

# An Alternative Representation of the Negative Binomial–Lindley Distribution. New Results and Applications

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## Abstract

In this paper we present an alternative representation of the Negative Binomial–Lindley distribution recently proposed by Zamani and Ismail (2010) which shows some advantages over the latter model. This new formulation provides a tractable model with attractive properties which makes it suitable for application not only in insurance settings but also in other fields where overdispersion is observed. Basic properties of the new distribution are studied. A recurrence for the probabilities of the new distribution and an integral equation for the probability density function of the compound version, when the claim severities are absolutely continuous, are derived. Estimation methods are discussed and a numerical application is given.

**Keywords:** Lindley Distribution, Mixture, Negative Binomial Distribution, EM algorithm, Insurance.

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# 1 Introduction

Distribution mixtures define one of the most important ways to obtain new probability distributions in applied probability and operational research. In this sense, and looking for a more flexible alternative to the Poisson distribution, especially under the overdispersion phenomena (variance larger than the mean), the negative binomial obtained as a mixture of Poisson and gamma distributions. In a similar fashion negative binomial–Pareto (Klugman et al. (2008)) and Poisson–inverse Gaussian distribution, also known as Sichel distribution (Willmot (1987)) have been proposed in actuarial contexts, particularly in the automobile insurance setting and other fields where the empirical data seems to show contagious or heterogeneity.

Recently, Zamani and Ismail (2010) proposed a mixture of the negative binomial distribution with parameters  $r > 0$  and  $0 < p < 1$ . For this purpose, they allow the parameter  $p = 1 - \exp(-\lambda)$ ,  $\lambda > 0$  to follow a Lindley distribution. The resulting mixture model has been also applied also recently in the context of accident analysis by Lord and Geedipally (2011).

In this paper we present an alternative representation of this mixture model which can be written in terms of the confluent hypergeometric function and the Pochhammer symbol. This representation shows some advantages over the one previously introduced in the literature. The new formulation provides a tractable model with attractive properties which makes it suitable for application not only in insurance settings but also in other fields where overdispersion is observed. Additionally, some basic properties of the new distribution that were not examined in Zamani and Ismail (2010) are introduced. Some of these features include the unimodality and overdispersion among other properties. A recurrence for the probabilities of the new distribution together with an integral equation for the probability density function of the compound version, when the claim severities are absolutely continuous, are also presented. Estimation methods are discussed by factorial moment and maximum likelihood methods. In addition to these methods, an EM type algorithm is introduced when the triple mixture Poisson–Gamma–Lindley is considered.

The remaining of the paper proceeds as follows. In Section 2 we introduce the basic distributions assumed, the negative binomial and Lindley distributions. Section 3 analyzes the basic properties of the model including the probability function, factorial and ordinary moments, recurrence, overdispersion and unimodality. Some methods of estimations are given in Section

4. Section 5 studies the compound negative binomial–Lindley distribution. An integral equation is derived for the probability density function of the compound version, when the claim severities are absolutely continuous, from the basic principles assumed in the collective risk model. Applications are provided in Section 6 and the work finishes with the conclusions.

## 2 Basic distributions

In this section we introduce the definition and some basic properties of the negative binomial and Lindley distributions. A classical negative binomial distribution with probability mass function

$$p_{r,\lambda}(x) = \binom{r+x-1}{x} \left(\frac{1}{1+\lambda}\right)^r \left(\frac{\lambda}{1+\lambda}\right)^x, \quad x = 0, 1, \dots \quad (1)$$

will be denoted as  $X \sim \mathcal{NB}(r, \lambda)$ , where  $r > 0$  and  $\lambda > 0$ . As they will be needed later, we remind some characteristics of this distribution. The mean, variance and the factorial moment,  $\mu_{[k]}(X) = E[X(X-1)\cdots(X-k+1)]$ , of a negative binomial distribution (see Balakrishnan and Nevzorov (2003)) are respectively given by,

$$\begin{aligned} E(X) &= r\lambda, \\ \text{var}(X) &= r\lambda(1-\lambda), \\ \mu_{[k]}(X) &= (r)_k \lambda^k, \quad k = 1, 2, \dots, \end{aligned} \quad (2)$$

where  $(a)_n = \Gamma(a+n)/\Gamma(a)$  represents the Pochhammer symbol and  $\Gamma(s) = \int_0^\infty \tau^{s-1} e^{-\tau} d\tau$  denotes the complete gamma function.

The probability generating function of a random variable  $X$  following the probability function (1) is given by

$$G_X(z) = (1 + \lambda(1-z))^{-r}, \quad |z| \leq 1.$$

Henceforward, we will use  $X \sim \mathcal{NB}(r, \lambda)$  to denote a random variable  $X$  that follows a negative binomial distribution with parameters  $r > 0$  and  $\lambda > 0$ .

Although the continuous one–parameter Lindley distribution, initially introduced by Lindley (1958), has not been widely used in the past, however it has become a popular probabilistic model in the last decade, in part as a

result of its simplicity and excellent performance in practice. In this regard, it has been chosen as mixing distribution when the parameter of the Poisson distribution is considered random (Sankaran (1971)). In that paper it is shown that the resulting distribution provided a better fit to the empirical set of data considered than the negative binomial and Hermite distributions. Recently, a good deal of attention has been given to this probability density function and new papers have been included in the statistical literature. See Ghitany et al. (2008) and references therein, Ghitany et al. (2013), Gómez-Déniz et al. (2013); among others. A random variable  $\Lambda$  has a Lindley distribution if its probability density function is given by,

$$g_{\theta}(\lambda) = \frac{\theta^2}{1 + \theta}(1 + \lambda) \exp(-\theta\lambda), \quad \lambda > 0 \quad (3)$$

where  $\theta > 0$ . In the following, we will denote as  $\Lambda \sim \mathcal{L}(\theta)$  for a random variable that follows a Lindley distribution.

### 3 Representation of the negative binomial–Lindley distribution

In this paper we introduce an alternative representation of the Negative Binomial–Lindley distribution recently proposed by Zamani and Ismail (2010) that has several advantages over the latter model. This new formulation provides a tractable model with attractive properties that makes it suitable for applications not only in insurance settings but also in other fields where the overdispersion phenomenon is observed.

**Definition 1** *We say that a random variable  $X$  has a negative binomial–Lindley distribution if it admits the stochastic representation:*

$$X|\lambda \sim \mathcal{NB}(r, \lambda), \quad (4)$$

$$\lambda \sim \mathcal{L}(\theta), \quad (5)$$

with  $r, \lambda, \theta > 0$ . We will denote this distribution by  $X \sim \mathcal{NB}\mathcal{L}(r, \theta)$ .

The next result provides closed-form expressions for the probability mass function and factorial moments.

**Theorem 1** Let  $X \sim \mathcal{NB}\mathcal{L}(r, \theta)$  be a negative binomial–Lindley distribution defined in (4)–(5). Some basic properties are:

(a) The probability mass function is given by

$$p_{r,\theta}(x) = \frac{\theta^2 (r)_x}{1 + \theta} \mathcal{W}(x + 1, 3 - r, \theta), \quad x = 0, 1, \dots, \quad (6)$$

where

$$\mathcal{W}(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty \tau^{a-1} (1 + \tau)^{b-a-1} \exp(-z\tau) d\tau,$$

is the confluent hypergeometric function (see Gradshteyn and Ryzhik (1994), p. 1085, formula 9211-4).

(b) The factorial moment of order  $k$  is given by

$$\mu_{[k]}(X) = \frac{(r)_k k! (k + \theta + 1)}{(1 + \theta) \theta^k}, \quad (7)$$

with  $k = 1, 2, \dots$

(c) The mean and the variance are given by,

$$\begin{aligned} E(X) &= \frac{2 + \theta r}{1 + \theta \theta}, \\ \text{var}(X) &= \frac{r(6(1 + \theta) + (4 + \theta)((1 + \theta)\theta + r\theta) + 2r)}{\theta^2(1 + \theta)^2}. \end{aligned}$$

**Proof:**

(a) The probability function of  $X$  can be obtained by using the well-known compound formula,

$$p_{r,\theta}(x) = \int_0^\infty p_{r,\lambda}(x) g(\lambda) d\lambda,$$

and rearranging parameters. Here,  $g(\lambda)$  is the probability density function of the Lindley distribution in (3).

(b) The factorial moments of order  $k$  are obtained making use of (2) and having into account that

$$E(\mu_{[k]}(X)) = E_\lambda(E(\mu_{[k]}(X|\lambda))) = (r)_k E_\lambda(\lambda^k).$$

(c) Finally, the mean and variance are straightforwardly derived from (7).

■

Now, by using the fact that

$$\mathcal{U}(a, b, z) = z^{1-b} \mathcal{U}(a - b + 1, 2 - b, z)$$

and for computational purposes it is convenient rewrite (6) as

$$p_{r,\theta}(x) = \frac{\theta^r (r)_x}{1 + \theta} \mathcal{U}(x + r - 1, r - 1, \theta). \quad (8)$$

Observe that the special case  $r = 1$  provides the geometric–Lindley distribution.

**Theorem 2** *The probability function of an  $\mathcal{NB}\mathcal{L}$  distribution can be evaluated by the recursive formula*

$$p_{r,\theta}(x) = \frac{r + x - 1}{x} p_{r,\theta}(x - 1) - \frac{r}{x} p_{r+1,\theta}(x - 1), \quad x = 1, 2, \dots \quad (9)$$

where  $p_r(0) = \theta^r \exp(\theta) \Gamma(2 - r, \theta) / (1 + \theta)$  and  $\Gamma(a, z) = \int_z^\infty \tau^{a-1} \exp(-\tau) d\tau$  is the incomplete gamma function.

**Proof:** For the negative binomial distribution with pmf

$$p_{r,\lambda}(x) = \binom{r + x - 1}{x} \left( \frac{1}{1 + \lambda} \right)^r \left( \frac{\lambda}{1 + \lambda} \right)^x, \quad x = 0, 1, \dots$$

we have the simple recursion

$$p_{r,\lambda}(x) = \frac{\lambda}{1 + \lambda} \frac{r + x - 1}{x} p_{r,\lambda}(x - 1), \quad x = 1, 2, \dots \quad (10)$$

Using the definition of a  $\mathcal{NB}\mathcal{L}$  distribution and (10) we get

$$\begin{aligned} p_{r,\theta}(x) &= \int_0^\infty p_{r,\lambda}(x) g(\lambda) d\lambda \\ &= \frac{r + x - 1}{x} \int_0^\infty \frac{\lambda}{1 + \lambda} p_{r,\lambda}(x - 1) g(\lambda) d\lambda \\ &= \frac{r + x - 1}{x} \int_0^\infty \left( 1 - \frac{1}{1 + \lambda} \right) p_{r,\lambda}(x - 1) g(\lambda) d\lambda \\ &= \frac{r + x - 1}{x} \left[ p_{r,\theta}(x - 1) - \int_0^\infty \frac{1}{1 + \lambda} p_{r,\lambda}(x - 1) g(\lambda) d\lambda \right]. \end{aligned}$$

Now, since

$$\begin{aligned} \int_0^\infty \frac{1}{1+\lambda} p_{r,\lambda}(x-1)g(\lambda) d\lambda &= \frac{r}{r+x-1} \int_0^\infty p_{r+1,\lambda}(x-1)g(\lambda) d\lambda \\ &= \frac{r}{r+x-1} p_{r+1,\theta}(x-1), \end{aligned}$$

we obtain (9). ■

Calculation of the probabilities are now easy and they do not require the use of the confluent hypergeometric function. This result can be also obtained by using expression (12) in Willmot (1993).

**Proposition 1** *Let  $Z$  a positive and continuous random variate following a gamma distribution with probability density function  $f(z) \propto \lambda^{-r} \exp(-z/\lambda)$ ,  $r > 0$ ,  $\lambda > 0$ , and assume that  $\lambda$  is random following a Lindley distribution with parameter  $\theta > 0$ . Then, the unconditional probability density function of  $Z$  results,*

$$f(z) = \frac{2\theta^{r/2+1} z^{r/2-1}}{(1+\theta)\Gamma(r)} \left( zK_{r-2}(2\sqrt{\theta z}) + \sqrt{\theta z} K_{r-1}(2\sqrt{\theta z}) \right), \quad z > 0, \quad (11)$$

where  $K_n(\cdot)$  is the modified Bessel function of the second kind.

**Proof:** The result follows by computing the integral

$$\begin{aligned} f(z) &= \frac{\theta^2 z^{r-1}}{(1+\theta)\Gamma(r)} \int_0^\infty \lambda^{-r} (1+\lambda) \exp(-z/\lambda - \lambda\theta) d\lambda \\ &= \frac{\theta^2 z^{r-1}}{(1+\theta)\Gamma(r)} \left( \int_0^\infty \lambda^{-r} \exp(-z/\lambda - \lambda\theta) d\lambda \right. \\ &\quad \left. + \int_0^\infty \lambda^{1-r} \exp(-z/\lambda - \lambda\theta) d\lambda \right). \end{aligned}$$

Hence the result. ■

**Proposition 2** *The probability density function in (11) is log-concave for  $r \geq 1$  and therefore unimodal.*

**Proof:** It is well-known that the gamma distribution defined above is log-concave for  $r \geq 1$ . Now the result follows by using Prekopa's Theorem (see Lynch (1999)) and having into account that the probability density function (11) is obtained as a mixture of a gamma distribution with the Lindley distribution which is also log-concave. ■

As a consequence of the last Proposition, we have the following result.

**Proposition 3** *The discrete distribution with probability function given in (6) is unimodal for  $r \geq 1$ .*

**Proof:** It is direct consequence of a result provided in Holgate (1970). ■

Next result shows that the  $\mathcal{NB}\mathcal{L}$  discrete distribution is a Poisson mixture distribution.

**Proposition 4** *The discrete distribution with probability function given in (6) is a Poisson ( $\mathcal{P}(\sigma)$ ,  $\sigma > 0$ ) mixture distribution with mixing distribution given in (11).*

**Proof:** Using  $Z \sim \mathcal{GL}(r, \theta)$  to denote a random variate which follows the probability density function (11) and  $\Sigma \sim \mathcal{G}(r, \lambda)$  when  $\Sigma$  follows a gamma probability density function, it is obvious that the mixture  $\mathcal{NB}(r, \lambda) \wedge_{\lambda} \mathcal{L}(\theta)$  can be written as

$$\begin{aligned} p_{r,\theta}(x) &= \mathcal{NB}(r, \lambda) \wedge_{\lambda} \mathcal{L}(\theta) = \left( \mathcal{P}(\sigma) \wedge_{\sigma} \mathcal{G}(r, \lambda) \right) \wedge_{\lambda} \mathcal{L}(\theta) \\ &= \mathcal{P}(\sigma) \wedge_{\sigma} \left( \mathcal{G}(r, \lambda) \wedge_{\lambda} \mathcal{L}(r, \theta) \right). \end{aligned} \quad (12)$$

Hence the proposition. ■

In the following, we state two more results (without proof) addressing the calculation of the posterior expectations and overdispersion of the  $\mathcal{NB}\mathcal{L}$  distribution.

By using Proposition 10 in Karlis and Xekalaki (2005) the posterior expectation of  $\lambda^r$  given  $x$  can be computed as follows

$$E(\lambda^s|x) = \frac{\Gamma(x+s) p_{r,\theta}(x+s)}{\Gamma(x+1) p_{r,\theta}(x)},$$

for  $s$  taking positive or negative values.

Furthermore, since the  $\mathcal{NB}\mathcal{L}$  distribution arises from a mixture of a Poisson distribution, the variance-to-mean ratio is greater than one (see Karlis and Xekalaki (2005) and Sundt and Vernic (2009), p.66) which implies that the new distribution is overdispersed (variance larger than mean).



## 4 Estimation of parameters

Let  $\tilde{x} = (x_1, \dots, x_n)$  be a random sample from model (6). A simple polynomial equation can be obtained by equating the first two sample and theoretical factorial moments derived from (7). Let  $\tilde{f}_1 = m_{[1]}(X)$  and  $\tilde{f}_2 = m_{[2]}(X)$  the sample version of the factorial moments. Then, we have the following system of equations,

$$\tilde{f}_1 = \mu_{[1]}(X), \quad (13)$$

$$\frac{\tilde{f}_2}{\tilde{f}_1} = \frac{m_{[2]}(X)}{m_{[1]}(X)}. \quad (14)$$

After some computations we have the expression

$$\theta(2 + \theta)^2 \tilde{f}_2 - 2\tilde{f}_1(3 + \theta) [\theta(1 + \tilde{f}_1(1 + \theta)) + 2] = 0,$$

that depends solely on the parameter  $\theta$  and it can be solved numerically. Finally, by plugging this estimated parameter into (13), the estimate of the parameter  $r$  is obtained.

These moment estimates can be used as starting values in the calculation of the maximum likelihood estimates. The maximum likelihood estimates can be obtained directly by maximizing the log-likelihood function, which is straightforwardly derived from (8), is given by

$$\begin{aligned} \ell(\tilde{x}; r, \theta) &= n [r \log(\theta) - \log(1 + \theta) - \log(\Gamma(r))] \\ &\quad + \sum_{i=1}^n [\log(\Gamma(r + x_i)) + \log(\mathcal{U}(x_i + r - 1, r - 1, \theta))]. \end{aligned} \quad (15)$$

Since the global maximum of the log-likelihood surface is not guaranteed, different initial values of the parametric space can be considered as a seed point. In this sense, by using the `FindMaximum` function of Mathematica software package v.11.0 (Wolfram (2003)) and comparing by using other different methods such as Newton, `PrincipalAxis` and `QuasiNewton` (all of them available in that package) the same result is obtained. Finally, the standard errors of the parameter estimates have been approximated by inverting the Hessian matrix. These also can be obtained by approximating the Hessian matrix and recovering it from the Cholesky factors.

## 4.1 Estimation by EM type algorithm

Maximum likelihood estimates can also be achieved by means of the EM algorithm to avoid to use the confluent hypergeometric function when maximizing the log-likelihood function (15). The algorithm could be implemented by using the fact that the  $\mathcal{NB}\mathcal{L}$  discrete distribution arises as a mixture of the Poisson distribution where the Poisson parameter  $\sigma$  follows the distribution given by (11). However as the latter probabilistic family is not a member of the exponential family of probability distributions, the conditional expectations require in the Expectation E-step do not coincide with their sufficient statistics. For that reason, to put into action this algorithm we make use of the mixture representation given in (12). Given the observations  $\tilde{x}$  and the missing observations  $\tilde{\sigma} = (\sigma_1, \dots, \sigma_n)^\top$  and  $\tilde{\lambda} = (\lambda_1, \dots, \lambda_n)^\top$ , the complete probability mass function is

$$\begin{aligned} f(\tilde{x}, \tilde{\sigma}, \tilde{\lambda}|r, \theta) &= \prod_{i=1}^n f(x_i, \sigma_i, \lambda_i|r, \theta) \\ &= \prod_{i=1}^n f(x_i|\sigma_i) \times f(\sigma_i|\lambda_i, r) \times f(\lambda_i|\theta) \end{aligned}$$

and the complete likelihood is

$$\ell(r, \theta; \tilde{x}, \tilde{\sigma}, \tilde{\lambda}) \propto \sum_{i=1}^n \log f(\sigma_i|\lambda_i, r) + \sum_{i=1}^n \log f(\lambda_i|\theta). \quad (16)$$

In the E-step, the expectation of (16), conditional of the observations  $\tilde{x}$  and given the parameter estimates  $r$  and  $\theta$  is given by

$$\begin{aligned} E(\ell(r, \theta; \tilde{x}, \tilde{\sigma}, \tilde{\lambda})|\tilde{x}, \hat{r}, \hat{\theta}) &\propto E\left(\sum_{i=1}^n \log f(\sigma_i|\lambda_i, r)|\tilde{x}, \hat{r}, \hat{\theta}\right) \\ &+ E\left(\sum_{i=1}^n \log f(\lambda_i|\theta)|\tilde{x}, \hat{r}, \hat{\theta}\right). \end{aligned} \quad (17)$$

where  $\hat{r}$  and  $\hat{\theta}$  denotes estimates of parameter  $r$  and  $\theta$  respectively and  $E \equiv E_{\tilde{\lambda}|\tilde{x}, \tilde{\sigma}, \hat{\theta}}$  with  $\hat{\theta} = (r, \theta)$ .

In the M-step, the updated parameter estimates are obtained from maximizing the quantity (17) with respect to  $r$  and  $\theta$ . Particularly, by conditional independence, we have

$$\begin{aligned}
\mathcal{A}(\theta) &= E\left(\sum_{i=1}^n \log f(\lambda_i|\theta)|\tilde{x}, \hat{r}, \hat{\theta}\right) \\
&= \sum_{i=1}^n E(\log f(\lambda_i|\theta)|\tilde{x}, \hat{r}, \hat{\theta}) \\
&= 2n \log \theta - n \log(1 + \theta) + \sum_{i=1}^n E(\log(1 + \lambda_i)|\tilde{x}, \hat{r}, \hat{\theta}) + \theta \sum_{i=1}^n E(\lambda_i|\tilde{x}, \hat{r}, \hat{\theta}) \\
&\propto 2n \log \theta - n \log(1 + \theta) + \theta \sum_{i=1}^n E(\lambda_i|\tilde{x}, \hat{r}, \hat{\theta}). \tag{18}
\end{aligned}$$

In a similar fashion we have

$$\begin{aligned}
\mathcal{B}(r) &= E\left(E_{\tilde{\sigma}|\tilde{x}, \hat{\theta}}\left(\sum_{i=1}^n \log f(\sigma_i|\lambda_i, r)|\tilde{x}, \hat{r}, \hat{\theta}\right)\right) \\
&= \sum_{i=1}^n E\left(E_{\tilde{\sigma}|\tilde{x}, \hat{\theta}}\left(\log f(\sigma_i|\lambda_i, r)|\tilde{x}, \hat{r}, \hat{\theta}\right)\right) \\
&\propto r \sum_{i=1}^n E(\log \lambda_i|\tilde{x}, \hat{r}, \hat{\theta}) - n \log \Gamma(r) \\
&\quad + (r - 1) \sum_{i=1}^n E(E_{\tilde{\sigma}|\tilde{x}, \hat{\theta}}(\log \sigma_i|\tilde{x}, \hat{r}, \hat{\theta})). \tag{19}
\end{aligned}$$

From these expressions we proceed as follows:

- at the E-step the conditional expectation of some functions of  $\lambda_i$  are calculated. From the current estimates,  $\hat{r}^{(j)}$ ,  $\hat{\theta}^{(j)}$ , we calculate the pseudo-values,

$$\begin{aligned}
r_i &= E(\lambda_i|x_i, \hat{r}^{(j)}, \hat{\theta}^{(j)}) = \frac{(1 + x_i)\mathcal{U}(2 + x_i, 4 - \hat{r}^{(j)}, \hat{\theta}^{(j)})}{\mathcal{U}(1 + x_i, 3 - \hat{r}^{(j)}, \hat{\theta}^{(j)})}, \\
s_i &= E(\log \lambda_i|x_i, \hat{r}^{(j)}, \hat{\theta}^{(j)}) \\
&= \frac{\int_0^\infty \log \lambda_i \binom{r + x_i - 1}{x_i} \left(\frac{1}{1 + \lambda_i}\right)^r \left(\frac{\lambda_i}{1 + \lambda_i}\right)^{x_i} \frac{\theta^2}{1 + \theta} (1 + \lambda_i) \exp(-\theta \lambda_i) d\lambda_i}{\int_0^\infty \binom{r + x_i - 1}{x_i} \left(\frac{1}{1 + \lambda_i}\right)^r \left(\frac{\lambda_i}{1 + \lambda_i}\right)^{x_i} \frac{\theta^2}{1 + \theta} (1 + \lambda_i) \exp(-\theta \lambda_i) d\lambda_i},
\end{aligned}$$

$$\begin{aligned}
t_i &= E(\log(\lambda_i + x_i) | x_i, \hat{r}^{(j)}, \hat{\theta}^{(j)}) \\
&= \frac{\int_0^\infty \log(\lambda_i + x_i) \binom{r + x_i - 1}{x_i} \left(\frac{1}{1 + \lambda_i}\right)^r \left(\frac{\lambda_i}{1 + \lambda_i}\right)^{x_i} \frac{\theta^2}{1 + \theta} (1 + \lambda_i) \exp(-\theta \lambda_i) d\lambda_i}{\int_0^\infty \binom{r + x_i - 1}{x_i} \left(\frac{1}{1 + \lambda_i}\right)^r \left(\frac{\lambda_i}{1 + \lambda_i}\right)^{x_i} \frac{\theta^2}{1 + \theta} (1 + \lambda_i) \exp(-\theta \lambda_i) d\lambda_i}.
\end{aligned}$$

- At the M-step, one maximizes the likelihood of the complete model which reduces to maximization of the mixing distribution. Then, the updated values of the parameters are

$$\begin{aligned}
\hat{\theta}^{(j+1)} &= \frac{n - \sum_{i=1}^n r_i + \sqrt{(\sum_{i=1}^n r_i)^2 + 6n \sum_{i=1}^n r_i + n^2}}{2 \sum_{i=1}^n r_i}, \\
\hat{r}^{(j+1)} &= \Psi^{-1} \left( \sum_{i=1}^n s_i - \sum_{i=1}^n t_i + \sum_{i=1}^n \Psi(r + x_i) \right),
\end{aligned}$$

where  $\Psi(\cdot)$  is the digamma function and  $\Psi^{-1}(\cdot)$  is the inverse of the digamma function.

- If some convergence condition is satisfied then stop iterating, otherwise move back to the E-step for another iteration.

## 5 Compound model

Let  $X$  be the number of claims in a portfolio of policies in a time period. Let  $Y_i$ ,  $i = 1, 2, \dots$  be the amount of the  $i$ -th claim and  $S = \sum_{i=1}^X Y_i$  the aggregate claims generated by the portfolio in the period under consideration. As usual, two fundamental assumptions are made in risk theory: (1) the random variables  $Y_1, Y_2, \dots$  are independent and identically distributed with cumulative distribution function  $F(y)$  and probability density function  $f(y)$  and (2) the random variables  $X, Y_1, Y_2, \dots$  are mutually independent. When an  $\mathcal{NBL}$  is chosen for  $X$  (in actuarial setting this is called the primary distribution), the distribution of the aggregate claims  $S$  is called compound negative binomial-Lindley distribution. The cdf of  $S$  is:

$$F_S(y) = \sum_{k=0}^{\infty} F^{*k}(y) \Pr(X = k)$$

where  $F^{*k}(\cdot)$  denotes the  $k$ -fold convolution of  $F(\cdot)$  and  $\Pr(X = k)$  is given in (6). The main result is given in the next theorem.

**Theorem 3** *If the claim sizes are absolutely continuous random variables with pdf  $f(y)$  for  $y > 0$ , then the pdf  $g_s(y; r)$  of the compound  $\mathcal{NB}\mathcal{L}$  distribution satisfies the integral equation,*

$$g_s(y; r) = p_r(0) + \int_0^y \frac{rs + y - s}{y} g_s(y - s; r) f(s) ds - \int_0^y \frac{rs}{y} g_s(y - s; r + 1) f(s) ds. \quad (20)$$

**Proof:** We have that the aggregated claims distribution is given by

$$g_s(y; r) = \sum_{k=0}^{\infty} p_r(k) f^{k*}(y) = p_r(0) f^{0*}(y) + \sum_{k=1}^{\infty} p_r(k) f^{k*}(y),$$

where  $f^{k*}$  denotes the  $k$ -fold convolution of  $f(x)$ . Now, using (9) we have that

$$p_r(k) = \left( \frac{r-1}{k} + 1 \right) p_r(k-1) - \frac{r}{k} p_{r+1}(k-1), \quad k = 1, 2, \dots$$

Then,

$$\begin{aligned} \sum_{k=1}^{\infty} p_r(k) f^{k*}(y) &= \sum_{k=1}^{\infty} \frac{r-1}{k} p_r(k-1) f^{k*}(y) + \sum_{k=1}^{\infty} p_r(k-1) f^{k*}(y) \\ &\quad - \sum_{k=1}^{\infty} \frac{r}{k} p_{r+1}(k-1) f^{k*}(y). \end{aligned}$$

Now, after some straightforward calculations and using the identities:

$$f^{k*}(y) = \int_0^y f^{(k-1)*}(y-s) f(s) ds, \quad k = 1, 2, \dots \quad (21)$$

$$\frac{f^{k*}(y)}{k} = \int_0^y \frac{s}{y} f^{(k-1)*}(y-s) f(s) ds, \quad k = 1, 2, \dots \quad (22)$$

we obtain the result. ■

Integral equation (20) must be solved numerically. There are several implementations and algorithms to solve Volterra integral equation of the second kind but, however, they need to be modified in order to be used in (20). Finally, it is simple to show that if the claim amount distribution is discrete, expressions (21) and (22) are verified by interchanging  $\int_0^y$  by  $\sum_{s=1}^y$  (see Rolski et al. (1999), p.119). Then, the recursion of compound  $\mathcal{NB}\mathcal{L}$  distribution is

$$g_s(y; r) = p_r(0) + \sum_{s=1}^y \frac{rs + y - s}{y} g_s(y - s; r) f(s) - \sum_{s=1}^y \frac{rs}{y} g_s(y - s; r + 1) f(s).$$

## 6 Numerical application

In order to test the performance in practice of the  $\mathcal{NB}\mathcal{L}$  distribution, a simple example dealing where the model introduced in this paper is fitted to an insurance dataset that concerns to the number of automobile liability policies in Zaire (1974) for private cars (Willmot (1987)) is discussed. This dataset appears in Table 1 (first and second columns). As it can be seen, these data are heavily skewed to the right and overdispersed since the sample variance,  $s^2 = 0.12$ , is greater than the sample mean,  $\bar{x} = 0.08$ . Therefore, it is sensible to use an overdispersed (i.e.  $\mathcal{NB}\mathcal{L}$  distribution) discrete distribution to fit this dataset.

By taking as starting values the factorial moment estimates, the maximum likelihood method have been calculated for this dataset by using the  $\mathcal{NB}\mathcal{L}$  distribution. For the sake of comparison, others two-parameter discrete models, the negative binomial ( $\mathcal{NB}$ ) and Poisson-inverse Gaussian ( $\mathcal{PIG}$ ) distributions have been used to describe this dataset. By using the chi-squared test to test the adherence to data of the aforementioned models with test statistic given by  $\chi^2 = \sum_x (p_x - \hat{p}_x)^2 / \hat{p}_x$ . In order to comply with the rule of five, the last three rows were combined. The  $\mathcal{NB}\mathcal{L}$  distribution provides the lowest value for the test statistics. By assuming that the theoretical distribution of the test statistics is  $\chi_2^2$ , the  $p$ -values are easily derived. Based on these  $p$ -values, there exists enough statistical evidence to not reject the null hypothesis that the data come from any of the models considered at the usual significance levels and therefore, there exists statistical evidence to not reject the data come none of the models. However, the test reject the  $\mathcal{NB}$  and  $\mathcal{PIG}$  distributions earlier than the  $\mathcal{NB}\mathcal{L}$  distribution. Additionally, by using the maximum of the likelihood function  $\ell_{max}$  as criterion of comparison the  $\mathcal{NB}\mathcal{L}$  is preferable to  $\mathcal{NB}$  and  $\mathcal{PIG}$  distributions. The parameter estimates for  $\mathcal{NB}\mathcal{L}$  distribution, obtained by maximum likelihood estimation, are  $\hat{r} = 0.486$  and  $\hat{\theta} = 6.381$  with standard errors given by 0.12 and 1.50, respectively. The estimated values by the other distributions can be viewed in Willmot (1987). The maximum likelihood estimates were also obtained by using the EM type algorithm introduced in this papers by using as the values mentioned above as starting values. In this case 155 iterations were needed to obtain the estimates  $\hat{\theta} = 6.663$  and  $\hat{r} = 0.509$  when the relative change of the log-likelihood function was smaller than  $1 \times 10^{-10}$  obtaining a value for  $\ell_{max} = -1183.45$ .

Table 1: Observed and expected claim counts versus different models. See Willmot (1987).

Counts	Observed	Fitted		
		$\mathcal{NB}$	$\mathcal{PI}$	$\mathcal{NBL}$
0	3719	3719.22	3718.58	3718.82
1	232	229.90	234.54	232.98
2	38	39.91	34.86	36.59
3	7	8.42	8.32	8.21
4	3	1.93	2.45	2.26
5	1	0.46	0.80	0.72
Total	4000	4000	4000	4000
$\chi_2^2$		1.17	0.54	0.06
$p$ -value		55.70%	76.20%	80.33%
$\ell_{\max}$		-1183.550	-1183.524	-1183.430

## 7 Conclusion

In this article an alternative representation of the Negative–Binomial–Lindley distribution has been proposed to explain positively skewed and overdispersed count data. The formulation of the model introduced in this work is more tractable the one presented in Zamani and Ismail (2010). Additionally, it includes some attractive properties such as the unimodality and overdispersion. A recurrence for the probabilities of the new distribution together with an integral equation for the probability density function of the compound version, when the claim severities are absolutely continuous, were presented. Finally, an EM type algorithm was also introduced when the triple mixture Poisson–Gamma–Lindley is considered to estimate the parameters of the model.

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