

## FOLLOWING THE STEPS OF SPANISH MATHEMATICAL ANALYSIS: FROM CAUCHY TO WEIERSTRASS BETWEEN 1880 AND 1914

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### Abstract

Rigorous Mathematical Analysis in the Cauchy style was not accepted in a straightforward manner by the European mathematical community of the central years of the 19<sup>th</sup> Century. In average, only around forty years after the 1821 *Cours d'Analyse* did Cauchy's treatment become a standard in the more mathematically advanced countries, as a paradigm that remained in use until the arithmetisation of Analysis by Weierstrass replaced it before the end of the century. In this paper the authors show how rigorous Mathematical Analysis *à la* Cauchy was adopted in Spain quite late -around 1880- and how in some more forty years, the Weierstrassian formulation became the usual presentation in Spanish texts.

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### 1 Introduction

It is known that the definitive introduction of rigorous Mathematical Analysis in Spain was achieved by Julio Rey Pastor (1888-1962) when he gave to print his two basic books *Elementos de Análisis Algebraico* (Rey Pastor, 1917) and *Teoría de las Funciones Reales* (Rey Pastor, 1918) after two stages in Germany: with Fröbenius, Schottky and Schwartz in Berlin, 1911-12, and with Caratheodory, Courant, Hölder and Koebe in Göttingen, 1913-1914. Inspired on the original theories by Cauchy, Weierstrass, Cantor, and Dedekind, these two books incorporated to Spanish Mathematics the most rigorous standards of the German mathematical schools.

Nevertheless, several attempts had been made in the presentation of Mathematical Analysis in Spain to introduce rigour before Rey's books: The aim of this paper is to provide a critical description of these mathematical activities.

According to Belhoste (Belhoste, 1991), Cauchy's viewpoints on Mathematical Analysis were not accepted in a straightforward manner, neither in France nor elsewhere. When Cauchy fled into exile the year 1830, his followers Navier, Sturm, Liouville and Duhamel maintained his ideas and used them in teaching and in mathematical writing, as a rule in a less accurate way and even mixing them with other mathematical traditions (Grattan-

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Guinness, 2000, p.67). It was Duhamel who became one of the forerunners of the introduction of the Cauchy style in Spain around 1880 through his *Cours d'Analyse*, a very popular book in this country. The second step towards rigour, the Weierstrassian revolution, was accepted in France when Camille Jordan (1838-1922) published his own *Cours d'Analyse de l'École Polytechnique* (Jordan, 1893-96) and adopted the  $\varepsilon$ - $\delta$  style. In spite of it, the older Cauchy presentation was still in use for several years: Jesper Lützen points out that Sturm's text was used in Copenhagen as late as 1915 (Lützen, 2003).

Around 1840, the Spanish educational system was gradually recovering from the long reign of Fernando VII, where most universities and higher education centres were either closed or dismantled. Soon the basic works of the mathematical scene were translated and/or adapted into Spanish and the newer ideas of Gauss, Cauchy, and Abel on Mathematical Analysis, as well as the birth of non-euclidean geometries, the development of projective geometry, and the modern *Funktionentheorie* according to Riemann became available to the small Spanish mathematical community, although in those days the books selected for translation or adaptation did not yet include the ideas by Cauchy. In a previous paper (Pacheco et al., 2007) the authors have shown how notions such as limits, functions, infinitesimals, etc. were introduced in texts by authors like Vallejo and Feliu, but those terms and techniques were not used in their proofs or would-be proofs.

As Grabiner points out, insistence in proof is a characteristic feature of the development and foundations of Analysis according to the Cauchy style (Grabiner, 1981), and 1880 was the year when the older pre-Cauchy Analysis really disappeared from Spanish higher education and proofs actually entered the Spanish mathematical literature. The mathematician Simón Archilla and the civil engineer Antonio Portuondo wrote the books where Cauchy's vision of Analysis was presented in Spanish words for the first time. They elaborated on the new ideas, but through Duhamel's books rather than by studying the original work by Cauchy, even though these last ones were available at some learned libraries, like the one at the *Real Observatorio de la Armada* (Royal Navy Observatory) in San Fernando, close to Cádiz. This paper does not consider Portuondo's contribution (Portuondo, 1880), essentially the small book *Tratado sobre el infinito*, which is the object of another forthcoming paper by the same authors (Pacheco et al., 2009). More on other mathematical activities of Portuondo can be found in (Pacheco 2008).

## 2 Archilla's *Principios del Cálculo Diferencial*

Simón Archilla (1836-1890) taught Mathematics at the Universities of Barcelona and Madrid. His courses comprised Higher Algebra, Analytic Geometry, and Differential and Integral Calculus. In order to cater for these last topics, he published in 1880 the fundamental book *Principios del Cálculo Diferencial* (The Principles of Differential Calculus, *Principios* from now on). For availability reasons, in this paper the second edition of 1894 compiled by his son Faustino Archilla will be used as the standard reference (Archilla y Espejo, 1894). The authors also checked the first edition and realised that the only difference between them are the page numbers: Not a single change was introduced in the second edition which strictly speaking was only a second printing.

Archilla was elected to the *Real Academia de Ciencias Exactas, Físicas y Naturales*, (Royal Academy for the Exact, Physical, and Natural Sciences), and he read his inaugural dissertation on June 10<sup>th</sup> 1888 with the title *Sobre el concepto y principios fundamentales del Cálculo Infinitesimal* (On the concept and fundamental principles of Infinitesimal Calculus) (Archilla y Espejo and Vicuña y Lazcano, 1888) where he made a most interesting historical report on the idea of infinity, ranging from Archimedes to the date the discourse was delivered. In page 61 of this report, Archilla acknowledges the role of Cauchy:

“(…) there was a need to achieve what seemed so difficult from the very beginning: To force the notion of infinity to serve the needs of Analysis. Opposing to this aim were not only the special status of infinite quantities when considered as tools in mathematical applications to number and distance, but also the philosophical criteria upon which the legitimate intervention of infinity in Analysis would be based” (note 1).

Moreover, he points out how

“(…) under the light of the new doctrine, it has become common knowledge that the ratio of certain infinitesimally small, although it is always finite quantity, does not tend to any limit whatsoever; it is plain to see that the continuity of a function does not imply that the orders of its increment and that of the variable be the same; and it is possible to conceive and to determine, as Weierstrass has shown, continuous functions without a derivative at any point: Such things, if not unconceivable, were difficult to understand and to explain with the old ideas” (note 2).

In the foreword to *Principios* the author presents the basics of the discourse in his book through a quotation from the preface of Hoüel’s *Cours de Calcul Infinitésimal*, a book translated into Spanish in 1878:

“There is only one rigorous method to present Infinitesimal Calculus: It is the method of the infinitely small, or of the limits, the method of Cauchy and Duhamel…”

This opinion is clearly detailed in the introduction to *Principios*:

“Our aim in this book is to summarise the most important principles of Infinitesimal Calculus, trying to explain their natural interrelations, and to study the intimate relationships between the fundamental notions upon which they are based and those that are legitimately obtained from them, according to the doctrine first expounded by Cauchy and then developed by Duhamel…” (note 3).

The difference between this text and its predecessors is finally featured in the treatment given to the ideas of continuity and differentiability in the light of infinitesimals (Introduction, page VI):

“In the study of functions we have focused on the notion of continuity, by showing the difficulties of directly studying it, and by relating it indirectly with the idea of infinitesimal quantities through the limit notion…”

## 2.1 Archilla on infinitesimals

Archilla takes infinitesimals as the fundamental notion in his book, and he introduces the concept of a variable in page 1: A *variable* quantity, or simply a variable, is a quantity that can take a series of *successive values* according to any prescribed law. He goes on by explaining how to operate with variables according to the successive values that determine them, and he defines infinitely small and infinitely large quantities in page 4:

“A variable quantity that *can* take values smaller than any given quantity and can indefinitely satisfy this condition is an *infinitely small quantity*, or simply an *infinitely small*. On the other hand, if a variable quantity *can* take values larger than any given quantity and can indefinitely satisfy this condition is an *infinitely large quantity*, or simply an *infinitely large*. Finally, quantities that are constant or are neither infinitely small nor infinitely large are called *finite quantities*”.

Limits are presented in a way that directly matches the definition of an infinitesimally small, the limit of a variable quantity being a constant quantity such that the varying one approaches it in the following sense: The difference between the constant and the successive values of the variable becomes smaller than any *given* quantity, but never equals zero. Therefore, from this definition it follows that:

- (1) The difference between a variable and its limit is an infinitesimally small quantity.
- (2) Infinitesimally small quantities have zero as their limit.
- (3) Infinitely large quantities have no limit.
- (4) The limit of a variable is a constant that can not be found among the successive values of the variable.

This last observation is directly inherited from Cauchy and introduces a certain lack of generality, for sequences tending to zero like  $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \dots$  would not comply with such a definition. Nevertheless, this observation can be considered a minor flaw. More interesting is the idea of an extended real line expounded in this paragraph, obtained from page 6:

“Our aim has been to give more generality to this doctrine by encompassing in a common classification finite quantities, infinitely small and infinitely large ones, thus establishing a general theory of the *orders of quantity*.”

Then Archilla proceeds to the definition of infinitely small, or large, or constant quantities via the ratios between two quantities:

“When the ratio  $\frac{x}{y}$  is an infinitely small  $\alpha$ , then  $x$  is said to be infinitely small *with respect* to  $y$ .  
When the ratio  $\frac{x}{y}$  is an infinitely large  $A$ , then  $x$  is said to be infinitely large *with respect* to  $y$ .  
Finally, when the ratio  $\frac{x}{y}$  is some constant  $a$ , then  $x$  is said to be *of the same order* of  $y$ .”

And he goes on by specifying a comparison unit:

“Once a fundamental infinitesimally small quantity  $\alpha$  has been arbitrarily chosen, we shall deem every infinitesimal of the same order as *first-order infinitesimals*”.

Thus, if  $\beta$  is any other first order infinitesimal, then some finite quantity  $a$  will exist such that  $\frac{\beta}{\alpha} = a + \omega$ , where  $\omega$  is an infinitesimally small, from where  $\beta = \alpha(a + \omega)$ . In the same way the definition of an  $n$ -th order infinitesimal with respect to the basic one is obtained as

$$\beta_n = \alpha^n(a + \omega)$$

Moreover, acknowledging that  $A = \frac{1}{\alpha}$  is an infinitely large quantity, the orders for infinities are likewise defined through the following equalities:

$$B_n = A^n(a + \omega) = \left(\frac{1}{\alpha}\right)^n(a + \omega) = \alpha^{-n}(a + \omega),$$

and a whole scale of infinitesimals, ranging from the infinitely small to the infinitely large through finite quantities, is obtained by allowing the exponent  $n$  to run over the real numbers. In the above computations there is the implicit assumption that the product of an infinitely small  $\omega$  and a finite quantity  $a$  is, again, an infinitely small. The proof of this fact is found in page 15: In order to prove it, it suffices to show that  $\alpha\omega < \frac{1}{\delta}$ , where  $\frac{1}{\delta}$  is an arbitrarily small quantity. But this inequality can be also written in the form  $\alpha\delta < \frac{1}{\omega}$  and since  $\omega$  is an infinitesimally small, its inverse is larger than any given number. Therefore  $a\omega$  is an infinitely small quantity. With the above remarks, Archilla presents the usual algebraic rules for infinitesimals and establishes the relationships between limits and infinitesimals by noticing in page 20 that:

“If the difference between a variable  $x$  and a constant  $a$  is infinitely small, then this constant is the limit of the variable. Therefore, the equation  $x = a + \omega$  necessarily implies  $\lim x = a$ ”

## 2.2 Archilla on continuous functions

The notion of continuity is presented by Archilla in two stages. First, a general idea of continuity for any variable quantity, is introduced (page 56):

“A variable quantity  $x$  varies in a continuous manner if it necessarily takes every intermediate value when going from  $a$  to  $b$ , and, moreover, the same property holds for any subinterval  $[a_1, b_1] \subseteq [a, b]$ ”.

This definition has a geometric flavour, and the precision about the restriction to any subinterval is a most interesting one. In a second stage the notion of continuous function is introduced (page 57):

“A function  $f(x)$  of a variable  $x$  is said to be continuous between the values  $x = a$  and  $x = b$  when  $x$  varies in a continuous way between these values, if the function values can not pass

from some value  $m$  to any other one  $n \neq m$ , both between  $f(a)$  and  $f(b)$ , without taking all intermediate values between  $m$  and  $n$ ".

This is Medvedev's "continuity in the sense of Dirichlet-Lobachevskii" (Medvedev, 1991, p. 60-62), and it obviously needs one more condition, *i.e.* monotonicity of the function, which is implicitly assumed as shown by the definition of discontinuity explained by the figure in page 58. Nevertheless, when coming to practical questions, Archilla does not forget his programme on infinitesimals: In the same page 58 he recalls that

"(...) a function is continuous on a given interval if, to any infinitely small increment of the variable in that interval, there exists an associated infinitely small increment of the function".

Next let us observe how the continuity of the logarithm is assessed: Let the increment of the logarithm be  $\log(x+h) - \log(x) = \log\frac{x+h}{x} = \log(1 + \frac{h}{x})$ . Now it is known -because it had been proved earlier in the book- that for any fixed  $x$ ,  $\log(1 + \frac{h}{x})$  and  $\frac{h}{x}$  differ in an infinitely small  $\omega$  of order larger than  $\frac{h}{x}$ , namely  $\log(x+h) - \log(x) = \frac{h}{x} + \omega$ . Therefore it is plain that Archilla considers that both the Cauchy and the Dirichlet-Lobachevskii viewpoints on continuous functions one are equivalent ones, without taking care of proving it: To him, that was simply true.

### 2.3 Archilla on derivable functions and on differentials

Differential calculus deals on how to establish infinitesimal relationships between the increments of the independent variable(s) and that of the dependent variable or function. Archilla denotes those increments by  $\Delta x$  and  $\Delta\varphi(x)$ , and he writes (pages 65-66):

"Direct consideration of the limit of  $\frac{\Delta\varphi(x)}{\Delta x}$  when  $\Delta x \rightarrow 0$  greatly simplifies the research whose aim is to determine the infinitesimal relationships between  $\Delta x$  and  $\Delta\varphi(x)$ , and it introduces one of the most important objects in the Mathematical Analysis. The limit is called the derivative, or derived function, of  $\varphi(x)$ ".

To show how *Principios* is still a mixture of old and new ideas, it is enough to notice that Archilla sticks to the old idea that the variable can not take the value of the limit when approaching it and, moreover, he insists on a function being continuous for it to be differentiable, even acknowledging that this condition is not sufficient. He explains it by considering the infinitesimal orders of the two increments.

As a remark, the authors realise that there is no reference in *Principios* either to Weierstrass or to the existence of continuous functions without a derivative at any point. Nevertheless, Archilla might have known about these functions after the first 1880 edition of *Principios*, since he spoke about them in his 1888 discourse for the *Academia*. It has been impossible to know whether he thought of including such a topic in a possible second edition, as the 1894 one is simply a reprint, most possibly prepared by his son for monetary reasons.

The treatment of the mean value theorems shows clearly the influence of Cauchy. In a rather complicated paragraph (see note 4) in page 76, the author explains that if the derivative  $\varphi'(x)$  is a continuous function for all  $x$  between  $x_0$  and  $x_0 + h$ , then it will take finite values, their mean value  $\mu(\varphi'(x))$  being somewhere between the maximum and the minimum of the derivative on that interval. Therefore, due to the assumed continuity of the derivative there, exists a value  $x_d$  such that  $\varphi'(x_d) = \mu(\varphi'(x_d))$  and the equation he obtained previously as an “interesting property” of the derivative:

$$\varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_d),$$

can be rewritten as:

$$\varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_h) = h\varphi'(x + \theta h),$$

from which a completely modern version of Rolle’s Theorem follows.

It must be noticed that the “interesting property” is simply an integral-free version of the mean value for integrals as applied to the continuous derivative:

$$\varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_h) = \int_{x_0}^{x_0+h} \varphi'(s) ds,$$

obtained by the author through a rather obscure argument in pages 74 and 75.

The largest part of *Principios* extends from page 115 until the end of the treatise, under the title “Book III: Differential Calculus”, and it starts with the concept of the differential of a function. Thus:

“When studying the form and general properties of the infinitely small increment of functions of one or several variables, we realised that among the infinite number of quantities differing infinitely little from the increment of the function, there existed one simpler than any other quantity, and it was expressed through the derivative or partial derivatives of the function; moreover, we saw that this unique form, in the case of a function of a single variable  $y = \varphi(x)$ , was  $\varphi'(x)\Delta x$ . This infinitely small quantity, which is unique by his form among all quantities that differ infinitely little from  $\Delta y$ , will be called the differential of  $y = \varphi(x)$ , and will be represented by the special notation  $dy$  or  $d\varphi(x)$ ...”

This surprisingly modern definition lacks only the words “linear function” to be a fully current one. Nevertheless, Archilla is well aware of the linearity of the differential as a function of the increment of the independent variable:

“The values of the differential  $d\varphi(x)$  for some definite  $x$  are proportional to those of  $\Delta x = dx$  and this is a property exclusive to the differential among all quantities infinitely close to the increment  $\Delta y$ .”

*Principios* ends without any consideration of the Integral Calculus. The authors do not know whether the author intended to publish a sequel on this topic, but the fact is that no evidence has been found to support this view.

### 3 Clariana and his lecture notes on Mathematical Analysis: Complex Analysis

Lauro Clariana Ricart (1842-1916) spent his life in Catalonia, mostly in Barcelona and Tarragona, the capital city of the neighbouring province of the same name. He studied in Barcelona the courses needed to become *Licenciado* (1872) in the so called *Ciencias Exactas*, the name given in Spain to Mathematics for more than a century. One year later he obtained the Doctorate in the exact Sciences at the same university. Clariana was, for a long period between 1865 and 1916, a prolific writer on many topics ranging from Mathematical Analysis and Rational Mechanics to Music and the construction of several machines. A detailed biographical account and a complete list of Clariana's writings –some of them unpublished- can be found in (Clariana-Clarós, 1993).

Clariana taught Integral and Differential Calculus, and Rational Mechanics, and was one of the few Spaniards who participated in several International Congresses and Meetings. He travelled to Paris, Brussels, München, and Freiburg between 1888 and 1900, and in 1888 he was awarded in Paris a prize for his Memoir “On the spirit of Mathematics in the modern times”. Clariana was the author of two books on Mathematical Analysis for the use of students at the *Escuela Superior de Ingeniería Industrial* in Barcelona. They were published in 1892 and 1893 under the titles *Resumen de las lecciones de Cálculo Diferencial e Integral* (Clariana Ricart, 1892b) (*Resumen* from now on), and *Complemento a los elementos de los Cálculos* (Clariana Ricart, 1892a) (*Complemento* in what follows). They appeared as lithographed handwritten lecture notes, and a joint edition under the title *Conceptos fundamentales de Análisis Matemático* appeared in a more normal printing style the year 1903 (Clariana Ricart, 1903). A small favourable review of this last book appeared in the Bulletin of the American Mathematical Society the year 1904 (McFarlane, 1904).

*Resumen* is a general introduction to Analysis, and at the very beginning, in page 5, Clariana makes his position clear (note 5):

“Because the infinitely small and the infinitely large are the only elements that can become the basis of quantity in Mathematics, synthesised in the finite ones, we shall admit three categories of quantity under the following forms:

- (1) That of the infinitely small.
- (2) That of the finite.
- (3) That of the infinitely large.

And these are the only true concepts of quantity that are directly connected to the Leibnizian idea of a differential”

A very long introduction (*Prolegómenos*) of 45 pages is offered on the various classes of numbers and functions, as well as on the foundations and the history of the infinitesimal method, where the author summons Descartes, Johann Bernouilli and Cournot, and indeed Newton, Leibniz, and D'Alembert. The rest of the book is a classical treatise on the usual topics on Differential and Integral Calculus presented in a straightforward way. Theorems



are not highlighted and proofs are not distinctly offered. The emphasis is on the succession of useful and applicable formulas, with a few examples spread over the text. This makes the book very readable and surprisingly modern even to today's standards. *Complemento* has a different flavour: It is a compilation of loosely knit topics in higher Analysis. Clariana declares in a brief introduction:

“The aim of this book is to present what we should call ‘modern theories of the infinitesimal and integral calculus’, not because some of them are recent ones, but because they have not yet been presented in the Spanish education”.

The first two chapters are devoted to “Infinitesimally Small Triangles” and “Orders of Comparison for Curves”, where the fundamental Leibnizian triangle is explained and applied in depth, as well as the idea of the order of an infinitesimal is deeply applied to the study of different elements of curves.

With this equipment, the book follows with a study of Classical Differential Geometry. On the remaining chapters, a variety of topics is included. There are the Euler-McLaurin summation formula, special functions and elliptic integrals... To summarise; it is a simplified version of the usual second volume in the classical French treatises that clearly inspired the author, and the style is the same of *Resumen*. In basic questions, Clariana holds the same opinions of Archilla. As an example, the definition of a continuous function on an interval reads:

“The function  $y = F(x)$  is continuous between the values  $a$  and  $b$  attributed to  $x$  if its values pass from one value to another through values that differ between them as little as desired.”

But the main feature in Clariana's work is that he is the first author to introduce in print Complex Analysis in Spain. He plainly states that “a complex quantity has the form  $x + yi$ , where  $x$  and  $y$  are real quantities” and goes on by explaining that Gauss was the first to speak of  $yi$  as imaginary and that Cauchy denoted as imaginary the whole complex quantity. Of course he points out that when  $x$  and  $y$  are variable quantities, then  $x + yi$  is a complex variable and that a complex quantity is infinitesimally small (large) if the real part  $x$  and the imaginary part  $y$  are infinitesimally small (large) quantities. Continuity of a complex quantity is, of course, assessed from the continuity of its component real variables. Then, a standard theory follows.

It must be noted that before Clariana no Spanish mathematician had studied complex quantities as the object of Analysis. Only algebraic, geometric or arithmetic considerations had been made in Spain on these numbers, and for nearly forty years the source book was the rather obscure *Teoría trascendental de las cantidades imaginarias*, the posthumously edited work (1865) of José María Rey Heredia (1818-1861) who inspired several developments, especially in the presentation of Analytic Geometry. The work of Rey Heredia and some of his followers has been extensively studied elsewhere by the authors (Pacheco et al., 2006).

#### 4 Villafañe and the *Tratado de Análisis Matemático*

José María Villafañe Viñals (1830-1915) was of Cuban origin and taught Mathematics at different Universities: Barcelona, Valencia, and Madrid. His only academic degree was obtained at a very early age from a secondary school in his native Santiago de Cuba, and once in metropolitan Spain he took advantage of various legal shortcuts and academic tricks until he became a *Catedrático* (full professor) at the University of Valencia, where he was a friend of the Spanish Nobel Prize winner for Medicine (1906) Santiago Ramón y Cajal. A most complete biography of Villafañe can be found in (Llompart and Lorenzo, 2001).

In 1892 Villafañe published a treatise on Mathematical Analysis (Villafañe y Viñals, 1892) in three parts, which he afterwards enlarged to four. The first one deals with the basic techniques of Analysis: Functions, continued fractions, numerical congruencies, complex numbers... The second and most interesting part of the book contains the Infinitesimal Analysis: limits, infinitesimals, continuous functions, derivatives, integrals, and infinite series. Equation solving and algebraic forms conform the respective contents of the third and fourth parts. Although Villafañe's book studies the same topics presented in the above analysed books, it must be noted that its presentation is much more formal and rigorous. The authors shall dwell only in some remarkable differences. A first one is observed in the definition of a limit (note 6):

“A fixed quantity will be called the limit of a variable quantity if this last one indefinitely approaches the first one until the difference between them is smaller than any arbitrarily small chosen quantity.”

Indeed this formulation is much closer to the familiar  $\varepsilon - \delta$  definition, although the absolute value is not employed. Nevertheless, a few pages onwards the absolute value is used when trying to prove that  $\frac{a}{0}$  is larger than any finite quantity. In Part II, Chapter III, this definition of continuity is found (note 7):

“A function is continuous if (infinitely) small increments of the variable there yield (infinitely) small increments *-in absolute value-* of the function”.

Villafañe does not make the distinction between continuity and uniform continuity, established by Heinrich Heine (1821-1881) (Heine, 1872), and already dealt with by Dirichlet as early as 1854. What Villafañe really tried to do was to prove the equivalence between the continuity definitions of Cauchy and of Dirichlet-Lobachevskii. Complex Analysis is also considered in this treatise, as in the books by Clariana. But Villafañe goes further by offering for the first time in a Spanish text a fairly good approximation to the correct definition of continuity of a function of a complex variable, when he writes:

“(...)the function  $u = f(z)$  of the imaginary variable  $z$  is continuous on the (plane) surface limited by a closed contour if for each value of  $z$  within these bounds the modulus of the functional increment  $\Delta u = f(z + \Delta z) - f(z)$  tends to zero if the modulus of  $\Delta z$  tends to zero”.

Moreover, holomorphic functions are defined, in page 194 of Part II, before the derivative of a complex function as continuous, monotropic and monogenic functions, or equivalently admitting a well determined derivative. Some bulk errors are found in Villafañe's treatise, and here two of them are considered. When dealing with derivatives (Part II, page 211) this paragraph is found:

"There exists a finite limit for the ratio of the infinitely small increments  $k$  and  $h$  of a continuous variable and its independent variable, and only some exceptional values of the variable can make the ratio grow indefinitely towards infinity or indefinitely decrease towards zero".

This is obviously wrong; since the statement affirms that every continuous function is differentiable but for a set of exceptional points, and it contradicts the condensation principle introduced by Hermann Hankel (1839-1873). With this method Hankel managed to present in 1870 a construction of a continuous function without a derivative at any rational point. Strangely enough, Villafañe knew, if not this construction, at least the 1872 example by Weierstrass of a continuous function without a derivative at any point. Nevertheless, Part II, page 216 it reads:

"We do not deem it necessary to take into account the theorem of Weierstrass where he affirms the existence of continuous functions without a derivative at any point, a matter still under discussion. Even if we admitted the existence of such functions, it would not interfere with what we are going to expound on the derivatives of continuous functions".

The calculus of complex functions is found from page 391 onwards, and the first result is the assertion that the Cauchy conditions are necessary and sufficient conditions for a function to have a correctly determined derivative with respect to the complex variable  $z$ , i.e. to be a monogenic function. The proof is essentially the same given by Cauchy (Cauchy, 1882-1974, Sér. 1, Tome I, p. 330) and continuity of the partial derivatives is used without postulating it beforehand. Indeed, with this addition, the Cauchy conditions become sufficient ones.

## 5 Pérez de Muñoz and his *Elementos de Cálculo Infinitesimal*

Ramón Pérez de Muñoz was a professor of Mathematics at the *Escuela Superior de Ingenieros de Minas* (School of Mining Engineers) in Madrid. This Pérez de Muñoz should not be mistaken with his brother Francisco, a civil engineer who promoted the study of quaternions in Spain and was a Professor at the University of Manila in the Philippines.

Pérez de Muñoz taught Calculus and Mechanics, and was the author of a book published in 1914 entitled *Elementos de Cálculo Infinitesimal* (Pérez de Muñoz, 1914). According to the author's foreword, the book is inspired in the works of Archilla, Villafañe, La Vallée-Poussin, Duhamel, Cantor and others. In addition, he explains that his aim is to present a rigorous and scientifically clear exposition, even at the cost of overflowing the usual contents of books for engineers. Both Pérez de Muñoz and Rey Pastor were members of the steering committee for the year 1912 of the newly founded *Sociedad de Matemáticas*, as shown in the February 1912 number of the *Revista* of the Society (pp. 223-233), when Rey Pastor was about to come back to Spain after his first German period. Although the authors

have not found documentary evidence of any written mathematical collaboration between them, they hypothesise that the influence of Rey Pastor was determinant in Pérez de Muñoz writing in the new mathematical style in Spain. A comparison with (Rey Pastor, 1918) clearly points out in that direction.

In his text several novelties are found. The first chapters deal with number systems, being this book the first Spanish one to offer a construction of the real field following Cantor and Dedekind. It starts with the notions of a set and its cardinality, and a proof is offered of the fact that between any two (rational) numbers there are infinitely many ones. After considering these “cuts” in the sense of Dedekind, the continuity and non denumerability of the real line are established, though in a rather rhetoric manner. The theorem on the existence of a least upper bound for any bounded set is also presented. Limits are defined in a modern way, although inaccessibility of the limit is still preserved. Nevertheless, the proof that a monotonic and bounded sequence has a unique limit is a completely rigorous one based on the previous ideas on real numbers. For the first time, the theorem asserting that any function having a derivative at some point is also continuous at that point is proved, although the hypothesis of continuity in some neighbourhood of the point is not stated.

To summarise, most possibly due to the influential ideas brought from Germany by Rey Pastor, there is a large qualitative leap forward in this book when compared with its predecessors by Archilla, Clariana or Villafaña, thus paving the way to the mathematical 20<sup>th</sup> Century in Spain.

## 6 Conclusions and views

In this paper the authors show that the work of several mathematicians and engineers must be acknowledged in the enterprise of introducing rigour in the teaching and spreading of Mathematical Analysis in Spanish universities and engineering schools. The most representative four personalities and books have been dealt with, highlighting their achievements:

(1) Archilla is the first Spanish mathematician to introduce the Cauchy style in his book *Principios de Cálculo Diferencial*.

(2) Clariana was the first to present complex analysis, as well as the definition of continuity of a complex function.

(3) Villafaña made a most interesting effort when he presented

- The total derivative of a complex function.
- Several definitions related with complex functions, among them that of a holomorphic function.
- The relationship between the existence of a derivative and the Cauchy-Riemann conditions for continuous functions.
- Absolute values as a tool in proofs and definitions.

(4) Pérez de Muñoz presented for the first time in Spanish a construction of the real field and the proof of the continuity of a derivable function.

Indeed, the texts by these authors are not milestones in the road to Mathematical knowledge, but they must be acknowledged since they represent the steps towards introducing Spain in the mainstream of Mathematical Analysis during the first decades of the 20<sup>th</sup> century, in particular between the years 1915 and 1935.

## Notes

1. (...) *era necesario conseguir lo que desde un principio tan difícil aparecía: había necesidad de obligar la noción del infinito a someterse dócil y servir de instrumento a las necesidades del análisis; y a esto se oponía no sólo el carácter y sello especial con que la noción del infinito se mostraba a la consideración matemática, cuando se aplicaba al número y a la distancia, sino también el criterio filosófico que había de servir de fundamento a la legítima intervención del infinito en el análisis.*

2. (...) *a la luz de la nueva doctrina, es ya noción vulgar la de la razón de ciertos infinitamente pequeños, que, aunque siempre finita, no tiende a límite alguno; se ve claramente que la continuidad de una función no implica que los incrementos de ésta y de su variable hayan de ser del mismo orden; y se conciben y determinan, como lo ha hecho Weierstrass, funciones continuas que no tienen derivada: cosas, si no inconcebibles, difíciles de entender y de explicar en el antiguo orden de ideas.*

3. *Nos proponemos resumir en este libro los principios más importantes del Cálculo Diferencial, procurando establecer su natural subordinación y dependencia, y estudiar las íntimas relaciones que existen entre las nociones fundamentales que les sirven de base y las que de éstas legítimamente se derivan, conforme a la doctrina propuesta primero por Cauchy, y desarrollada después por Duhamel.*

4. *Si la derivada  $\varphi'(x)$  de la función de que se trata es también continua para todos los valores de  $x$  comprendidos entre  $x_0$  y  $x_0 + h$ , los valores de la derivada serán finitos en este intervalo, y su media  $\mu(\varphi'(x))$  tendrá siempre un valor comprendido entre el mayor y menor de dichos valores de la derivada; por consiguiente, hay un valor  $x_d$  de la variable para el cual la derivada  $\varphi'(x_d)$  es igual a dicha media, y la ecuación  $\varphi(x_0 + h) - \varphi(x_0) = h\mu(\varphi'(x))$  puede escribirse con otra forma*

$$(B) \varphi(x_0 + h) - \varphi(x_0) = h\mu'(x_d)$$

*y como  $x_d$  es uno de los valores de la variable comprendidos entre  $x_0$  y  $x_0 + h$ , sera  $x_d = x_0 + \theta h$ , en cuya expresión es  $\theta$  un número positivo, en general comprendido entre cero y la unidad, que también en casos particulares podrá tomar uno de estos dos valores. La ecuación (B) puede, por lo tanto, escribirse como sigue:*

$$(C) \varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_0 + \theta h)$$

5. Siendo los indefinidamente pequeños, así como los indefinidamente grandes, los únicos elementos que pueden constituir la base de la cantidad en matemáticas, sintetizados en lo finito, admitiremos en esta ciencia tres categorías de cantidad, bajo la forma siguiente:

(1) Correspondiente a los indefinidamente pequeños.

(2) Correspondiente a lo finito.

(3) Correspondiente a los indefinidamente grandes.

Estos son los únicos y verdaderos conceptos de cantidad que se enlazan directamente con la diferencial de Leibniz.

6. Se llama límite de una variable la cantidad fija, a que esta variable se aproxima indefinidamente hasta poder ser la diferencia entre la variable y la cantidad fija menor que toda magnitud tan pequeña como se quiera.

7. Una función es continua, cuando para incrementos pequeños de la variable, los incrementos correspondientes de de la función son también infinitamente pequeños en valor absoluto.

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