x,y)\in C(\bar\Omega)\cap C^2(\Omega^-),\ u_{xx}\in C(\Omega^+),\ {}_C D_{oy}^{\alpha} u\in C(\Omega^+)\},$$

that satisfies the boundary conditions:

$$u(x,y)\big|_{A_1A_2}=\varphi(y), \quad 0\le y\le h; \tag{1.4}$$

$$u(x,y)\big|_{B_1B_2}=\psi(y), \quad 0\le y\le h; \tag{1.5}$$

$$\frac{d}{dx}u(\theta(x)) = a(x)u_y(x,0)+b(x)u_x(x,0)+c(x)u(x,0)+d(x), \quad x\in I_1, \tag{1.6}$$

and gluing condition

$$\lim_{y\to+0} y^{1-\alpha}u_y(x,y) = \lambda_1(x)u_y(x,-0)+\lambda_2(x)\int_x^1 r(t)u(t,0)dt, \quad 0<x<1 \tag{1.7}$$

where $\varphi(y)$, $\psi(y)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, and $\lambda_j(x)$, are given functions, such that $\sum_{j=1}^2 \lambda_j^2(x)\neq 0$.

2. Uniqueness of solution for Problem I

It is known that equation (1.1) on the characteristics coordinate $\xi = x + y$ and $\eta = x - y$ at $y \le 0$ has the form

$$u_{\xi\eta} = \frac{q(\xi)}{4} \int_{\xi}^{1} (t-\xi)^{\gamma-1} u(t,0)dt. \tag{2.1}$$

We introduce the notation: $u(x,0) = \tau(x)$, $0 \le x \le 1$; $u_y(x,-0) = \nu^-(x)$, $0 < x < 1$;

$$\lim_{y\to+0} y^{1-\alpha} u_y(x,y) = \nu^+(x), \quad 0 < x < 1.$$

A solution of the Cauchy problem for the equation (1.1) in the domain Ω^- can be represented as

$$\begin{aligned} u(x,y) &= \frac{\tau(x+y)+\tau(x-y)}{2} - \frac{1}{2}\int_{x+y}^{x-y} \nu^-(t)dt \\ &\quad + \frac{1}{4}\int_{x+y}^{x-y} q(\xi)d\xi \int_{\xi}^{x-y} d\eta \int_{\xi}^{1} (t-\xi)^{\gamma-1}\tau(t)dt. \end{aligned} \tag{2.2}$$

After using condition (1.6) and taking (1.3) into account, from (2.2) we obtain

$$\begin{aligned} &(2a(x)-1)\nu^-(x) \\ &= \frac{1-x}{2}\Gamma(\gamma)q(x)D_{x\,1}^{-\gamma}\tau(x) + (1-2b(x))\tau'(x) - 2c(x)\tau(x) - 2d(x). \end{aligned} \tag{2.3}$$

Considering above notation and gluing condition (1.7) we have

$$\nu^+(x) = \lambda_1(x)\nu^-(x) + \lambda_2(x)\int_x^1 r(t)\tau(t)dt, \tag{2.4}$$

Further from (1.1) at $y \to +0$ considering (1.2), (2.4) and

$$\lim_{y\to 0} D_{0y}^{\alpha-1} f(y) = \Gamma(\alpha)\lim_{y\to 0} y^{1-\alpha} f(y),$$

we obtain k1:

$$\tau''(x) - \Gamma(\alpha)\lambda_1(x)\nu^-(x) - \Gamma(\alpha)\lambda_2(x)\int_x^1 r(t)\tau(t)dt + \Gamma(\beta)p(x,0)D_{x1}^{-\beta}\tau(x) = 0. \tag{2.5}$$

Main results.

Theorem 2.1. *If the given functions satisfy conditions*

$$\Big(\frac{\lambda_2(x)}{r(x)}\Big)' \ge 0, \quad \frac{\lambda_1(0)q(0)}{2a(0)-1} \ge 0, \quad p(0,0) \le 0, \;\; p'(x,0) \le 0, \quad \frac{\lambda_2(0)}{r(0)} \ge 0; \tag{2.6}$$

$$\Big(\frac{(1-x)q(x)\lambda_1(x)}{(2a(x)-1)}\Big)' \ge 0, \quad \frac{c(x)\lambda_1(x)}{(2a(x)-1)} \le 0, \quad \Big(\frac{(1-2b(x))\lambda_1(x)}{(2a(x)-1)}\Big)' \le 0. \tag{2.7}$$

then, the solution $u(x,y)$ of the Problem I is unique.

Proof. Known that if homogeneous problem has only trivial solution, then we can state that original problem has unique solution. For this aim, we assume that the Problem I has two solutions, then denoting difference of these as $u(x,y)$, we get appropriate homogenous problem. Equation (2.5) we multiply to $\tau(x)$ and integrate from 0 to 1:

$$\begin{aligned}&\int_0^1 \tau''(x)\tau(x)dx-\Gamma(\alpha)\int_0^1 \lambda_1(x)\tau(x)\nu^-(x)dx\\&-\Gamma(\alpha)\int_0^1 \lambda_2(x)\tau(x)dx\int_x^1 r(t)\tau(t)dt+\Gamma(\beta)\int_0^1 \tau(x)p(x,0)D_{x1}^{-\beta}\tau(x)dx=0.\end{aligned}$$

We investigate the integral

$$\begin{aligned}I=\Gamma(\alpha)&\int_0^1 \lambda_1(x)\tau(x)\nu^-(x)dx+\Gamma(\alpha)\int_0^1 \lambda_2(x)\tau(x)dx\int_x^1 r(t)\tau(t)dt\\&-\Gamma(\beta)\int_0^1 \tau(x)p(x,0)D_{x1}^{-\beta}\tau(x)dx.\end{aligned}$$

Taking (2.3) into account at $d(x)=0$, we obtain

$$\begin{aligned}I&=\frac{\Gamma(\alpha)\Gamma(\gamma)}{2}\int_0^1 \frac{(1-x)q(x)}{2a(x)-1}\lambda_1(x)\tau(x)D_{x1}^{-\gamma}\tau(x)dx\\&\quad+\Gamma(\alpha)\int_0^1 \frac{(1-2b(x))\lambda_1(x)}{2a(x)-1}\tau(x)\tau'(x)dx-2\Gamma(\alpha)\int_0^1 \frac{\lambda_1(x)c(x)}{2a(x)-1}\tau^2(x)dx\\&\quad+\Gamma(\alpha)\int_0^1 \lambda_2(x)\tau(x)dx\int_0^x r(t)\tau(t)dt-\Gamma(\beta)\int_0^1 \tau(x)p(x,0)D_{x1}^{-\beta}\tau(x)dx\\&=\frac{\Gamma(\alpha)}{2}\int_0^1 \frac{(1-x)q(x)}{2a(x)-1}\lambda_1(x)\tau(x)dx\int_x^1 (t-x)^{\gamma-1}\tau(t)dt\\&\quad+\frac{\Gamma(\alpha)}{2}\int_0^1 \frac{1-2b(x)}{2a(x)-1}\lambda_1(x)d(\tau^2(x))\\&\quad-2\Gamma(\alpha)\int_0^1 \frac{\lambda_1(x)c(x)}{2a(x)-1}\tau^2(x)dx-\frac{\Gamma(\alpha)}{2}\int_0^1 \frac{\lambda_2(x)}{r(x)}d(\int_x^1 r(t)\tau(t)dt)^2\\&\quad-\int_0^1 \tau(x)p(x,0)dx\int_x^1 (t-x)^{\beta-1}\tau(t)dt.\end{aligned}\tag{2.8}$$

Considering $\tau(1)=0$, $\tau(0)=0$ (which deduced from the conditions (1.4), (1.5) in homogeneous case) and on a base of the formula [29]:

$$|x-t|^{-\gamma}=\frac{1}{\Gamma(\gamma)\cos\frac{\pi\gamma}{2}}\int_0^\infty z^{\gamma-1}cos[z(x-t)]dz,\quad 0<\gamma<1$$

After some simplifications from (2.8) we obtain

$$\begin{aligned}I&=\frac{\Gamma(\alpha)q(0)\lambda_1(0)}{4(2a(0)-1)\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}}\int_0^\infty z^{-\gamma}[M^2(0,z)+N^2(0,z)]dz\\&\quad+\frac{\Gamma(\alpha)}{4\Gamma(1-\gamma)\sin\frac{\pi\gamma}{2}}\int_0^\infty z^{-\gamma}dz\int_0^1 [M^2(x,z)+N^2(x,z)]d\big[\lambda_1(x)\frac{(1-x)q(x)}{2a(x)-1}\big]\\&\quad-\frac{\Gamma(\alpha)}{2}\int_0^1 \tau^2(x)\Big(\lambda_1(x)\frac{1-2b(x)}{2a(x)-1}\Big)'dx-2\Gamma(\alpha)\int_0^1 \frac{\lambda_1(x)c(x)}{2a(x)-1}\tau^2(x)dx\\&\quad+\frac{\Gamma(\alpha)}{2}\frac{\lambda_2(0)}{r(0)}\Big(\int_0^1 r(t)\tau(t)dt\Big)^2+\frac{\Gamma(\alpha)}{2}\int_0^1 \Big(\frac{\lambda_2(x)}{r(x)}\Big)'\Big(\int_x^1 r(t)\tau(t)dt\Big)^2dx\end{aligned}$$

$$
\begin{aligned}
&-\frac{p(0,0)}{2\Gamma(1-\beta)\sin\frac{\pi\beta}{2}}\int_0^\infty z^{-\beta}[M^2(0,z)+N^2(0,z)]dz\\
&-\frac{1}{2\sin\frac{\pi\beta}{2}\Gamma(1-\beta)}\int_0^\infty z^{-\beta}dz\int_0^1\frac{\partial}{\partial x}[p(x,0)][M^2(x,z)+N^2(x,z)]dx \qquad (2.9)
\end{aligned}
$$

where $M(x,z)=\int_x^1\tau(t)\cos ztdt$, $N(x,z)=\int_x^1\tau(t)\sin ztdt$.

Thus, owing to (2.6),(2.7) from (2.9) it is concluded, that $\tau(x)\equiv 0$. Hence, based on the solution of the first boundary problem for the (1.1) [11, 25] owing to account (1.4) and (1.5) we obtain $u(x,y)\equiv 0$ in $\overline{\Omega}^+$. Further, from functional relations (2.3), taking into account $\tau(x)\equiv 0$ we obtain that $\nu^-(x)\equiv 0$. Consequently, based on the solution (2.2) we obtain $u(x,y)\equiv 0$ in closed domain $\overline{\Omega}^-$. □

3. Existence of solutions for Problem I

Theorem 3.1. *If conditions* (2.6), (2.7) *and*

$$p(x,y)\in C(\overline{\Omega^+})\cap C^2(\Omega^+),\quad q(x+y)\in C(\overline{\Omega^-})\cap C^2(\Omega^-); \qquad (3.1)$$

$$\varphi(y),\psi(y)\in C(\overline{I_2})\cap C^1(I_2); a(x),b(x),c(x),d(x)\in C^1(\overline{I_1})\cap C^2(I_1) \qquad (3.2)$$

hold, then the solution of the investigating problem exists.

Proof. Taking (2.3) into account from (2.5) we obtain

$$\tau''(x)-A(x)\tau'(x)=f(x)-B(x)\tau(x) \qquad (3.3)$$

where

$$
\begin{aligned}
f(x)=&\frac{\Gamma(\alpha)\Gamma(\gamma)(1-x)\lambda_1(x)q(x)}{2(2a(x)-1)}D_{x1}^{-\gamma}\tau(x)-\Gamma(\beta)p(x,0)D_{x1}^{-\beta}\tau(x)\\
&+\Gamma(\alpha)\lambda_2(x)\int_x^1 r(t)\tau(t)dt-\frac{2\Gamma(\alpha)\lambda_1(x)d(x)}{2a(x)-1}
\end{aligned} \qquad (3.4)
$$

$$A(x)=\frac{\Gamma(\alpha)\lambda_1(x)(1-2b(x))}{2a(x)-1},\quad B(x)=\frac{2\Gamma(\alpha)\lambda_1(x)c(x)}{1-2a(x)} \qquad (3.5)$$

The solution of (3.3) with conditions

$$\tau(0)=\psi(0),\quad \tau(1)=\varphi(0) \qquad (3.6)$$

has the form

$$
\begin{aligned}
\tau(x)=&\psi(0)+A_1(x)\Big(\int_x^1(B(t)\tau(t)-f(t))A'_1(t)dt+\frac{\varphi(0)-\psi(0)}{A_1(1)}\Big)\\
&-\frac{A_1(x)}{A_1(1)}\int_0^1(B(t)\tau(t)-f(t))\frac{A_1(t)}{A'_1(t)}dt\\
&+\int_0^x(B(t)\tau(t)-f(t))\frac{A_1(t)}{A'_1(t)}dt
\end{aligned} \qquad (3.7)
$$

where

$$A_1(x)=\int_0^x\exp\Big(\int_0^t A(z)dz\Big)dt. \qquad (3.8)$$

Further, considering (3.4) and using (1.3) from (3.7) we obtain

$\tau(x)$

$$
\begin{aligned}
&= f_1(x) + A_1(x)\int_x^1 B(t)\tau(t)dA_1(t) \\
&- \frac{\Gamma(\alpha)}{2}\int_x^1 \frac{(1-t)\lambda_1(t)q(t)}{2a(t)-1}dA_1(t)\int_t^1 \frac{\tau(s)ds}{(s-t)^{1-\gamma}} \\
&+ A_1(x)\int_x^1 A'_1(t)dt\int_t^1 [p(t,0)(s-t)^{\beta-1} - \Gamma(\alpha)\lambda_2(t)r(s)]\tau(s)ds \\
&- \frac{A_1(x)}{A_1(1)}\int_0^1 \frac{A_1(t)}{A'_1(t)}B(t)\tau(t)dt + \int_0^x \frac{A_1(t)}{A'_1(t)}p(t,0)dt\int_t^1 (s-t)^{\beta-1}\tau(s)ds \\
&+ \Gamma(\alpha)\frac{A_1(x)}{A_1(1)}\int_0^1 \frac{A_1(t)}{A'_1(t)}dt\int_t^1 [\frac{(1-t)\lambda_1(t)q(t)(s-t)^{\gamma-1}}{2(2a(t)-1)} - \lambda_2(t)r(s)]\tau(s)ds \\
&- \frac{A_1(x)}{A_1(1)}\int_0^1 \frac{A_1(t)}{A'_1(t)}p(t,0)dt\int_t^1 (s-t)^{\beta-1}\tau(s)ds + \int_0^x \frac{A_1(t)}{A'_1(t)}B(t)\tau(t)dt \\
&- \Gamma(\alpha)\int_0^x \frac{A_1(t)}{A'_1(t)}dt\int_t^1 [\frac{(1-t)\lambda_1(t)q(t)}{2(2a(t)-1)}(s-t)^{\gamma-1} + \lambda_2(t)r(s)]\tau(s)ds \qquad (3.9)
\end{aligned}
$$

where

$$
\begin{aligned}
&f_1(x) \\
&= (1-\frac{A_1(x)}{A_1(1)})\int_0^x \frac{2\Gamma(\alpha)d(t)A_1(t)\lambda_1(t)}{A'_1(t)(2a(t)-1)}dt \\
&\quad + 2\Gamma(\alpha)A_1(x)\int_x^1 \frac{d(t)A'_1(t)\lambda_1(t)}{2a(t)-1}dt \qquad (3.10)\\
&\quad - \frac{A_1(x)}{A_1(1)}\int_x^1 \frac{2\Gamma(\alpha)d(t)A_1(t)\lambda_1(t)}{A'_1(t)(2a(t)-1)}dt - \frac{A_1(x)}{A_1(1)}(\psi(0)-\varphi(0)) + \psi(0).
\end{aligned}
$$

After some simplifications we rewrite (3.9) in the form

$$
\begin{aligned}
&\tau(x) \\
&= A_1(x)\int_x^1 \tau(s)ds\int_x^s [p(t,0)(s-t)^{\beta-1} - \Gamma(\alpha)\lambda_2(t)r(s)]A'_1(t)dt \\
&- \Gamma(\alpha)A_1(x)\int_x^1 \tau(s)ds\int_x^s (s-t)^{\gamma-1}\frac{(1-t)\lambda(t)q(t)}{2(2a(t)-1)}A'_1(t)dt \\
&- \Gamma(\alpha)\int_x^1 \tau(s)ds\int_0^x [\frac{(1-t)\lambda_1(t)q(t)}{2(2a(t)-1)}(s-t)^{\gamma-1} + \lambda_2(t)r(s)]\frac{A_1(t)}{A'_1(t)}dt \\
&+ \int_x^1 \tau(s)ds\int_0^x (s-t)^{\beta-1}\frac{A_1(t)}{A'_1(t)}p(t,0)dt + A_1(x)\int_x^1 A'_1(t)B(t)\tau(t)dt \\
&+ \Gamma(\alpha)\frac{A_1(x)}{A_1(1)}\int_0^1 \tau(s)ds\int_0^s \Big[\frac{(1-t)\lambda_1(t)q(t)(s-t)^{\gamma-1}}{2(2a(t)-1)} - \lambda_2(t)r(s)\Big]\frac{A_1(t)}{A'_1(t)}dt \\
&- \frac{A_1(x)}{A_1(1)}\int_0^1 \tau(s)ds\int_0^s (s-t)^{\beta-1}\frac{A_1(t)}{A'_1(t)}p(t,0)dt - \frac{A_1(x)}{A_1(1)}\int_0^1 \frac{A_1(t)}{A'_1(t)}B(t)\tau(t)dt \\
&+ \int_0^x \tau(s)ds\int_0^s (s-t)^{\beta-1}\frac{A_1(t)}{A'_1(t)}p(t,0)dt + \int_0^x \frac{A_1(t)}{A'_1(t)}B(t)\tau(t)dt \\
&- \Gamma(\alpha)\int_0^x \tau(s)ds\int_0^s \Big[\frac{(1-t)\lambda_1(t)q(t)}{2(2a(t)-1)}(s-t)^{\gamma-1} + \lambda_2(t)r(s)\Big]\frac{A_1(t)}{A'_1(t)}dt + f_1(x)
\end{aligned}
$$

i.e., we have the integral equation:

$$\tau(x)=\int_0^1 K(x,t)\tau(t)dt+f_1(x), \tag{3.11}$$

where

$$K(x,t)=\begin{cases} K_1(x,s), & 0\le t\le x,\\ K_2(x,s), & x\le t\le 1.\end{cases} \tag{3.12}$$

$$\begin{aligned} K_1(x,s)=&\Big(\frac{A_1(x)}{A_1(1)}-1\Big)\frac{\Gamma(\alpha)}{2}\int_0^s (s-t)^{\gamma-1}\frac{(1-t)A_1(t)\lambda(t)q(t)}{(2a(t)-1)A'_1(t)}dt\\ &-\Gamma(\alpha)\Big(\frac{A_1(x)}{A_1(1)}+1\Big)r(s)\int_0^s \lambda_2(t)\frac{A_1(t)}{A'_1(t)}dt\\ &-\Big(\frac{A_1(x)}{A_1(1)}-1\Big)\Big[\int_0^s (s-t)^{\beta-1}\frac{A_1(t)}{A'_1(t)}p(t,0)dt+\frac{A_1(s)}{A'_1(s)}B(s)\Big]\end{aligned} \tag{3.13}$$

$$\begin{aligned} &K_2(x,s)\\ &=A_1(x)\Big(A'_1(s)B(s)-\frac{\Gamma(\alpha)}{2}\int_x^s (s-t)^{\gamma-1}\frac{(1-t)\lambda(t)q(t)}{2a(t)-1}A'_1(t)dt\Big)\\ &\quad-\Gamma(\alpha)A_1(x)\int_x^s \lambda_2(t)A'_1(t)dt+A_1(x)\int_x^s (s-t)^{\beta-1}A'_1(t)p(t,0)dt\\ &\quad-\Gamma(\alpha)r(s)\int_0^x \frac{A_1(t)\lambda_2(t)}{A'_1(t)}dt\\ &\quad+\frac{\Gamma(\alpha)}{2}\frac{A_1(x)}{A_1(1)}\int_0^s (s-t)^{\gamma-1}\frac{(1-t)A_1(t)\lambda(t)q(t)}{(2a(t)-1)A'_1(t)}dt\\ &\quad-\frac{A_1(x)}{A_1(1)}\frac{A_1(s)}{A'_1(s)}B(s)-\Gamma(\alpha)\frac{A_1(x)}{A_1(1)}r(s)\int_0^s \frac{A_1(t)\lambda_2(t)}{A'_1(t)}dt\\ &\quad-\frac{\Gamma(\alpha)}{2}\int_0^x \frac{(1-t)A_1(t)\lambda(t)q(t)}{A'_1(t)(2a(t)-1)}(s-t)^{\gamma-1}dt\\ &\quad+(1-\frac{A_1(x)}{A_1(1)})\int_0^x \frac{A_1(t)(s-t)^{\beta-1}}{A'_1(t)}p(t,0)dt.\end{aligned} \tag{3.14}$$

Owing to class (3.1), (3.2) of the given functions and after some evaluations from (3.13), (3.14) and (3.10), (3.12) we conclude that

$$|K(x,t)|\le \text{const},\quad |f_1(x)|\le \text{const}.$$

Since kernel $K(x,t)$ is continuous and function in right-side $F(x)$ is continuously differentiable, solution of integral equation (3.11) we can write via resolvent-kernel:

$$\tau(x)=f_1(x)-\int_0^1 \Re(x,t)f_1(t)dt, \tag{3.15}$$

where $\Re(x,t)$ is the resolvent-kernel of $K(x,t)$. Unknown functions $\nu^-(x)$ and $\nu^+(x)$ we found accordingly from (2.3) and (2.4):

$$\begin{aligned}\nu^-(x)=&\frac{(x-1)q(x)}{2(2a(x)-1)}\int_x^1 (t-x)^{\gamma-1}dt\int_0^1 \Re(t,s)f_1(s)ds\\ &+\frac{(1-x)q(x)}{2(2a(x)-1)}\int_x^1 (t-x)^{\gamma-1}f_1(t)dt\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-2b(x)}{2a(x)-1}f_1'(x)-\frac{1-2b(x)}{2a(x)-1}\int_0^1\frac{\partial\Re(x,t)}{\partial x}f_1(t)dt-\frac{2c(x)}{2a(x)-1}f_1(x)\\
&+\frac{2c(x)}{2a(x)-1}\int_0^1\Re(x,t)f_1(t)dt-\frac{2d(x)}{2a(x)-1}
\end{aligned}
$$

and $\nu^+(x)=\lambda(x)\nu^-(x)$.

The solution of Problem I in the domain Ω^+ can be written as [11],

$$
\begin{aligned}
&u(x,y)\\
&=\int_0^y G_\xi(x,y,0,\eta)\psi(\eta)d\eta-\int_0^y G_\xi(x,y,1,\eta)\varphi(\eta)d\eta\\
&\quad+\int_0^1 G_0(x-\xi,y)\tau(\xi)d\xi-\int_0^y\int_0^1 G(x,y,0,\eta)p(\xi)d\xi d\eta\int_\xi^1(t-\xi)^{\beta-1}\tau(t)dt
\end{aligned}
$$

where

$$
\begin{gathered}
G_0(x-\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y\eta^{-\alpha}G(x,y,\xi,\eta)d\eta,\\
G(x,y,\xi,\eta)=\frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{\infty}\Big[e^{1,\alpha/2}_{1,\alpha/2}\Big(-\frac{|x-\xi+2n|}{(y-\eta)^{\alpha/2}}\Big)\\
-e^{1,\alpha/2}_{1,\alpha/2}\Big(-\frac{|x+\xi+2n|}{(y-\eta)^{\alpha/2}}\Big)\Big]
\end{gathered}
$$

Is the Green's function of the first boundary problem (1.1) in the domain Ω^+ with the Riemanne-Liouville fractional differential operator instead of the Caputo ones [26],

$$
e^{1,\delta}_{1,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\delta n)}
$$

is the Wright type function [26]. Solution of the Problem I in the domain Ω^- will be found by the formula (2.2). Hence, the proof is complete. $\square$

We remark that if $a(x)=1/2$, then from (2.3) we find $\tau(x)$ as a solution Volterra type integral equation. After that we can find $\nu^+(x)$ from the first boundary value problem problem for the (1.1), and $\nu^-(x)$ will be defined from the gluing condition (2.4).

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