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Research Article

A Fixed Point Theorem for Mappings Satisfying a Contractive Condition of Rational Type on a Partially Ordered Metric Space

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The purpose of this paper is to present a fixed point theorem using a contractive condition of rational type in the context of partially ordered metric spaces.

1. Introduction

In [1], Jaggi proved the following fixed point theorem.

Theorem 1.1. *Let T be a continuous selfmap defined on a complete metric space (X, d). Suppose that T satisfies the following contractive condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta \cdot d(x, y), \tag{1.1}$$

for all $x, y \in X$, $x \neq y$, and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, then T has a unique fixed point in X.

The aim of this paper is to give a version of Theorem 1.1 in partially ordered metric spaces.

Existence of fixed point in partially ordered sets has been considered recently in [2–15]. Tarski's theorem is used in [7] to show the existence of solutions for fuzzy equations and in [9] to prove existence theorems for fuzzy differential equations. In [5, 6, 8, 11, 14], some applications to matrix equations and to ordinary differential equations are presented.

In [3, 6, 16], it is proved that some fixed theorems for a mixed monotone mapping in a metric space endowed with a partial order and the authors apply their results to problems of existence and uniqueness of solutions for some boundary value problems.

In the context of partially ordered metric spaces, the usual contractive condition is weakened but at the expense that the operator is monotone. The main idea in [8, 14] involves combining the ideas in the contraction principle with those in the monotone iterative technique [16].

2. Main Result

Definition 2.1. Let $(X, \leq)$ be a partially ordered set and $T : X \to X$. We say that T is a nondecreasing mapping if for $x, y \in X$, $x \leq y \Rightarrow Tx \leq Ty$.

In the sequel, we prove the following theorem which is a version of Theorem 1.1 in the context of partially ordered metric spaces.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(Tx, Ty) \leq \alpha \cdot \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta \cdot d(x, y), \quad \text{for } x, y \in X,\ x \geq y,\ x \neq y, \tag{2.1}$$

with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$, then the proof is finished. Suppose that $x_0 < Tx_0$. Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 \leq T^2x_0 \leq \cdots \leq T^n x_0 \leq T^{n+1}x_0 \leq \cdots \tag{2.2}$$

Put $x_{n+1} = Tx_n$. If there exists $n \geq 1$ such that $x_{n+1} = x_n$, then from $x_{n+1} = Tx_n = x_n$, x_n is a fixed point and the proof is finished. Suppose that $x_{n+1} \neq x_n$ for $n \geq 1$.

Then, from (2.1) and as the elements x_n and x_{n-1} are comparable, we get, for $n \geq 1$,

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\leq \alpha \frac{d(x_n, Tx_n) \cdot d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} + \beta d(x_n, x_{n-1}) \\ &= \alpha \frac{d(x_n, x_{n+1}) \cdot d(x_{n-1}, x_n)}{d(x_n, x_{n-1})} + \beta d(x_n, x_{n-1}) \\ &= \alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_n, x_{n-1}). \end{aligned} \tag{2.3}$$

The last inequality gives us

$$d(x_{n+1}, x_n) \leq \frac{\beta}{1 - \alpha} d(x_n, x_{n-1}). \tag{2.4}$$

Again, using induction

$$d(x_{n+1}, x_n) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(x_1, x_0), \tag{2.5}$$

Put $k = \beta/(1-\alpha) < 1$.

Moreover, by the triangular inequality, we have, for $m \geq n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) d(x_1, x_0) \leq \left(\frac{k^n}{1-k}\right) d(x_1, x_0), \end{aligned} \tag{2.6}$$

and this proves that $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

So, $\{x_n\}$ is a Cauchy sequence and, since X is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Further, the continuity of T implies

$$Tz = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z, \tag{2.7}$$

and this proves that z is a fixed point.

This finishes the proof. □

In what follows, we prove that Theorem 2.2 is still valid for T, not necessarily continuous, assuming the following hypothesis in X:

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \longrightarrow x, \text{ then } x = \sup\{x_n\}. \tag{2.8}$$

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies (2.8). Let $T : X \to X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y), \quad \textit{for } x, y \in X,\ x \geq y,\ x \neq y, \tag{2.9}$$

with $\alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. Following the proof of Theorem 2.2, we only have to check that $Tz = z$.

As $\{x_n\}$ is a nondecreasing sequence in X and $x_n \to z$, then, by (2.8), $z = \sup\{x_n\}$. Particularly, $x_n \leq z$ for all $n \in \mathbb{N}$.

Since T is a nondecreasing mapping, then $Tx_n \leq Tz$, for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \leq Tz$ for all $n \in \mathbb{N}$. Moreover, as $x_0 < x_1 \leq Tz$ and $z = \sup\{x_n\}$, we get $z \leq Tz$.

Suppose that $z < Tz$. Using a similar argument that in the proof of Theorem 2.2 for $x_0 \leq Tx_0$, we obtain that $\{T^n z\}$ is a nondecreasing sequence and $\lim_{n\to\infty} T^n z = y$ for certain $y \in X$.

Again, using (2.8), we have that $y = \sup\{T^n z\}$.

Moreover, from $x_0 \leq z$, we get $x_n = T^n x_0 \leq T^n z$ for $n \geq 1$ and $x_n < T^n z$ for $n \geq 1$ because $x_n \leq z < Tz \leq T^n z$ for $n \geq 1$.

As x_n and $T^n z$ are comparable and distinct for $n \geq 1$, applying the contractive condition we get

$$\begin{aligned} d\left(x_{n+1}, T^{n+1}z\right) = d(Tx_n, T(T^n z)) &\leq \alpha \frac{d(x_n, Tx_n) \cdot d\left(T^n z, T^{n+1}z\right)}{d(x_n, T^n z)} \\ &+ \beta d(x_n, T^n z) = \alpha \frac{d(x_n, x_{n+1}) \cdot d\left(T^n z, T^{n+1}z\right)}{d(x_n, T^n z)} + \beta d(x_n, T^n z). \end{aligned} \tag{2.10}$$

Making $n \to \infty$ in the last inequality, we obtain

$$d(z, y) \leq \beta d(z, y). \tag{2.11}$$

As $\beta < 1$, $d(z, y) = 0$, thus, $z = y$.

Particularly, $z = y = \sup\{T^n z\}$ and, consequently, $Tz \leq z$ and this is a contradiction.

Hence, we conclude that $z = Tz$. □

Now, we present an example where it can be appreciated that hypotheses in Theorem 2.2 do not guarantee uniqueness of the fixed point. This example appears in [8].

Let $X = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x, y) \leq (z, t) \Longleftrightarrow x \leq z, \ \ y \leq t. \tag{2.12}$$

Thus, $(X, \leq)$ is a partially ordered set whose different elements are not comparable. Besides, (X, d_2) is a complete metric space considering, d_2, the Euclidean distance. The identity map $T(x, y) = (x, y)$ is trivially continuous and nondecreasing and assumption (2.1) of Theorem 2.2 is satisfied since elements in X are only comparable to themselves. Moreover, $(1, 0) \leq T(1, 0)$ and T has two fixed points in X.

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 2.2 and 2.3. This condition appears in [14] and

$$\text{for } x, y \in X, \text{ there exists a lower bound or an upper bound.} \tag{2.13}$$

In [8], it is proved that the above-mentioned condition is equivalent,

$$\text{for } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.14}$$

Theorem 2.4. *Adding condition* (2.14) *to the hypotheses of Theorem 2.2 (or Theorem 2.3) one obtains uniqueness of the fixed point of T.*

Proof. Suppose that there exists $z, y \in X$ which are fixed point.

We distinguish two cases.

Case 1. If y and z are comparable and $y \neq z$, then using the contractive condition we have

$$\begin{aligned} d(y,z) = d(Ty,Tz) &\leq \alpha \frac{d(y,Ty)\cdot d(z,Tz)}{d(y,z)} + \beta d(y,z) \\ &= \alpha \frac{d(y,y)\cdot d(z,z)}{d(y,z)} + \beta d(y,z) = \beta d(y,z). \end{aligned} \tag{2.15}$$

As $\beta < 1$ is the last inequality, it is a contradiction. Thus, $y = z$.

Case 2. If y is not comparable to z, then by (2.14) there exists $x \in X$ comparable to y and z. Monotonicity implies that $T^n x$ is comparable to $T^n y = y$ and $T^n z = z$ for $n = 0, 1, 2, \ldots$.

If there exists $n_0 \geq 1$ such that $T^{n_0} x = y$, then as y is a fixed point, the sequence $\{T^n x : n \geq n_0\}$ is constant, and, consequently, $\lim_{n\to\infty} T^n x = y$.

On the other hand, if $T^n x \neq y$ for $n \geq 1$, using the contractive condition, we obtain, for $n \geq 2$,

$$\begin{aligned} d(T^n x, y) = d(T^n x, T^n y) &\leq \alpha \frac{d(T^{n-1}x, T^n x)\cdot d(T^{n-1}y, T^n y)}{d(T^{n-1}x, T^{n-1}y)} \\ &\quad + \beta d\left(T^{n-1}x, T^{n-1}y\right) \leq \alpha \frac{d(T^{n-1}x, T^n x)\cdot d(y,y)}{d(T^{n-1}x, y)} + \beta d\left(T^{n-1}x, y\right) \\ &= \beta d\left(T^{n-1}x, y\right). \end{aligned} \tag{2.16}$$

Using induction,

$$d(T^n x, y) \leq \beta^n \cdot d(x,y), \quad \text{for } n \geq 2, \tag{2.17}$$

and as $\beta < 1$, the last inequality gives us $\lim_{n\to\infty} T^n x = y$.

Hence, we conclude that $\lim_{n\to\infty} T^n x = y$.

Using a similar argument, we can prove that $\lim_{n\to\infty} T^n x = z$.

Now, the uniqueness of the limit gives us $y = z$.

This finishes the proof. □

Remark 2.5. It is easily proved that the space $C[0,1] = \{x : [0,1] \to \mathbb{R}, \text{continuous}\}$ with the partial order given by

$$x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0,1], \tag{2.18}$$

and the metric given by

$$d(x,y) = \sup\{|x(t) - y(t)| : t \in [0,1]\} \tag{2.19}$$

satisfies condition (2.8). Moreover, as for $x, y \in C[0,1]$, the function $\max(x,y)(t) = \max\{x(t), y(t)\}$ is continuous, $(C[0,1], \leq)$ satisfies also condition (2.14).

3. Some Remarks

In this section, we present some remarks.

Remark 3.1. In [8], instead of condition (2.8), the authors use the following weaker condition:

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \longrightarrow x, \text{ then } x_n \leq x \quad \forall n \in \mathbb{N}. \tag{3.1}$$

We have not been able to prove Theorem 2.3 using (3.1).

Remark 3.2. If, in Theorems 2.2, 2.3, and 2.4, $\alpha = 0$, then we obtain Theorems 2.1, 2.2, and 2.3 of [8].

If in the theorems of Section 2, $\beta = 0$, we obtain the following fixed point theorem in partially ordered complete metric spaces.

Theorem 3.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists $\alpha \in [0,1)$ satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x,y)}, \quad \text{for } x, y \in X \text{ with } x \geq y,\ x \neq y. \tag{3.2}$$

Suppose also that either T is continuous or X satisfies condition (2.8)*. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.*

Besides, if $(X, \leq)$ satisfies condition (2.14)*, then one obtains uniqueness of the fixed point.*

Finally, we present an example where Theorem 2.2 can be applied and this example cannot be treated by Theorem 1.1.

Example 3.4. Let $X = \{(0,1), (1,0), (1,1)\}$ and consider in X the partial order given by $R = \{(x,x) : x \in X\}$. Notice that elements in X are only comparable to themselves. Besides, (X, d_2) is a complete metric space considering d_2 the Euclidean distance. Let $T : X \to X$ be defined by

$$T(0,1) = (1,0), \qquad T(1,0) = (0,1), \qquad T(1,1) = (1,1). \tag{3.3}$$

T is trivially continuous and nondecreasing, and assumption (2.1) of Theorem 2.2 is satisfied since elements in X are only comparable to themselves. Moreover, $(1,1) \leq T(1,1) = (1,1)$ and, by Theorem 2.2, T has a fixed point (obviously, this fixed point is $(1,1)$).

On the other hand, for $x = (0,1)$, $y = (1,0) \in X$, we have

$$d(Tx, Ty) = \sqrt{2}, \qquad d(x, Tx) = \sqrt{2}, \qquad d(y, Ty) = \sqrt{2}, \qquad d(x,y) = \sqrt{2}, \tag{3.4}$$

and the contractive condition of Theorem 1.1 is not satisfied because

$$d(Tx,Ty) = \sqrt{2} \le \alpha \frac{d(x,Tx)\cdot d(y,Ty)}{d(x,y)} + \beta d(x,y)$$
$$= \alpha \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}} + \beta\sqrt{2} = (\alpha+\beta)\sqrt{2}, \tag{3.5}$$

and thus $\alpha + \beta \ge 1$.

Consequently, this example cannot treated by Theorem 1.1.

Moreover, notice that in this example we have uniqueness of fixed point and $(X, \le)$ does not satisfy condition (2.14). This proves that condition (2.14) is not a necessary condition for the uniqueness of the fixed point.

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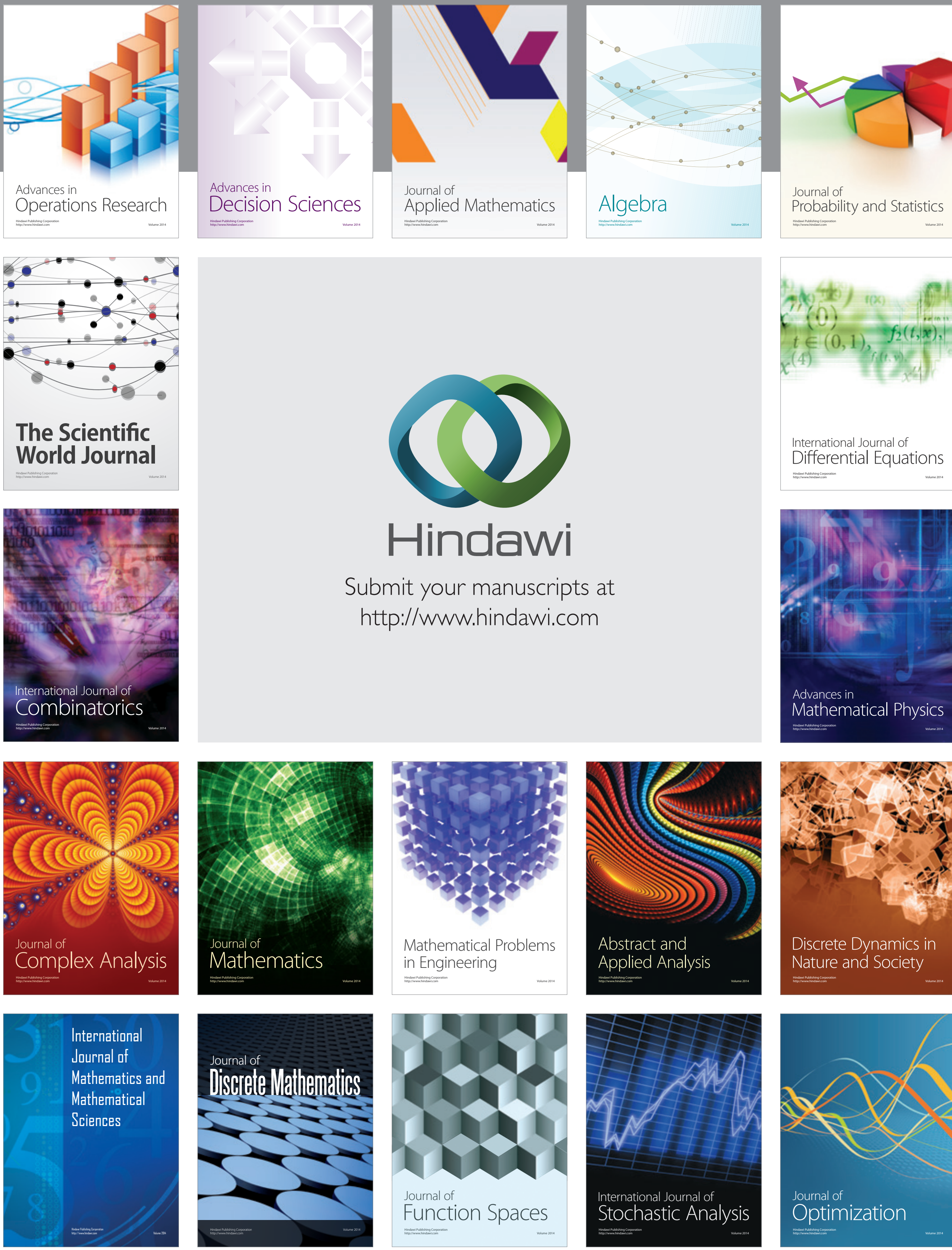
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