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EXISTENCE OF SOLUTIONS FOR MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this article, we give some results concerning the continuity of the nonlinear Volterra and Fredholm integral operators on the space $L^1[0,\infty)$. Then by using the concept of measure of weak noncompactness, we prove an existence result for a functional integral equation which includes several classes of nonlinear integral equations. Our results extend some previous works.

1. Introduction

Integral Equations occur in mechanics and many related fields of engineering and mathematical physics [6, 7, 8, 11, 12, 13, 14, 17, 22, 24, 25, 26, 27]. They also form one of useful mathematical tools in many branches of pure analysis such as functional analysis [21, 26]. Recently many papers have been devoted to the existence of solutions of nonlinear functional integral equations [1, 2, 4, 5, 8, 11]. Our main purpose is to prove an existence theorem for a class of functional integral equations which contains many integral or functional integral equations. For example, we can mention the nonlinear Volterra integral equations, mixed Volterra-Fredholm integral equations and Fredholm integral equations on the unbounded interval $[0,\infty)$.

The concept of measure of weak noncompactness was developed by De Blasie [16]. Banaś and Knap [6] introduced a measure of weak noncompactness in the space of real Lebesgue integrable functions on an interval which is convenient for our purpose. In the proof of main result we will use a measure of weak noncompactness given by Banaś and Knap to find a special subset of $L^1[0,\infty)$ and also by applying the Schauder fixed point theorem on this set, the existence result which generalizes several previous works [3, 7, 8, 9, 11, 13, 17, 27] will be proven.

Organization of this article: Section 2 gives some definitions and preliminary results about continuous operator on $L^1(\mathbb{R}_+)$, Section 3 describes the concept of measure of weak noncompactness and weakly compact sets in $L^1(\mathbb{R}_+)$ and finally in Section 4 we give our main result and some examples.

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2. Notations and auxiliary results

In this paper, $\mathbb{R}_+$ indicates the interval $[0,\infty)$ and for the Lebesgue measurable subset D of $\mathbb{R}$, $m(D)$ implies the Lebesgue measure of D. Also, let $L^1(D)$ be the space of all Lebesgue integrable functions y on D equipped with the standard norm $\|y\|_{L^1(D)} = \int_D |y(x)|dx$.

Lemma 2.1 ([20])**.** *Let Ω be a Lebesgue measurable subset of $\mathbb{R}$ and $1 \le p \le \infty$. If $\{f_n\}$ is convergent to $f \in L^p(\Omega)$ in the L^p-norm, then there is a subsequence $\{f_{n_k}\}$ which converges to f a.e., and there is $g \in L^p(\Omega)$, $g \ge 0$, such that*

$$|f_{n_k}(x)| \le g(x), \quad a.e.\ x \in \Omega.$$

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \times \mathbb{R} \to \mathbb{R}$ is said to have the Carathéodory property if

(M) for all $x \in \mathbb{R}$ the function $t \mapsto f(t,x)$ is Lebesgue measurable on I;
(C) for almost all $t \in I$ the function $x \mapsto f(t,x)$ is continuous on $\mathbb{R}$.

One of the most important nonlinear mappings is the so-called Nemytski operator which is also called the substitution (or superposition) operator [6, 8, 20, 27]. By substituting the function $x : I \to \mathbb{R}$ into the function f the Nemytski operator $F : x \to f(.,x(.))$ has been obtained which acts on a space containing functions x. Krasnosel'skii [22] and Appell and Zabreiko [3] have proven the following assertion when I is a bounded and an unbounded domain respectively.

Theorem 2.2. *The superposition operator F generated by function $f(t,x)$ maps the space $L^1(I)$ continuously into itself if and only if $|f(t,x)| \le g(t) + c|x|$ for all t in an interval I, and $x \in \mathbb{R}$, where g is a function from the space $L^1(I)$ and c is a nonnegative constant.*

Remark 2.3. The Carathéodory property can be generalized to functions $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ where Ω is a measurable subset of $\mathbb{R}^n$. Theorem 2.2 holds similarly if and only if there exist $c \in \mathbb{R}$ and $g \in L^1(\Omega)$ such that

$$|f(x,y)| \le g(x) + c\sum_{i=1}^{m} |y_i|, \tag{2.1}$$

for almost all $x \in \Omega$ and all $y = (y_1,\dots,y_m) \in \mathbb{R}^m$.

Now we are going to review a theorem from [6] about the continuity of the linear Volterra integral operator on the space $L^1 = L^1(\mathbb{R}_+)$. Let $\Delta = \{(t,s) : 0 \le s \le t\}$ and $k : \triangle \to \mathbb{R}$ be a measurable function with respect to both variables. Consider

$$(Kx)(t) = \int_0^t k(t,s)x(s)ds, \quad t \in \mathbb{R}_+,\ x \in L^1(\mathbb{R}_+).$$

We notice that K is a linear Volterra integral operator generated by k.

Theorem 2.4. *Let k be measurable on Δ and such that*

$$\operatorname{ess\,sup}_{s\ge 0} \int_s^\infty |k(t,s)|dt < \infty. \tag{2.2}$$

Then the Volterra integral operator K generated by k maps (continuously) the space $L^1(\mathbb{R}_+)$ into itself and the norm $\|K\|$ of this operator is majorized by the number $\operatorname{ess\,sup}_{s\ge 0} \int_s^\infty |k(t,s)|dt$.

Assume that A is a measurable subset of $\mathbb{R}_+$, we denote by $\|K\|_A$ the norm of linear Volterra operator $K : L^1(A) \to L^1(A)$.

Now we give a result concerning the continuity of the nonlinear Volterra operator on $L^1(\mathbb{R}_+)$. In what follows we suppose that $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies:

(a) $t \to u(t, s, x)$ is measurable for all $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$;
(b) $(s, x) \to u(t, s, x)$ is continuous for almost all $t \in \mathbb{R}_+$.

Theorem 2.5. *Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function such that*

$$|u(t, s, x)| \le k_1(t, s) + k_2(t, s)|x|, \quad t, s \in \mathbb{R}_+ \; x \in \mathbb{R}, \tag{2.3}$$

where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ (i=1,2) are measurable functions. Moreover, the integral operator K_2 generated by k_2 is a continuous map from $L^1(\mathbb{R}_+)$ into itself and $\int_0^t k_1(t, s)ds \in L^1(\mathbb{R}_+)$. Then the operator

$$(Ux)(t) = \int_0^t u(t, s, x(s))ds,$$

maps $L^1(\mathbb{R}_+)$ continuously into itself

Proof. Let $\{x_n\}$ be an arbitrary sequence in $L^1 = L^1(\mathbb{R}_+)$ which converges to $x \in L^1$ in the L^1-norm. By using Lemma 2.1 there is a subsequence $\{x_{n_k}\}$ which converges to x a.e., and there is $g \in L^1$, $g \ge 0$, such that

$$|x_{n_k}(s)| \le g(s), \quad \text{a.e. on } \mathbb{R}_+. \tag{2.4}$$

Since $x_{n_k} \to x$ almost everywhere in $\mathbb{R}_+$, it readily follows from (b) that

$$u(t, s, x_{n_k}(s)) \to u(t, s, x(s)), \quad \text{for almost all } s, t \in \mathbb{R}_+. \tag{2.5}$$

From inequalities (2.3) and (2.4), we infer that

$$|u(t, s, x_{n_k}(s))| \le k_1(t, s) + k_2(t, s)g(s), \text{ for almost all } s, t \in \mathbb{R}_+. \tag{2.6}$$

As a consequence of the Lebesgue's Dominated Convergence Theorem, (2.5) and (2.6) yield

$$\int_0^t u(t, s, x_{n_k}(s))ds \to \int_0^t u(t, s, x(s))ds,$$

for almost all $t \in \mathbb{R}_+$. Inequality (2.6) implies that

$$|(Ux_{n_k})(t)| \le \int_0^t |u(t, s, x_{n_k}(s))|ds \le \int_0^t k_1(t, s)ds + \int_0^t k_2(t, s)g(s)ds, \tag{2.7}$$

for almost all $t \in \mathbb{R}_+$. Regarding the assumptions on k_1 and k_2, we obtain

$$\int_0^\infty \int_0^t k_1(t, s)\, ds\, dt + \int_0^\infty \int_0^t k_2(t, s)g(s)\, ds\, dt < \infty. \tag{2.8}$$

Then inequalities (2.7)-(2.8) and the Lebesgue's Dominated Convergence Theorem imply

$$\|Ux_{n_k} - Ux\|_{L^1} \to 0.$$

Since any sequence $\{x_n\}$ converging to x in L^1 has a subsequence $\{x_{n_k}\}$ such that $Ux_{n_k} \to Ux$ in L^1, we can conclude that $U : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ is a continuous operator. $\square$

Similar to the above theorem, we can prove the following theorem for the nonlinear Fredholm integral operators.

Theorem 2.6. *Let $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function satisfying* (a)–(b) *such that*

$$|v(t,s,x)| \le k_1(t,s) + k_2(t,s)|x|, \quad t,s \in \mathbb{R}_+ \; x \in \mathbb{R}, \tag{2.9}$$

where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ are measurable functions. Moreover, the integral operator $(K_2x)(t) = \int_0^\infty k_2(t,s)|x(s)|ds$ maps $L^1(\mathbb{R}_+)$ continuously into itself and $k_1(t,s) \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$. Then the operator

$$(Vx)(t) = \int_0^\infty v(t,s,x(s))ds,$$

maps $L^1(\mathbb{R}_+)$ continuously into itself.

3. Measure of weak noncompactness

Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space with zero element θ. We write $B(x,r)$ to denote the closed ball centered at x with radius r and $\operatorname{conv} X$ to denote the closed convex hull of X. Further let:

$\mathbf{m}_E$ be the family of all nonempty bounded subsets of E, $\mathbf{n}_E^w$: the subfamily of $\mathbf{m}_E$ consisting of all relatively weakly compact sets, and $\overline{X}^w$: the weak closure of a set X.

In this paper, we use the following definition of the measure of weak noncompactness [9].

Definition 3.1. *A mapping $\mu : \boldsymbol{m}_E \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:*

1. *The family* $\ker \mu = \{X \in \boldsymbol{m}_E : \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset \boldsymbol{n}_E^w$,
2. $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$,
3. $\mu(\operatorname{conv} X) = \mu(X)$,
4. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$,
5. *If $X_n \in \boldsymbol{m}_E$, $X_n = \overline{X}_n^w$ for $n = 1, 2, \dots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.*

In the sequel, we will use a measure of weak noncompactness represented by a convenient formula in the space $L^1(\mathbb{R}_+)$ [10]. For $X \in \mathbf{m}_{L^1(\mathbb{R}_+)}$ define:

$$c(X) = \lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup[\int_D |x(t)|dt : D \subset \mathbb{R}_+,\ m(D) \le \varepsilon]\}\},$$
$$d(X) = \lim_{T\to\infty}\{\sup[\int_T^\infty |x(t)|dt : x \in X]\},$$
$$\mu(X) = c(X) + d(X).$$

In [10], it is shown that μ is a measure of weak noncompactness on $L^1(\mathbb{R}_+)$. By using the following theorem [19], we can infer that $\ker \mu = \mathbf{n}_{L^1(\mathbb{R}_+)}^w$.

Theorem 3.2. *A bounded set X is relatively weakly compact in $L^1(\mathbb{R}_+)$ if and only if the following two conditions are satisfied:*

1. *for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(D) \le \delta$ then $\int_D |x(t)|dt \le \varepsilon$ for all $x \in X$,*
2. *for any $\varepsilon > 0$ there exists $T > 0$ such that $\int_T^\infty |x(t)|dt \le \varepsilon$ for any $x \in X$.*

4. Main results

In this section, we study the existence of solutions to the functional integral equation

$$x(t) = f\Big(t, \int_0^t u(t,s,x(s))ds, \int_0^\infty a_2(t)v(s,x(s))ds\Big), \quad t \geq 0. \tag{4.1}$$

This equation is a general form of many integral equations, such as the mixed Volterra-Fredholm integral equation

$$x(t) = g(t) + \int_0^t k(t,s)u(s,x(s))ds + a(t)\int_0^\infty v(s,x(s))ds, \quad t \geq 0. \tag{4.2}$$

Equations like (4.2) have been considered by many authors; see for example [12, 15, 18, 23, 25] and references cited therein. Moreover, (4.1) contains the nonlinear Volterra and Fredholm integral equations on $\mathbb{R}_+$ such as:

$$x(t) = g(t) + \int_0^t u(t,s,x(s))ds, \quad t \geq 0,$$
$$x(t) = f(t) + a(t)\int_0^\infty v(s,x(s))ds, \quad t \geq 0.$$

We consider equation (4.1) under the following assumptions:

(i) The function $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions and there exist constant $B \in \mathbb{R}_+$ and function $a_1 \in L^1(\mathbb{R}_+)$ such that

$$|f(t,x,y)| \leq a_1(t) + B(|x| + |y|), \quad t \in \mathbb{R}_+,\ x, y \in \mathbb{R}. \tag{4.3}$$

(ii) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies (a)–(b) and $|u(t,s,x)| \leq k_1(t,s) + k_2(t,s)|x|$ for $(t,s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ (i=1,2) satisfies Carathéodory conditions. Moreover, the integral operator K_2 generated by k_2 i.e.

$$(K_2x)(t) = \int_0^t k_2(t,s)x(s)ds, \tag{4.4}$$

is a continuous map from $L^1(\mathbb{R}_+)$ into itself and $\int_0^t k_1(t,s)ds \in L^1(\mathbb{R}_+)$.

(iii) $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and $|v(s,x)| \leq n(s) + b|x|$ for $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$ where $n \in L^1(\mathbb{R}_+)$ and b is a positive constant.

(iv) $a_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function in $L^1(\mathbb{R}_+)$.

(v) $B(b\|a_2\| + \|K_2\|) < 1$, where $\|K_2\|$ denotes the norm of operator K_2.

To prove the main result of this paper, we need the next lemma.

Let X be a nonempty, closed, convex, bounded and weakly compact subset of $L^1 = L^1(\mathbb{R}_+)$ and $I = [0,a]$ where $a > 0$. Moreover, we define the operator F on $L^1 = L^1(\mathbb{R}_+)$ as follows:

$$(Fx)(t) = f\Big(t, \int_0^t u(t,s,x(s))ds, \int_0^\infty a_2(t)v(s,x(s))ds\Big). \tag{4.5}$$

Lemma 4.1. *Suppose that assumptions (i)–(iv) hold and the operator F takes X into itself. Then for any $\varepsilon > 0$ there exists $D_\varepsilon \subset I$ with $m(I \setminus D_\varepsilon) \leq \varepsilon$ such that $F(\operatorname{conv} FX)$ on D_ε is a relatively compact subset of $C(D_\varepsilon)$.*

Proof. Consider an arbitrary but fixed $\varepsilon > 0$. Then using Lusin theorem and generalized version of Scorza-Dragoni theorem [14] we can find the closed set $D_\varepsilon \subset I$ with $m(I \setminus D_\varepsilon) \le \varepsilon$, such that the functions $a_i\big|_{D_\varepsilon}$, $k\big|_{D_\varepsilon\times\mathbb{R}_+}$, $u\big|_{D_\varepsilon\times\mathbb{R}_+\times\mathbb{R}}$ and $f\big|_{D_\varepsilon\times\mathbb{R}_+\times\mathbb{R}}$ are continuous. Let us take an arbitrary $x \in X$. Then for $t \in D_\epsilon$ we have

$$\begin{aligned} \Big|\int_0^t u(t,s,x(s))ds\Big| &\le \int_0^t k_1(t,s) + \int_0^t k_2(t,s)|x(s)|ds \\ &\le \bar{k}_1 a + \bar{k}_2\|x\| \le \bar{k}_1 a + \bar{k}_2\|X\| =: U_\varepsilon, \end{aligned} \tag{4.6}$$

and

$$\Big|\int_0^\infty a_2(t)v(s,x(s))ds\Big| \le \bar{a}_2(\|n\| + b\|x\|) \le \bar{a}_2(\|n\| + b\|X\|) =: V_\varepsilon, \tag{4.7}$$

where $\|X\| = \sup\{\|x\| : x \in X\}$, $\bar{a}_i = \sup\{|a_i(t)| : t \in D_\varepsilon\}$ and $\bar{k}_i = \sup\{|k_i(t,s)| : (t,s) \in D_\varepsilon \times I\}$ for $i = 1,2$. Now let $y \in FX$. Then there exists $x \in X$ such that $y = Fx$. Using the inequalities (4.6) and (4.7) for $t \in D_\varepsilon$ we obtain

$$\begin{aligned} |y(t)| = |(Fx)(t)| &\le a_1(t) + \Big|\int_0^t u(t,s,x(s))ds\Big| + \Big|\int_0^\infty a_2(t)v(s,x(s))ds\Big| \\ &\le \bar{a}_1 + U_\varepsilon + V_\varepsilon =: Y_\varepsilon. \end{aligned} \tag{4.8}$$

We can easily deduce that the inequality (4.8) is true, for any $y \in Y = \operatorname{conv} FX$. Now assume that $\{y_n\}$ is a sequence in Y and let $t_1, t_2 \in D_\varepsilon$. Without loss of generality we can assume that $t_1 \le t_2$. Relatively weakly compactness of the set $\{y_n\}$ implies that for $\varepsilon_0 = t_2 - t_1$ there exists $0 < \delta_0 \le \varepsilon_0$ such that for any measurable subset D of $[0,t_1]$ with $m(D) \le \delta_0$, we have:

$$\int_D |y_n(t)|dt \le \varepsilon_0 \quad \text{for } n = 1,2,\dots. \tag{4.9}$$

We see that the estimate (4.8) does not depend on the choice of ε. Thus for $\varepsilon = \delta_0$ there exists a closed set $D_{\delta_0} \subset [0,t_1]$ with $m([0,t_1] \setminus D_{\delta_0}) \le \delta_0$ such that

$$|y_n(t)| \le Y_{\delta_0} \quad \text{for } t \in D_{\delta_0},\ n = 1,2,\dots. \tag{4.10}$$

Hence from (4.9) and uniform continuity of $u\big|_{D_\varepsilon\times D_{\delta_0}\times[-Y_{\delta_0},Y_{\delta_0}]}$ and $k_i\big|_{D_\varepsilon\times[0,a]}$ ($i = 1,2$) we infer that

$$\begin{aligned} &\int_0^{t_1} |u(t_1,s,y_n(s)) - u(t_2,s,y_n(s))|ds \\ &\le \int_{D_{\delta_0}} |u(t_1,s,y_n(s)) - u(t_2,s,y_n(s))|ds \\ &\quad + \int_{[0,t_1]\setminus D_{\delta_0}} |u(t_1,s,y_n(s)) - u(t_2,s,y_n(s))|ds \\ &\le O(|t_1 - t_2|) + 2\bar{k}_1 m([0,t_1]\setminus D_{\delta_0}) + 2\bar{k}_2 \int_{[0,t_1]\setminus D_{\delta_0}} |y_n(t)|dt \\ &\le O(|t_1 - t_2|) + 2(\bar{k}_1 + \bar{k}_2)|t_1 - t_2|. \end{aligned} \tag{4.11}$$

Here O is a function which $O(\eta) \to 0$ as $\eta \to 0$. Thus from (4.11) we have:

$$\begin{aligned}
&|\int_0^{t_1} u(t_1, s, y_n(s))ds - \int_0^{t_2} u(t_2, s, y_n(s))ds| \\
&\leq \int_0^{t_1} |u(t_1, s, y_n(s)) - u(t_2, s, y_n(s))|ds + |\int_{t_1}^{t_2} u(t_2, s, y_n(s))ds| \\
&\leq O(|t_1 - t_2|) + 2(\bar{k}_1 + \bar{k}_2)|t_1 - t_2| + \int_{t_1}^{t_2} k_1(t_2, s)ds + \int_{t_1}^{t_2} k_2(t_2, s)|y_n(s)|ds \\
&\leq O(|t_1 - t_2|) + 2(\bar{k}_1 + \bar{k}_2)|t_1 - t_2| + \bar{k}_1|t_1 - t_2| + \bar{k}_2 \int_{t_1}^{t_2} |y_n(s)|ds.
\end{aligned} \tag{4.12}$$

Weakly compactness of the set $\{y_n\}$ implies that $\int_{t_1}^{t_2} |y_n(s)|ds$ is arbitrary small uniformly with respect to $n \in \mathbb{N}$ if $t_2 - t_1$ is small enough. Then from (4.12) and (4.6) the sequence $\{Uy_n\}$ which

$$(Uy_n)(t) = \int_0^t u(t, s, y_n(s))ds,$$

is equibounded and equicontinuous on the set D_ε. Obviously from assumption (iii) and inequality (4.7) we can easily infer that the sequence $\{Vy_n\}$ is equibounded and equicontinuous on D_ε where

$$(Vy_n)(t) = \int_0^\infty a_1(t)v(s, y_n(s))ds.$$

Hence, uniform continuity of $f\big|_{D_\varepsilon\times[-U_\varepsilon,U_\varepsilon]\times[-V_\varepsilon,V_\varepsilon]}$ implies that the sequence $\{Fy_n\}$ is equibounded and equicontinuous on D_ε. Then, by Ascoli theorem the sequence $\{Fy_n\}$ has a convergent subsequence in the norm $C(D_\varepsilon)$. Therefore, $F(\operatorname{conv} FX)$ is a relatively compact subset of $C(D_\varepsilon)$. $\square$

Now we present our main result.

Theorem 4.2. *Under assumptions* (i)–(v)*, the functional integral equation* (4.1) *has at least one solution* $x \in L^1(\mathbb{R}_+)$.

Proof. At first we define the operator F on $L^1 = L^1(\mathbb{R}_+)$ by

$$(Fx)(t) = f\Big(t, \int_0^t u(t, s, x(s))ds, \int_0^\infty a_2(t)v(s, x(s))ds\Big).$$

We prove the theorem in the following steps.

Step 1. $F : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ is a continuous operator.
Using Theorems 2.5 and 2.6, the operators

$$(Ux)(t) = \int_0^t u(t, s, x(s))ds, \quad (Vx)(t) = \int_0^\infty v(t, s, x(s))ds,$$

map $L^1(\mathbb{R}_+)$ continuously into itself. Also by assumptions (i)–(iv) and Remark 2.3, the Nemytski operator generated by f is a continuous operator from $L^1(\mathbb{R}_+)$ into $L^1(\mathbb{R}_+)$. Thus the operator F is continuous.

Step 2. There exists a positive number r such that the operator F takes the ball $B(\theta, r)$ into itself. Let $x \in L^1(\mathbb{R}_+)$. Then

$$\begin{aligned}
\|Fx\| &= \int_0^\infty |(Fx)(t)|dt \\
&= \Big| \int_0^\infty f\big(t, \int_0^t u(t,s,x(s))ds, \int_0^\infty a_2(t)v(s,x(s))ds\big)dt\Big| \\
&\le \|a_1\| + B\int_0^\infty \Big(\int_0^t |u(t,s,x(s))|ds + \int_0^\infty |a_2(t)v(s,x(s))|ds\Big)dt \\
&\le \|a_1\| + B(K_1 + \|K_2\|\|x\|) + B\|a_2\|(\|n\| + b\|x\|),
\end{aligned} \tag{4.13}$$

where

$$K_1 = \int_0^\infty \int_0^t k_1(t,s)\,ds\,dt.$$

From (4.13) and assumption (v), one can deduce that for $r = \frac{\|a_1\| + B(K_1 + \|a_2\|\|n\|)}{1 - B(\|K_2\| + \|a_2\|b)}$, the operator F takes $B_r = B(\theta, r)$ into itself.

Step 3. There exists a weakly compact subset Y such that the operator F maps Y into itself. Let X be a nonempty subset of B_r. Let $\varepsilon > 0$ be an arbitrary number and $D \subset \mathbb{R}_+$ be a measurable subset with $m(D) \le \varepsilon$. Then for $x \in X$ we have:

$$\begin{aligned}
&\int_D |(Fx)(t)|dt \\
&\le \int_D a_1(t)dt + B\int_D \Big(\int_0^t |u(t,s,x(s))|ds + \int_0^\infty |a_2(t)v(s,x(s))|ds\Big)dt \\
&\le \int_D a_1(t)dt + B\int_D\int_0^t k_1(t,s)\,ds\,dt \\
&\quad + B\int_D |(K_2x)(t)|dt + B(\|n\| + br)\int_D |a_2(t)|dt \\
&\le \int_D a_1(t)dt + B\int_D\int_0^t k_1(t,s)\,ds\,dt \\
&\quad + B\|K_2\|_D \int_D |x(t)|dt + B(\|n\| + br)\int_D |a_2(t)|dt
\end{aligned} \tag{4.14}$$

Further, as a simple consequence of the fact that a single set in $L^1(\mathbb{R}_+)$ is weakly compact, for $\gamma(t) = a_1(t), \int_0^t k_1(t,s)ds$ or $a_2(t)$, we have:

- (C1) $\lim_{\varepsilon\to 0}\{\sup[\int_D |\gamma(t)|dt : D \subset \mathbb{R}_+,\ m(D) \le \varepsilon]\} = 0$,
- (C2) $\lim_{T\to\infty} \int_T^\infty |\gamma(t)|dt = 0$.

Then from (4.14) and (C1) we conclude that

$$c(FX) \le B\|K_2\|c(X). \tag{4.15}$$

By similar calculations we obtain:

$$\begin{aligned}
\int_T^\infty |(Fx)(t)|dt \le \int_T^\infty a_1(t)dt + B\int_T^\infty\int_0^t k_1(t,s)ds\,dt \\
+ B\|K_2\|\int_T^\infty |x(t)|dt + B(\|n\| + br)\int_T^\infty |a_2(t)|dt.
\end{aligned} \tag{4.16}$$

Therefore, from (C2), we have

$$d(FX) \le B\|K_2\|d(X). \tag{4.17}$$

Hence by adding (4.15) and (4.17) we obtain

$$\mu(FX) \le B\|K_2\|\mu(X). \tag{4.18}$$

Assumption (v) implies that $B\|K_2\| < 1$. Thus inequality (4.18) yields that there exists a closed, convex and weakly compact set $X_\infty \subset B_r$ such that $FX_\infty \subset X_\infty$. Let $Y = \operatorname{conv} FX_\infty$. Obviously $FY \subset Y \subset X_\infty$. Thus FY and Y are relatively weakly compact.

Step 4. The set FY obtained in the Step 3 is a relatively compact subset of $L^1(\mathbb{R}_+)$. Suppose $\{y_n\} \subset Y$, and fix an arbitrary $\varepsilon > 0$. Applying Theorem 3.2 for relatively weakly compact set FY implies that there exists $T > 0$ such that for $m, n \in \mathbb{N}$

$$\int_T^\infty |(Fy_n)(t) - (Fy_m)(t)|dt \le \frac{\varepsilon}{2}. \tag{4.19}$$

Further, by using Lemma 4.1 for any $k \in \mathbb{N}$ there exists a closed set $D_k \subset [0,T]$ with $m([0,T] \setminus D_k) \le \frac{1}{k}$ such that $\{Fy_n\}$ is a relatively compact subset of $C(D_k)$. So for any $k \in \mathbb{N}$ there exists a subsequence $\{y_{k,n}\}$ of $\{y_n\}$ which is a Cauchy sequence in $C(D_k)$. Also these subsequences can be chosen such that $\{y_{k+1,n}\} \subseteq \{y_{k,n}\}$. Consequently the subsequence $\{y_{n,n}\}$ is a Cauchy sequence in each space $C(D_k)$ for any $k \in \mathbb{N}$ which for simplicity we call it again $\{y_n\}$.

From the relatively weakly compactness of $\{Fy_n\}$ we can find $\delta > 0$ such that for each closed subset D_δ with $m([0,T] \setminus D_\delta) \le \delta$ we obtain:

$$\int_{[0,T]\setminus D_\delta} |(Fy_n)(t) - (Fy_m)(t)|dt \le \frac{\varepsilon}{4}, \ m, n \in \mathbb{N}. \tag{4.20}$$

Considering the fact $\{Fy_n\}$ is Cauchy in $C(D_k)$ for each $k \in \mathbb{N}$ one can find k_0 such that $m([0,T] \setminus D_{k_0}) \le \delta$ and for $m, n \ge k_0$

$$\|(Fy_n) - (Fy_m)\|_{C(D_{k_0})} \le \frac{\varepsilon}{4(m(D_{k_0}) + 1)}, \tag{4.21}$$

consequently (4.20) and (4.21) imply that

$$\begin{aligned} &\int_0^T |(Fy_n)(t) - (Fy_m)(t)|dt \\ &= \int_{D_{k_0}} |(Fy_n)(t) - (Fy_m)(t)|dt + \int_{[0,T]\setminus D_{k_0}} |(Fy_n)(t) - (Fy_m)(t)|dt \\ &\le \frac{\varepsilon}{2}, \end{aligned} \tag{4.22}$$

for $m, n \ge k_0$. Now by considering (4.19) and (4.22) for $m, n \ge k_0$ we obtain the inequality

$$\|(Fy_n) - (Fy_m)\|_{L^1} = \int_0^\infty |(Fy_n)(t) - (Fy_m)(t)|dt \le \varepsilon, \tag{4.23}$$

which shows that the sequence $\{Fy_n\}$ is a Cauchy sequence in the Banach space $L^1(\mathbb{R}_+)$. Then $\{Fy_n\}$ has a convergent subsequence which implies that FY is a relatively compact subset of $L^1(\mathbb{R}_+)$.

Step 5. By the Step 4 there exists a bounded, closed, convex set $Y \subset L^1(\mathbb{R}_+)$ such that the operator $F : Y \to Y$ is continuous and compact. Then Schauder fixed point theorem completes the proof. $\square$

Next, by applying our theorem we prove the existence of solutions for some integral equations.

Example 4.3. Consider the Fredholm integral equation

$$x(t) = \frac{t^{2/3}}{t^3+1} + \int_0^\infty a_2(t) \tanh\big(\frac{s+|x(s)|}{(1+s^2)^2}\big) ds, \quad t \ge 0, \tag{4.24}$$

where

$$a_2(t) = \frac{t\pi}{8}\chi_{[0,1]} + \frac{1}{4(1+t^2)}\chi_{(1,\infty)},$$

in which for $A \subset \mathbb{R}_+$ and $\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in \mathbb{R}_+ \setminus A. \end{cases}$ Put

$$\begin{gathered} f(t,x,y) = \frac{t^{2/3}}{1+t^3} + y, \quad u(t,s,x) = 0, \\ v(s,x) = \tanh\big(\frac{s+|x|}{(1+s^2)^2}\big), \quad a_1(t) = \frac{t^{2/3}}{1+t^3}, \\ n(s) = \frac{s}{(1+s^2)^2}, \quad B = 1,\ b = 1. \end{gathered}$$

We know that $\tanh(\alpha) \le \alpha$, for $\alpha > 0$. Then

$$\begin{gathered} |v(s,x)| \le n(s) + b|x|, \\ |f(t,x,y)| \le \frac{t^{2/3}}{1+t^3} + B(|x|+|y|), \end{gathered}$$

Further

$$\|a_2\| = \int_0^\infty |a_2(t)| dt = \frac{\pi}{8}.$$

Since $u = 0$ we can choose $k_1 = k_2 = 0$ and then $\|K_2\| = 0$. Thus, $B(b\|a_2\| + \|K_2\|) = \frac{\pi}{8} < 1$. It is easy to see that assumptions (i)-(v) are fulfilled. Consequently Theorem 4.2 ensures that the equation (4.24) has at least one solution in $L^1(\mathbb{R}_+)$.

Example 4.4. Consider the mixed Volterra-Fredholm integral equation

$$\begin{aligned} x(t) = \frac{1+t^2}{\cosh(t)} &+ \int_0^t \frac{\lfloor t+s^2 \rfloor}{2} \exp(-t) \sin(x(s)) ds \\ &+ \int_0^\infty \frac{t \ln(1+sx^2(s))}{3(1+t^2)^2(s+1)} ds, \quad t \ge 0, \end{aligned} \tag{4.25}$$

where the symbol $\lfloor z \rfloor$ means the largest integer less than or equal to z. Let

$$\begin{gathered} f(t,x,y) = \frac{1+t^2}{\cosh(t)} + x + y, \\ u(t,s,x) = \frac{\lfloor t+s^2 \rfloor}{2} \exp(-t) \sin(x), \\ v(s,x) = \frac{\ln(1+sx^2)}{s+1}, \quad a_2(t) = \frac{t}{3(1+t^2)^2}, \end{gathered}$$

$$k_2(t,s) = \frac{\lfloor t+s^2 \rfloor}{2}\exp(-t), \quad B=1, \quad b=1.$$

We know that $\ln(1+\alpha^2) \le |\alpha|$ for $\alpha \in \mathbb{R}$. Then

$$|f(t,x,y)| \le \frac{1+t^2}{\cosh(t)} + B(|x|+|y|),$$

$$|u(t,s,x)| \le k_2(t,s)|x|, \quad |v(s,x)| \le b|x|,$$

$$\|a_2\| = \int_0^\infty |a_2(t)|dt = \int_0^\infty \frac{t}{3(1+t^2)^2}dt = \frac{1}{6},$$

$$\int_s^\infty |k_2(t,s)|dt = \int_s^\infty \frac{\lfloor t+s^2 \rfloor}{2}\exp(-t)dt \le \frac{(s^2+s+1)}{2}\exp(-s) \le \frac{3}{2}\exp(-1),$$

for $s,t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Therefore, from Theorem 2.4, we have that $B\|K_2\| \le \frac{3}{2}\exp(-1)$ and then $B(b\|a_2\| + \|K_2\|) \le \frac{1}{6} + \frac{3}{2}\exp(-1) < 1$. Using Theorem 4.2 we deduce that the equation (4.25) has at least one solution in $L^1(\mathbb{R}_+)$.

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