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Hartley transform: basic theory and applications in oceanographic time series analysis

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Abstract

The Hartley transform, a real valued alternative to the complex Fourier transform, is presented as an efficient tool for data analysis in physical oceanography. Basic theoretical properties of this real-valued transform are briefly reviewed. Similarities and differences between Fourier and Hartley integral transforms and their discrete versions, as well as computational benefits or disadvantages between numerical algorithms used to evaluate their discrete versions are presented. The Hartley transform is used to estimate the spectral density function of ocean surface waves and coastal current time series.

1 Introduction

In physical oceanography, as in many other areas of science and engineering, the spectral analysis of time series is an standard procedure to investigate the physics underlying the observed dynamical processes. The basic idea of spectral analysis rests on the method of Fourier series, which states that any periodic function satisfying certain conditions, chiefly those of convergence, may be represented by a series of complex exponencial functions. The generalization of this idea to non-periodic functions implies the substitution of Fourier series by the Fourier integral, leading to the concepts of Fourier transform and spectral analysis.

The Fourier transform utility lies in its ability to transform a time signal into the frequency domain to analize its frequency content in terms of amplitude and phase. This capability is due to the fact that the Fourier coefficients of the transformed function represent the contribution of each sine and cosine function at a given frequency.

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Spectral analysis has its roots in the early 1800s, with the Joseph Fourier works. However, it has a history filled of controversy, discoveries, and rediscoveries. Well documented analyses of this fascinating history have been given by various authors [1, 2, 3, 4]. The frequency spectrum concept was introduced in the field of oceanographic time series analysis during the period from the late 1940s to the early 1950s. It was first used in the study of ocean wind waves around 1950 [5, 6]. Over the last five decades, its use has developed and generalised quickly after the discovery of the Wiener-Kintchine theorem, establishing a relation between covariance and power spectrum [7], and also with the increasing availability of powerful digital computers and the development of fast algorithms [8], to become the most widely used data analysis method by oceanic scientists.

Nevertheless, the practical use of Fourier methods presents some drawbacks in the analysis of time series observed in nature. Particularly, while signals observed in most real-word applications are real-valued the Fourier transform uses complex arithmetic, transforming a sequence of real data from the time domain into a sequence of complex numbers in the frequency domain. Half of the numbers in the frequency domain corresponds to the information in the negative frequencies and repeat the information contained in the positive frequencies. Furthermore, the multiplication of two complex numbers require four real multiplications and to two real additions. Consequently, due to the amount of memory required, the redundant information, and the number of computations needed, it seems obvious that the Fourier transform is not the most efficient method to transform real time series into the frequency domain.

Hartley in 1942 [9] proposed an alternative transform to avoid the time and memory computation shortcomings related to the Fourier transform of real data. This transform was expressed in a more symmetrical form between the function of the real variable and its transform. However, while this transform works only with real numbers and is easy to compute, it remained little known among signal analysts. To understand this fact is necessary to take into account that the Hartley's work was published at about the same time as the massive use of spectral methods based on the complex Fourier transform was starting up. It was not until its rediscovery in 1983 by Bracewell [10] that the algorithm began to gain some attention. Bracewell contributions [10, 11] revived the interest in the Hartley transform and a large number of articles, over two hundred, have been published during the last ten years, mainly in the signal processing specialized literature. Nevertheless, the Hartley transform still has not gained the attention of the oceanographic data analysis community, even though it has been applied during the last decade in closely related areas such as geophysics [12, 13].

The fundamental purpose of the present work is to introduce the Hartley transform as an efficient tool for time series analysis in physical oceanography. The remainder of the paper is organized as follows. A brief introduction to the discrete Fourier transform and an overview of some basic properties of the discrete Hartley transform are provided in section 2. Relations between the Fourier and Hartley transforms and the corresponding fast algorithms are analysed in section 3. Section 4 presents some examples of spectral analysis of oceanic physical pro-

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cesses by exploting the advantages of the Hartley transform. These are followed by a summary of conclusions and suggestions for practical analysis of time series in physical oceanography in section 5.

2 Discrete Fourier and Hartley Transforms

To examine the properties of the Fourier and Hartley transforms let us consider a real-valued time series $x(t)$ of length T and digitized with a constant sampling period Δt,

$$x(t) = x(n\Delta t) = \Big\{x(0), x(\Delta t), x(2\Delta t), \cdots, x(N-1)\Big\} \tag{1}$$

where N is the number of samples and $T = N\Delta t$.

2.1 Discrete Fourier Transform

The well known integral Fourier transform of a continuous function of time $x(t)$ and its inverse transforms are given respectively by

$$F_x(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \tag{2}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_x(f)e^{i2\pi ft}df \tag{3}$$

where the kernel transform function is

$$e^{\pm i2\pi ft} = \cos(2\pi ft) \pm i\sin(2\pi ft) \tag{4}$$

In practice, the evaluation of equations (2) and (3) must be done using discrete samplings of the data over a finite range. Consequently, a discrete approximation of these equations must be used. The corresponding discrete versions are the well known discrete Fourier transform (DFT) and the inverse discrete Fourier transform (IDFT). The DFT can be expressed as

$$F_x(f) = F_x(k\Delta f) = \sum_{n=0}^{N-1} x(n\Delta t)e^{-i2\pi k\Delta fn\Delta t}\Delta t \qquad k = 0, \cdots, N-1 \tag{5}$$

whereas the original time series can be recovered by using the IDFT

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$$x(n\Delta t) = \frac{1}{N\Delta t}\sum_{k=0}^{N-1} F_x(k\Delta f)e^{i2\pi k\Delta f n\Delta t}\Delta f \qquad n = 0,\cdots,N-1 \tag{6}$$

It is worthy of note that the DFT and IDFT expressions seem almost identical, except for the scale factor and the reversed sing in the exponent.

2.2 Discrete Hartley Transform

The Hartley transform of a real-valued function $x(t)$ and its inverse are defined respectively as [10]

$$H_x(f) = \int_{-\infty}^{\infty} x(t)\,\mathrm{cas}(2\pi f t)dt \tag{7}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_x(f)\,\mathrm{cas}(2\pi f t)df \tag{8}$$

where the kernel transform function is

$$\mathrm{cas}(x) = \cos(x) + \sin(x)$$

The name cas( ) stands for "cosine-and-sine". It is interesting to note that the sum of the sine and cosine functions is just another sine function shifted by $\pi/4$. However, this is the key for the symmetry between the transform and its inverse exhibited by the Hartley transform.

As in the case of the Fourier transform, the practical evaluation of the Hartley transform pair (eqs. 7 and 8) from the real valued time series $x(t)$ requires the use of discrete approximations. The discrete Hartley transform (DHT) is given by

$$H_x(k\Delta f) = \sum_{n=0}^{N-1} x(n\Delta t)\,\mathrm{cas}(2\pi k\Delta f n\Delta t)\Delta t \qquad k = 0,\cdots,N-1 \tag{9}$$

and the corresponding inverse discrete Hartley transform (IDHT) can be written as

$$x(n\Delta t) = \frac{\Delta f}{N\Delta t}\sum_{k=0}^{N-1} H_x(k\Delta f)\,\mathrm{cas}(2\pi k\Delta f n\Delta t) \qquad n = 0,\cdots,N-1 \tag{10}$$

Note that equations (7-8), as well as their discrete counterparts (9-10), present the same form. There are no sign changes as in the traditional Fourier transform. Furthermore, while the Fourier transform of a real signal is a complex function, the Hartley transform of a real function is also real.

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3 Relations between the Fourier and Hartley transforms

In this section some relations between the Hartley and Fourier transform are presented. The analysis will be made based on the continuous time version of these transforms, although the discussion is obviously valid for the discrete time case too. A detailed study of both transforms can be found in [14].

Expanding the complex exponentials in the Fourier transform relations by using the Euler formulas and comparing the result to the Hartley transform relations, it is easy to obtain the following interesting results. The even part of the Hartley transform $E\left[H_x(f)\right]$ is the real part of the Fourier transform

$$E\left[H_x(f)\right] = \frac{H_x(f) + H_x(-f)}{2} = \int_{-\infty}^{\infty} x(t)\cos(2\pi f t)dt = \Re(F_x(f)) \quad (11)$$

Similarly, the odd part of the Hartley transform $O\left[H_x(f)\right]$ is the imaginary part of the Fourier transform

$$O\left[H_x(f)\right] = \frac{H_x(f) - H_x(-f)}{2} = \int_{-\infty}^{\infty} x(t)\sin(2\pi f t)dt = \Im(F_x(f)) \quad (12)$$

Thus, the Fourier transform of $x(t)$ can be readily extracted from $H_x(f)$ by simple reflections and additions

$$F_x(f) = E\left[H_x(f)\right] - iO\left[H_x(f)\right] \quad (13)$$

Conversely, given the Fourier transform $F_x(f)$, it is possible to obtain $H_x(f)$ by noting that

$$H_x(f) = \Re(F_x(f)) - \Im(F_x(f)) \quad (14)$$

It should be noted that Fourier and Hartley transforms are very similar. They are related to one another by equations (13) and (14). Also, it can be observed that both transforms may be expressed as combinations of the sine and cosine transforms. Another important fact is that Hartley and Fourier transform are invertible and consequently they carry the whole information about the original signal but in a different way.

3.1 Fast Fourier and Hartley transform algorithms

It is a well known fact that the practical evaluation of the Fourier integrals for a time series of N sample points through direct implementation of the DFT and IDFT is not an efficient procedure because it requires about N^2 arithmetic operations. Consequently, a large number of fast algorithms, generally named as fast

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Fourier transform (FFT) algorithms, have been developed over the years for the efficient computation of the DFT. The first major breakthrough was the Cooley & Tukey algorithm [8], which reduced the complexity of a DFT from $O[N^2]$ to $O[N \log N]$.

The implementation of the DHT requires a lower number of arithmetic operations than the DFT for the same record lenght. Nevertheless, it is also necessary the development of fast algorithms to improve its efficiency. A number of fast Hartley transform (FHT) algorithms have been developed to compute de DHT in just the same way as the FFT algorithms, by taking into account the similarity between both tranforms [11, 15, 16, 17]. The authors of the different FHT algorithms claim it to be computationally more efficient than the FFT, in terms of faster computing, simpler programming and identical direct and inverse transforms. This is evident in the comparison of the FHT with the complex-valued FFT. However, these assertions have caused controversy, but have also given rise to the development of a substantial number of efficient real valued fast Fourier transforms, specifically suited for the DFT computation of real signals.

To judge by the results reported by comparative studies, e.g. [18], it seems that the various real valued FFT and the FHT algorithms show a considerable similarity in terms of computational efficiency. An efficient real FFT algorithm or a FHT algorithm gives an increase in speed by approximately a factor of two. However, the main advantages of the FHT lie in its inherently real valued nature and equivalence of the forward and inverse transformations, making possible to apply just exactly the same algorithm to compute any of them.

4 Applications

As stated previously, when sampled data are in the real domain the FHT may be applied instead of the more commonly used FFT, designed for complex data, in virtually any application. In particular, the Fourier spectrum can be efficiently calculated via the FHT. The benefits of calculating the power spectra using the FHT are about 50% less data memory required, because there are no imaginary data, and about 40% faster program execution, since no complex operations are required, at no loss in accuracy.

It has been shown that for real signals the even and odd parts of the Hartley spectrum are the real and imaginary parts of the Fourier spectrum, respectively. Thus, the power spectrum in terms of the Hartley transform can be expressed as

$$P_x(f) = |\Re(F_x(f))|^2 + |\Im(F_x(f))|^2 = \Big(E\,[H_x(f)]\Big)^2 + \Big(O\,[H_x(f)]\Big)^2 \tag{15}$$

$$= \frac{[H_x(f) + H_x(-f)]^2}{4} + \frac{[H_x(f) - H_x(-f)]^2}{4} = \frac{H_x^2(f) + H_x^2(-f)}{2}$$

It is also straightforward to compute the phase spectrum from

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$$\phi_x(f) = \arctan\left[\frac{\Im(F_x(f))}{\Re(F_x(f))}\right] = \arctan\left[-\frac{O\left[H_x(f)\right]}{E\left[H_x(f)\right]}\right]$$
$$= \arctan\left[\frac{H_x(-f) - H_x(f)}{H_x(f) + H_x(-f)}\right] \quad (16)$$

Two examples of spectral density estimation via FHT in physical oceanography are presented below. Firstly, a synthetic record of wind waves is used to check the correctness of the FHT based methodology. Note that spectral density function is usually estimated form samples of an unknown function. In this case the true spectrum associated to the analysed time series is also unknown and uncertainty about the results exist. The use of numerically simulated time series from a known parent spectrum removes this uncertainty. The procedure applied to simulate the analysed wave record, shown in Fig. 1a, assums that the vertical displacement of the sea surface at a given point is due to the linear superposition of a finite but large

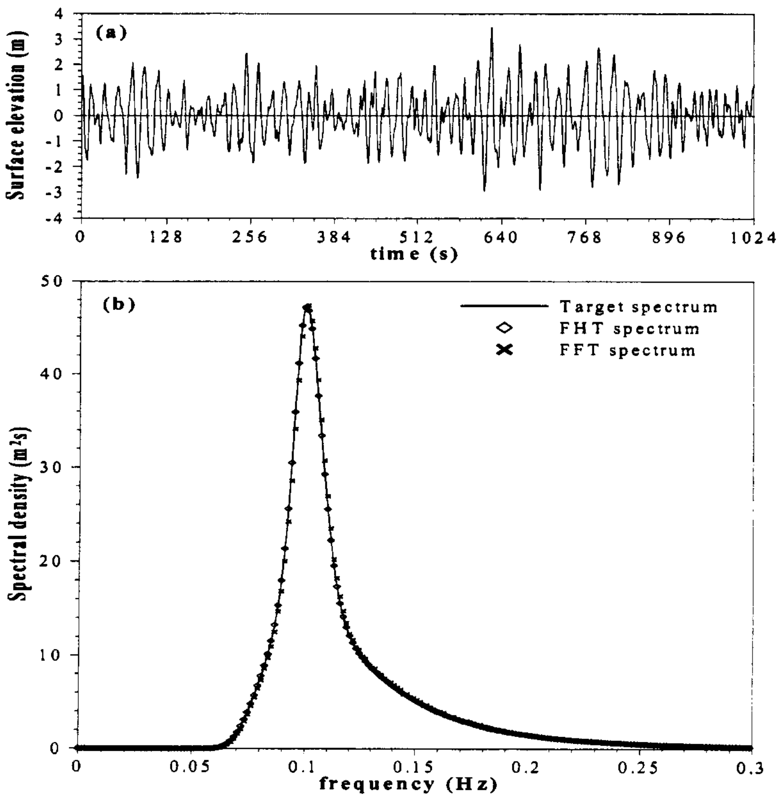

Figure 1: Simulated wave record (a), target spectrum and spectral density function estimated via FHT and FFT (b).

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number of Fourier components with fixed frequencies and random phases uniformly distributed over the range $(0 - 2\pi)$, whereas the amplitudes are related to the spectral density function in a deterministic way. This procedure ensures that the spectrum estimated from the simulated time series will reproduce the specified target spectrum.

The spectral density function of the numerically simulated wave record estimated via FHT is shown in Fig. 1b. Results obtained by using the FFT algorithm are also represented for comparison. It can be observed that both methods reproduce almost exactly the target spectrum. Thus, spectral densities computed through the FFT and FHT are equivalent, such as expected.

Figure 2 represents the spectral densities of the zonal (E-W) and meridional (N-S) components of a coastal current velocities record measured at the East coast of Gran Canaria island [19]. The examined time series is shown in Fig. 2a, as a vector stick diagram, and the spectra corresponding to the zonal and meridional components are represented in Fig. 2b. It can be observed that semidiurnal tidal motions strongly prevail, while diurnal currents are considerably weak, mainly in the zonal direction. It is also possible to detect some higher frequency tidal compo-

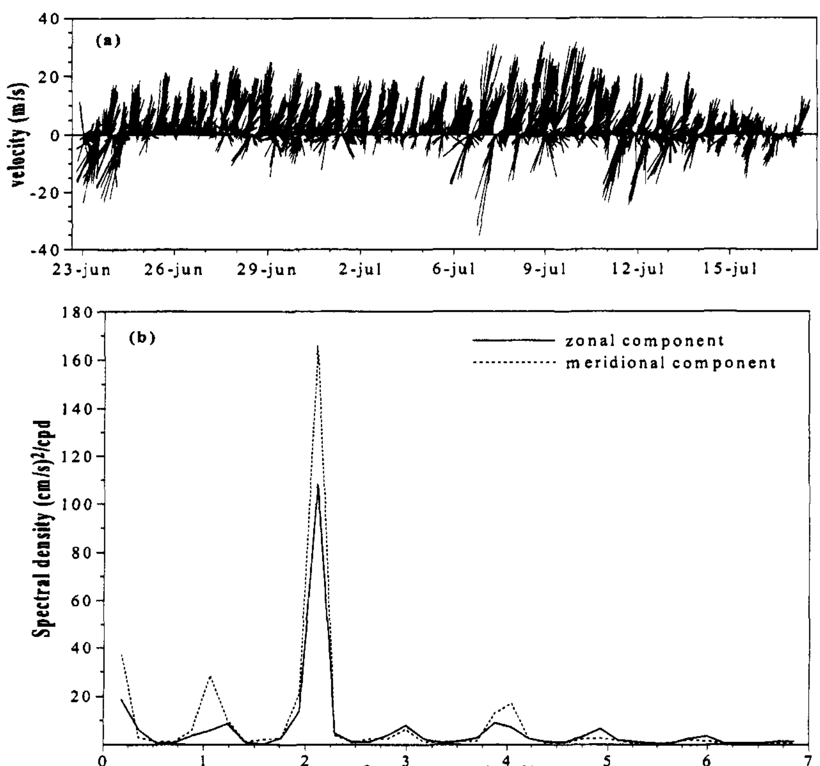

Figure 2: Measured coastal currents record (a) and spectral density functions estimated via FHT for the zonal and meridional velocity components (b).

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nents, such as the third-diurnal and fourth-diurnal constituents. Spectral densities represented in Fig. 2b have been obtained by averaging each five adjacent raw spectral estimations, to reduce the variability of the periodogram estimates.

5 Discussion

It has been shown that the fast Hartley transform represents an efficient alternative to the commonly used complex-valued FFT algorithms for the analysis of real-valued time series, avoiding redundant arithmetic operations and inefficient memory allocation. Furthermore, due to its real-valued nature, the transform of a real signal is also a single real signal that includes amplitude and phase information. A particularly remarkable property is the equivalence of the forward and inverse Hartley transforms.

Both, Fourier and Hartley transforms, satisfy similar theorems and can therefore be applied in an analogous manner. These facts ensure that FFT can be replaced by the FHT in virtually any oceanographic application, including estimation of the spectral density function and any other function or parameter that can be calculated by means of the Fourier transform.

For most oceanographic applications sampled data are limited to real-valued time series and procedures of frequency analysis of large data volumes, as for example wind-wave data bases, require large memory space and long computation times. As a consequence of the above commented properties, the fast Hartley transform may be considered as a computational tool specially suitable for processing large data bases of time series in physical oceanography. In particular, it has been shown that spectral analysis of oceanographic time series can be efficiently performed. Finally, it is worth emphasizing that the computational improvements are obtained without loss in accuracy.

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