Fixed point theory for cyclic \((\varphi - \psi)\)-contractions

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Abstract

In this article, the concept of cyclic \((\varphi - \psi)\)-contraction and a fixed point theorem for this type of mappings in the context of complete metric spaces have been presented. The results of this study extend some fixed point theorems in literature.

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1. Introduction and preliminaries

One of the most important results used in nonlinear analysis is the well-known Banach’s contraction principle. Generalization of the above principle has been a heavily investigated branch research. Particularly, in [1] the authors introduced the following definition.

Definition 1. Let \(X\) be a nonempty set, \(m\) a positive integer, and \(T: X \to X\) a mapping. \(X = \bigcup_{i=1}^{m} A_i\) is said to be a cyclic representation of \(X\) with respect to \(T\) if

(i) \(A_i, i = 1, 2, ..., m\) are nonempty sets.
(ii) \(T(A_1) \subseteq A_2, \ldots, T(A_{m-1}) \subseteq A_m, T(A_m) \subseteq A_1.\)

Recently, fixed point theorems for operators \(T\) defined on a complete metric space \(X\) with a cyclic representation of \(X\) with respect to \(T\) have appeared in the literature (see, e.g., [2-5]). Now, we present the main result of [5]. Previously, we need the following definition.

Definition 2. Let \((X, d)\) be a metric space, \(m\) a positive integer \(A_1, A_2, \ldots, A_m\) nonempty closed subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). An operator \(T: X \to X\) is said to be a cyclic weak \(\varphi\)-contraction if

(i) \(X = \bigcup_{i=1}^{m} A_i\) is a cyclic representation of \(X\) with respect to \(T\).
(ii) \(d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \) for any \(x \in A_i, y \in A_{i+1}, i = 1, 2, ..., m,\) where \(A_{m+1} = A_1\) and \(\varphi: [0, \infty) \to [0, \infty)\) is a nondecreasing and continuous function satisfying \(\varphi(t) < 0\) for \(t \in (0, \infty)\) and \(\varphi(0) = 0.\)

Remark 3. For convenience, we denote by \(F\) the class of functions \(\varphi: [0, \infty) \to [0, \infty)\) nondecreasing and continuous satisfying \(\varphi(t) > 0\) for \(t \in (0, \infty)\) and \(\varphi(0) = 0.\)

The main result of [5] is the following.
Theorem 4. ([5], Theorem 6) Let \((X, d)\) be a complete metric space, \(m\) a positive integer, \(A_1, A_2, ..., A_m\) nonempty closed subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Let \(T: X \to X\) be a cyclic weak \(\phi\)-contraction with \(\phi \in F\). Then \(T\) has a unique fixed point \(z \in \bigcap_{i=1}^{m} A_i\).

The main purpose of this article is to present a generalization of Theorem 4.

2. Main results

First, we present the following definition.

Definition 5. Let \((X, d)\) be a metric space, \(m\) a positive integer, \(A_1, A_2, ..., A_m\) nonempty subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). An operator \(T: X \to X\) is a cyclic weak \((\phi - \psi)\)-contraction if

\[
\begin{align*}
(i) \quad &X = \bigcup_{i=1}^{m} A_i \text{ is a cyclic representation of } X \text{ with respect to } T, \\
(ii) \quad &\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)), \text{ for any } X \in A_i, y \in A_{i+1}, i = 1, 2, ..., m, \text{ where } A_{m+1} = A_1 \text{ and } \phi, \psi \in F.
\end{align*}
\]

Our main result is the following.

Theorem 6. Let \((X, d)\) be a complete metric space, \(m\) a positive integer, \(A_1, A_2, ..., A_m\) nonempty subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Let \(T: X \to X\) be a cyclic \((\phi - \psi)\)-contraction. Then \(T\) has a unique fixed point \(z \in \bigcap_{i=1}^{m} A_i\).

Proof. Take \(x_0 \in X\) and consider the sequence given by

\[ x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots \]

If there exists \(n_0 \in \mathbb{N}\) such that \(x_{n+1} = x_n\) then, since \(x_{n+1} = Tx_n = x_{n+1}\), the part of existence of the fixed point is proved. Suppose that \(x_{n+1} \neq x_n\) for any \(n = 0, 1, 2, \ldots\). Then, since \(X = \bigcup_{i=1}^{m} A_i\), for any \(n > 0\) there exists \(i_n \in \{1, 2, ..., m\}\) such that \(x_{n-1} \in A_{i_n}\) and \(x_n \in A_{i_{n+1}}\). Since \(T\) is a cyclic \((\phi - \psi)\)-contraction, we have

\[
\phi(d(x_n, x_{n+1})) = \phi(d(Tx_{n-1}, Tx_n)) \leq \phi(d(x_{n-1}, x_n)) - \psi(d(x_{n-1}, x_n)) \leq \phi(d(x_{n-1}, x_n)) \quad (2.1)
\]

From 2.1 and taking into account that \(\phi\) is nondecreasing we obtain

\[
d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \quad \text{for any } n = 1, 2, \ldots
\]

Thus \([d(x_n, x_{n+1})]\) is a nondecreasing sequence of nonnegative real numbers. Consequently, there exists \(y \geq 0\) such that \(\lim_{n \to \infty} d(x_n, x_{n+1}) = y\). Taking \(n \to \infty\) in (2.1) and using the continuity of \(\phi\) and \(\psi\), we have

\[
\phi(y) \leq \phi(y) - \psi(y) \leq \phi(y),
\]

and, therefore, \(\psi(y) = 0\). Since \(\psi \in F, y = 0\), that is,

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \quad (2.2)
\]

In the sequel, we will prove that \([x_n]\) is a Cauchy sequence.

First, we prove the following claim.

Claim: For every \(\varepsilon > 0\) there exists \(n \in \mathbb{N}\) such that if \(p, q \geq n\) with \(p - q = 1(m)\) then \(d(x_p, x_q) < \varepsilon\).

In fact, suppose the contrary case. This means that there exists \(\varepsilon > 0\) such that for any \(n \in \mathbb{N}\) we can find \(p_n > q_n \geq n\) with \(p_n - q_n = 1(m)\) satisfying
\[ d(x_{q_n}, x_{p_n}) \geq \epsilon. \] (2.3)

Now, we take \( n > 2m \). Then, corresponding to \( q_n \geq n \) use can choose \( p_n \) in such a way that it is the smallest integer with \( p_n > q_n \) satisfying \( p_n - q_n \equiv 1(m) \) and \( d(x_{q_n}, x_{p_n}) \geq \epsilon \). Therefore, \( d(x_{q_n}, x_{p_n}) \leq \epsilon \). Using the triangular inequality

\[
\epsilon \leq d(x_{q_n}, x_{p_n}) \leq d(x_{q_n}, x_{p_{n-1}}) + \sum_{i=1}^{m} d(x_{p_{n-1}}, x_{p_{n-2}}) < \epsilon + \sum_{i=1}^{m} d(x_{p_{n-1}}, x_{p_{n-2}}).
\]

Letting \( n \to \infty \) in the last inequality and taking into account that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \), we obtain

\[
\lim_{n \to \infty} d(x_{q_n}, x_{p_n}) = \epsilon
\] (2.4)

Again, by the triangular inequality

\[
\epsilon \leq d(x_{q_n}, x_{p_n}) \leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n})
\]

\[
\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{p_{n+1}}, x_{p_n}) + d(x_{p_n}, x_{p_n})
\]

\[
= 2d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_n}, x_{p_n}) + 2d(x_{p_{n+1}}, x_{p_n})
\]

Letting \( n \to \infty \) in (2.4) and taking into account that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \) and (2.4), we get

\[
\lim_{n \to \infty} d(x_{q_{n+1}}, x_{p_n}) = \epsilon.
\] (2.6)

Since \( x_{q_n} \) and \( x_{p_n} \) lie in different adjacently labelled sets \( A_i \) and \( A_{i+1} \) for certain \( 1 \leq i \leq m \), using the fact that \( T \) is a cyclic \((\varphi, \psi)\)-contraction, we have

\[
\phi(d(x_{q_{n+1}}, x_{p_n})) = \phi(d(Tx_{q_n}, Tx_{q_n})) \leq \phi(d(x_{q_n}, x_n)) - \psi(d(x_{q_n}, x_n)) \leq \phi(d(x_{q_n}, x_{p_n}))
\]

Taking into account (2.4) and (2.6) and the continuity of \( \varphi \) and \( \psi \), letting \( n \to \infty \) in the last inequality, we obtain

\[
\phi(\epsilon) \leq \phi(\epsilon) - \psi(\epsilon) \leq \phi(\epsilon)
\]

and consequently, \( \psi(\epsilon) = 0 \). Since \( \psi \in F \), then \( \epsilon = 0 \) which is contradiction.

Therefore, our claim is proved.

In the sequel, we will prove that \((X, d)\) is a Cauchy sequence. Fix \( \epsilon > 0 \). By the claim, we find \( n_0 \in \mathbb{N} \) such that if \( p, q \geq n_0 \) with \( p - q \equiv 1(m) \)

\[
d(x_p, x_q) \leq \frac{\epsilon}{2}
\] (2.7)

Since \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \) we also find \( n_1 \in \mathbb{N} \) such that

\[
d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m}
\] (2.8)

for any \( n \geq n_1 \).

Suppose that \( r, s \geq \max(n_0, n_1) \) and \( r > s \). Then there exists \( k \in \{1, 2, \ldots, m\} \) such that \( s - r \equiv k(m) \). Therefore, \( s - r + j \equiv 1(m) \) for \( j = m - k + 1 \). So, we have \( d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j+1}) + \ldots + d(x_{s+1}, x_s) \). By (2.7) and (2.8) and from the last inequality,
we get
\[ d(x_n, x_m) \leq \frac{\varepsilon}{2} + \frac{j \varepsilon}{2m} \leq \frac{\varepsilon}{2} + \frac{m \varepsilon}{2m} = \varepsilon \]

This proves that \((x_n)\) is a Cauchy sequence. Since \(X\) is a complete metric space, there exists \(x \in X\) such that \(\lim_{n \to \infty} x_n = x\). In what follows, we prove that \(x\) is a fixed point of \(T\). In fact, since \(\lim_{n \to \infty} x_n = x\) and, as \(X = \bigcup_{i=1}^{m} A_i\) is a cyclic representation of \(X\) with respect to \(T\), the sequence \((x_n)\) has infinite terms in each \(A_i\) for \(i \in \{1, 2, \ldots, m\}\).

Suppose that \(x \in A_i\), \(Tx \in A_{i+1}\) and we take a subsequence \(x_{n_i}\) of \((x_n)\) with \(x_{n_i} \in A_{i-1}\) (the existence of this subsequence is guaranteed by the above-mentioned comment). Using the contractive condition, we can obtain
\[ \phi(d(x_{n_i}, Tx)) = \phi(d(Tx_{n_i}, Tx)) \leq \phi(d(Tx_{n_i}, x)) - \psi(d(x_{n_i}, x)) \leq \phi(d(x_{n_i}, x)) \]
and since \(x_{n_i} \to x\) and \(\phi\) and \(\psi\) belong to \(F\), letting \(k \to \infty\) in the last inequality, we have
\[ \phi(d(x, Tx)) \leq \phi(d(x, x)) = \phi(0) = 0 \]
or, equivalently, \(\phi(d(x, Tx)) = 0\). Since \(\phi \in F\), then \(d(x, Tx) = 0\) and, therefore, \(x\) is a fixed point of \(T\).

Finally, to prove the uniqueness of the fixed point, we have \(y, z \in X\) with \(y\) and \(z\) fixed points of \(T\). The cyclic character of \(T\) and the fact that \(y, z \in X\) are fixed points of \(T\), imply that \(y, z \in \bigcap_{i=1}^{m} A_i\). Using the contractive condition we obtain
\[ \phi(d(y, z)) = \phi(d(Ty, Tx)) \leq \phi(d(y, z)) - \psi(d(y, z)) \leq \phi(d(y, z)) \]
and from the last inequality
\[ \psi(d(y, z)) = 0 \]

Since \(\psi \in F\), \(d(y, z) = 0\) and, consequently, \(y = z\). This finishes the proof.

In the sequel, we will show that Theorem 6 extends some recent results.

If in Theorem 6 we take as \(\varphi\) the identity mapping on \([0, \infty)\) (which we denote by \(Id_{[0, \infty)}\)), we obtain the following corollary.

**Corollary 7.** Let \((X, d)\) be a complete metric space \(m\) a positive integer, \(A_1, A_2, \ldots, A_m\) nonempty subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Let \(T: X \to X\) be a cyclic \((Id_{[0, \infty)} \cdot \psi)\) contraction. Then \(T\) has a unique fixed point \(z \in \bigcap_{i=1}^{m} A_i\).

Corollary 7 is a generalization of the main result of [5] (see [[5], Theorem 6]) because we do not impose that the sets \(A_i\) are closed.

If in Theorem 6 we consider \(\varphi = Id_{[0, \infty)}\) and \(\psi = (1 - k)Id_{[0, \infty)}\) for \(k \in [0, 1)\) (obviously, \(\varphi, \psi \in F\)), we have the following corollary.

**Corollary 8.** Let \((X, d)\) be a complete metric space \(m\) a positive integer, \(A_1, A_2, \ldots, A_m\) nonempty subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Let \(T: X \to X\) be a cyclic \((Id_{[0, \infty)} \cdot (1 - k)Id_{[0, \infty)})\) contraction, where \(k \in [0, 1)\). Then \(T\) has a unique fixed point \(z \in \bigcap_{i=1}^{m} A_i\).

Corollary 8 is Theorem 1.3 of [1].

The following corollary gives us a fixed point theorem with a contractive condition of integral type for cyclic contractions.
Corollary 9. Let \((X, d)\) be a complete metric space, \(m\) a positive integer, \(A_1, A_2, \ldots, A_m\) nonempty closed subsets of \(X\) and \(X = \bigcup_{i=1}^{m} A_i\). Let \(T: X \to X\) be an operator such that

(i) \(X = \bigcup_{i=1}^{m} A_i\) is a cyclic representation of \(X\) with respect to \(T\).

(ii) There exists \(k \in [0, 1)\) such that

\[
\int_{0}^{d(Tx, Ty)} \rho(t) dt \leq k \int_{0}^{d(x, y)} \rho(t) dt
\]

for any \(x \in A_{i}, y \in A_{i+1}, i = 1, 2, \ldots, m\) where \(A_{m+1} = A_1\), and \(\rho: [0, \infty) \to [0, \infty)\) is a Lebesgue-integrable mapping satisfying \(\int_{0}^{\varepsilon} \rho(t) dt\) for \(\varepsilon > 0\).

Then \(T\) has unique fixed point \(z \in \bigcap_{i=1}^{m} A_i\).

Proof. It is easily proved that the function \(\phi: [0, \infty) \to [0, \infty)\) given by \(\phi(t) = \int_{0}^{t} \rho(s) ds\) satisfies that \(\phi \in F\). Therefore, Corollary 9 is obtained from Theorem 6, taking as \(\phi\) the above-defined function and as \(\psi\) the function \(\psi(t) = (1 - k)\phi(t)\).

If in Corollary 9, we take \(A_i = X\) for \(i = 1, 2, \ldots, m\) we obtain the following result.

Corollary 10. Let \((X, d)\) be a complete metric space and \(T: X \to X\) an operator such that for \(x, y \in X\),

\[
\int_{0}^{d(Tx, Ty)} \rho(t) dt \leq k \int_{0}^{d(x, y)} \rho(t) dt
\]

where \(\rho: [0, \infty) \to [0, \infty)\) is a Lebesgue-integrable mapping satisfying \(\int_{0}^{\varepsilon} \rho(t) dt\) for \(\varepsilon > 0\) and the constant \(k \in [0, 1)\). Then \(T\) has a unique fixed point.

Notice that this is the main result of [6]. If in Theorem 6 we put \(A_i = X\) for \(i = 1, 2, \ldots, m\) we have the result.

Corollary 11. Let \((X, d)\) be a complete metric space and \(T: X \to X\) an operator such that for \(x, y \in X\),

\[
\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)),
\]

where \(\phi, \psi \in F\). Then \(T\) has a unique fixed point.

This result appears in [7].

3. Example and remark

In this section, we present an example which illustrates our results. Throughout the article, we let \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\).

Example 12. Consider \(X = \left\{ \frac{1}{n} : n \in \mathbb{N}^* \right\} \cup \{0\}\) with the metric induced by the usual distance in \(\mathbb{R}\), i.e., \(d(x, y) = |x - y|\). Since \(X\) is a closed subset of \(\mathbb{R}\), it is a complete metric space. We consider the following subsets of \(X\):

\[A_1 = \left\{ \frac{1}{n} : n \ odd \right\} \cup \{0\}\]

\[A_2 = \left\{ \frac{1}{n} : n \ even \right\} \cup \{0\}\]
Obviously, \( X = A_1 \cup A_2 \). Let \( T: X \to X \) be the mapping defined by

\[
T_x = \begin{cases} 
\frac{1}{n+1} & \text{if } x = \frac{1}{n} \\
0 & \text{if } x = 0
\end{cases}
\]

It is easily seen that \( X = A_1 \cup A_2 \) is a cyclic representation of \( X \) with respect to \( T \).

Now we consider the function \( \rho: [0, \infty) \to [0, \infty) \) defined by

\[
\rho(t) = \begin{cases} 
0 & \text{if } t = 0 \\
\frac{1}{t^3 - 1} & \text{if } 0 < t < e \\
0 & \text{if } t \geq e
\end{cases}
\]

It is easily proved that \( \int_0^t \rho(s)ds = t^{1/3} \) for \( t \leq 1 \).

In what follows, we prove that \( T \) satisfies condition (ii) of Corollary 9.

In fact, notice that the function \( \rho(t) \) is a Lebesgue-integrable mapping satisfying

\[
\int_0^e \rho(t)dt > 0 \text{ for } \varepsilon > 0.
\]

We take \( m, n \in \mathbb{N}^* \) with \( m \geq n \) and we will prove

\[
\int_0^d(T(\frac{1}{n}), T(\frac{1}{m})) \rho(s)ds \leq \frac{1}{2} \int_0^d(T(\frac{1}{n}), T(\frac{1}{m})) \rho(s)ds
\]

Since \( \int_0^e \rho(s)ds = t^{1/3} \) for \( t \leq 1 \) and, as \( \text{diam}(X) \leq 1 \), the last inequality can be written as

\[
d(T(\frac{1}{n}), T(\frac{1}{m})) \leq \frac{1}{2} \left( \frac{1}{n} - \frac{1}{m} \right)
\]

or equivalently,

\[
\left| \frac{1}{n+1} - \frac{1}{m+1} \right| \leq \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|
\]

or equivalently,

\[
\left( \frac{m-n}{(n+1)(m+1)} \right) \left( \frac{(n+1)(m+1)}{m-n} \right) \leq \frac{1}{2} \left( \frac{m-n}{nm} \right)
\]

or equivalently,

\[
\left( \frac{m-n}{(n+1)(m+1)} \right) \frac{nm}{m-n} \times \left( \frac{nm}{(n+1)(m+1)} \right) \frac{nm}{m-n} \leq \frac{1}{2} \tag{3.1}
\]

In order to prove that this last inequality is true, notice that

\[
\frac{nm}{(n+1)(m+1)} < 1 \tag{3.2}
\]

and, therefore,

\[
\left( \frac{nm}{(n+1)(m+1)} \right) \frac{nm}{m-n} \leq 1
\]
On the other hand, from
\[ m - n \leq m \cdot n \]
\[ m - n \leq m + n \]
we obtain
\[ 2(m - n) \leq (n + 1)(m + 1) \]
and, thus,
\[ \left( \frac{m - n}{(n + 1)(m + 1)} \right) \leq \frac{1}{2} \]
Since \( \frac{n + m + 1}{m - n} \geq 1 \),
\[ \left( \frac{m - n}{(n + 1)(m + 1)} \right)^{\frac{n + m + 1}{m - n}} \leq \frac{1}{2} \]  \hspace{1cm} (3.3)
Finally, (3.2) and (3.3) give us (3.1).

Now we take \( x = \frac{1}{n} \) \( n \in \mathbb{N}^* \) and \( y = 0 \). In this case, condition (ii) of Corollary 9 for \( k = \frac{1}{2} \) has the form
\[
\begin{align*}
&d\left(T\left(\frac{1}{n}\right), T(0)\right) \leq \frac{1}{2}d\left(\frac{1}{n}, 0\right) \\
&= \frac{1}{2}\left(\frac{1}{n}\right)^n
\end{align*}
\]
The last inequality is true since
\[
\left(\frac{1}{n + 1}\right)^n < \left(\frac{1}{n}\right)^n
\]
and, then,
\[
\left(\frac{1}{n + 1}\right)^{n+1} = \left(\frac{1}{n + 1}\right)^n \frac{1}{n + 1} \leq \frac{1}{2} \left(\frac{1}{n + 1}\right)^n \leq \frac{1}{2} \left(\frac{1}{n}\right)^n
\]
Consequently, since assumptions of Corollary 9 are satisfied, this corollary gives us the existence of a unique fixed point (which is obviously \( x = 0 \).

This example appears in [6].

Now, we connect our results with the ones appearing in [3]. Previously, we need the following definition.

**Definition 13.** A function \( \phi : [0, \infty) \to [0, \infty) \) is a \((c)\)-comparison function if \( \sum_{k=0}^{\infty} \psi^k(t) \) converges for any \( t \in [0, \infty) \). The main result of [3] is the following.

**Theorem 14.** Let \( (X, d) \) be a complete metric space, \( m \) a positive integer, \( A_1, A_2, ..., A_m \) nonempty subsets of \( X, X = \bigcup_{i=1}^{m} A_i \) and \( \phi : [0, \infty) \to [0, \infty) \) a \((c)\)-comparison function.
Let $T: X \to X$ be an operator and we assume that

(i) $X = \bigcup_{i=1}^{m} A_{i}$ is a cyclic representation of $X$ with respect to $T$.

(ii) $d(Tx, Ty) \leq \phi(d(x, y)) - \varphi(d(x, y)) = (\phi - \varphi)(d(x, y))$

for any $x \in A_{i}$, $y \in A_{i+1}$, where $A_{m+1} = A$.

Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{m} A_{i}$.

Now, the contractive condition of Theorem 6 can be written as

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \varphi(d(x, y)) = (\phi - \varphi)(d(x, y))$$

for any $x \in A_{i}$, $y \in A_{i+1}$, where $A_{m+1} = A$, and $\varphi$, $\phi \in \mathcal{F}$.

Particularly, if we take $\varphi = Id(0, \infty)$ and $\varphi(t) = \frac{t^2}{1 + t}$, it is easily seen that $\varphi$, $\phi \in \mathcal{F}$. On the other hand,

$$(\phi - \varphi)(t) = t - \frac{t^2}{1 + t} = \frac{t}{1 + t}$$

and

$$(\phi - \varphi)^{(n)}(t) = \frac{t}{1 + nt}$$

Moreover, for every $t \in (0, \infty)$, $\sum_{k=0}^{\infty} (\phi - \varphi)^{(k)}(t)$ diverges. Therefore, $\varphi - \phi$ is not a $(c)$-comparison function. Consequently, our Theorem 6 can be applied to cases which cannot treated by Theorem 14.

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Authors’ contributions
The authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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