Journal of Applied Analysis and Computation
Volume 8, Number 1, February 2018, 331–343
Website:http://jaac-online.com/
DOI:10.11948/2018.331

ON URYSOHN-VOLTERRA FRACTIONAL QUADRATIC INTEGRAL EQUATIONS*

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Abstract In this paper the authors study a fractional quadratic integral equation of Urysohn-Volterra type. They show that the integral equation has at least one monotonic solution in the Banach space of all real functions defined and continuous on the interval $[0, 1]$. The main tools in the proof are a fixed point theorem due to Darbo and a monotonicity measure of noncompactness.

Keywords Fractional integral, quadratic integral equation, monotonic solutions, Darbo theorem, monotonicity measure of noncompactness.

MSC(2010) 45G10, 45M99, 47H09.

1. Introduction

In this paper, we consider the Urysohn-Volterra quadratic integral equation of fractional order

$$x(t) = h(t) + \frac{f(t, x(t))}{\Gamma(\beta)} \int_0^t \frac{m'(s)v(t, s, (\mathcal{A}x)(s))}{(m(t) - m(s))^{1-\beta}} ds, \ t \in I = [0, 1], \ 0 < \beta < 1. \quad (1.1)$$

Here, $\mathcal{A} : C(I) \to C(I)$ is an operator and $h, \ m : I \to \mathbb{R}$, $f : I \times \mathbb{R} \to \mathbb{R}$, and $v : I \times I \times \mathbb{R} \to \mathbb{R}$ are given functions that satisfy additional assumptions described below.

In the case $\beta = 1$, if we take $f(t, x) = -x$, $\mathcal{A}x = x$, $m(t) = t$, and $v(t, s, x) = \kappa(t, s)x$, then Eq.(1.1) reduces to the form

$$x(t) = h(t) - x(t) \int_0^t \kappa(t, s)x(s)ds, \ t \in I, \quad (1.2)$$

which is a generalization of a Volterra counterpart of a famous Chandrasekhar $H-$equation

$$x(t) = 1 + tx(t) \int_0^1 \frac{\varphi(s)x(s)}{t + s} ds, \ t \in I, \quad (1.3)$$

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[*]The third author was partially supported by the project MTM2013-44357-P.

where φ is a nonnegative characteristic function (see [8, 9, 20, 22, 28]). Moreover, many integral equations of Volterra and Uryshon-Volterra types are special cases of Eq.(1.1); see, for example, [6, 10, 23, 25, 29, 30] and references therein.

After the appearance of Darwish's paper [11], there has been significant interest in the study of the existence of solutions for fractional quadratic integral equations (see [3–5, 12–17]). In this paper, we establish a simple criteria for the existence of nondecreasing solutions of Eq.(1.1). The concept of measure of noncompactness related to monotonicity and a Darbo fixed point theorem are the main tools in proving our results.

2. Basic concepts

In this section we collect some definitions and results that will be needed later in the paper. First, we recall the definition of the Riemann-Liouville fractional integral (see [19, 21, 24, 26, 27]).

Definition 2.1. Let $f \in L_1(a, b)$, $0 \leq a < b < \infty$, and let $\beta > 0$ be a real number. The Riemann-Liouville fractional integral of order β of the function $f(t)$ is defined by

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(s)}{(t-s)^{1-\beta}}\, ds, \ a < t < b.$$

Now, let us assume that $(E, \|.\|)$ is a real infinite dimensional Banach space with zero element θ. Let $B(y, r)$ denote the closed ball centered at y with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If Y is a subset of E, then $\bar{Y}$ and $Conv\, Y$ denote the *closure* and *convex closure* of Y, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [1].

Definition 2.2. A mapping $\mu : \mathcal{M}_E \to [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\ker\mu = \{Y \in \mathcal{M}_E : \mu(Y) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$.
2) $Y \subset X$ implies $\mu(Y) \leq \mu(X)$.
3) $\mu(\bar{Y}) = \mu(ConvY) = \mu(Y)$.
4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ for $0 \leq \lambda \leq 1$.
5) If $Y_n \in \mathcal{M}_E$, $Y_n = \bar{Y}_n$, $Y_{n+1} \subset Y_n$ for $n = 1,\ 2,\ 3,\ ...$ and $\lim\limits_{n\to\infty} \mu(Y_n) = 0$, then $\cap_{n=1}^{\infty} Y_n \neq \phi$.

We will work in the Banach space $C(I)$ consisting of all real functions defined and continuous on I. The space $C(I)$ is equipped with the standard norm

$$\|y\| = \max\{|y(t)| : t \in I\}.$$

Next, we consider the construction of the measure of noncompactness that will be used in the next section (see [1, 2]).

Let Y be a nonempty and bounded subset of $C(I)$. For $y \in Y$ and $\varepsilon \geq 0$, denote by $\omega(y,\varepsilon)$, the modulus of continuity of the function y, i.e.,

$$\omega(y,\varepsilon) = \sup\{|y(t) - y(s)| : t,\ s \in I,\ |t-s| \leq \varepsilon\}.$$

In addition, we set

$$\omega(Y,\varepsilon) = \sup\{\omega(y,\varepsilon) : y \in Y\}$$

and

$$\omega_0(Y) = \lim_{\varepsilon \to 0} \omega(Y,\varepsilon).$$

Define

$$d(y) = \sup\{|y(s) - y(t)| - [y(s) - y(t)] : t, s \in I,\ t \leq s\}$$

and

$$d(Y) = \sup\{d(y) : y \in Y\}.$$

Clearly, all functions belonging to Y are nondecreasing on I if and only if $d(Y) = 0$.

Define the function μ on the family $\mathcal{M}_{C(J)}$ by

$$\mu(Y) = \omega_0(Y) + d(Y).$$

The function μ is a measure of noncompactness in the space $C(I)$.

We will make use of the following fixed point theorem due to Darbo [18]. To state this theorem, we need the following definition.

Definition 2.3. Let M be a nonempty subset of a Banach space E and let $\mathcal{P} : M \to E$ be a continuous operator that maps bounded sets onto bounded ones. We say that $\mathcal{P}$ satisfies the *Darbo condition* (with a constant $k \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset Y of M we have

$$\mu(\mathcal{P}Y) \leq k\ \mu(Y).$$

If $\mathcal{P}$ satisfies the Darbo condition with $k < 1$, then it is called a *contraction operator* with respect to μ.

Theorem 2.1. *Let Q be a nonempty, bounded, closed, and convex subset of the space E and let*

$$\mathcal{P} : Q \to Q$$

be a contraction with respect to the measure of noncompactness μ. Then $\mathcal{P}$ has a fixed point in the set Q.

Remark 2.1. Under the assumptions of the above theorem, it can be shown that the set $\mathrm{Fix}\,\mathcal{P}$ of fixed points of $\mathcal{P}$ belonging to Q is an element of $\ker\mu$.

3. Results

We consider Eq.(1.1) under the following assumptions.

(a_1) $h : I \to \mathbb{R}$ is continuous, nondecreasing, and nonnegative on I.

(a_2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$. Moreover, there is a constant $a \geq 0$ such that $|f(t,x) - f(t,y)| \leq a|x-y|$ for all $t \in I$ and x, $y \in \mathbb{R}$.

(a_3) The superposition operator F defined by $(Fx)(t) = f(t, x(t))$ satisfies that for any nonnegative function x, $d(Fx) \leq a\ d(x)$, where a is the same constant appearing in (a_2).

(a_4) $v : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $v : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$, and $v(t, s, y)$ is nondecreasing with respect to each variable t, s, and y, separately. Moreover, there exists a nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|v(t, s, y)| \leq \Phi(|y|)$ for all t, $s \in I$ and $y \in \mathbb{R}$.

(a_5) The operator $\mathcal{A}$ continuously maps the space $C(I)$ into itself and there exists a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|\mathcal{A}x\| \leq \phi(\|x\|)$ for any $x \in C(I)$. Moreover, for every nonnegative function $x \in C(I)$, the function $\mathcal{A}x$ is nondecreasing and nonnegative on I.

(a_6) The function $m : I \to \mathbb{R}$ belongs to $C^1(I)$ and is nondecreasing.

(a_7) There is a positive number r_0 satisfying

$$\|h\|\Gamma(\beta + 1) + (ar + f^*)(m(1) - m(0))^\beta \Phi(\phi(r)) \leq r\Gamma(\beta + 1), \tag{3.1}$$

where $a\Phi((\phi(r_0))(m(1) - m(0))^\beta < \Gamma(\beta + 1)$ and $f^* = \max_{0 \leq t \leq 1} f(t, 0)$.

We are now in a position to state and prove our main result in this paper.

Theorem 3.1. *If conditions (a_1)–(a_7) hold, then Eq.(1.1) has at least one solution that is continuous and nondecreasing on I.*

Proof. Let $\mathcal{T}$ denote the operator associated with the right-hand side of Eq.(1.1), i.e., $\mathcal{T}x = x$, where

$$(\mathcal{T}x)(t) = h(t) + (Fx)(t)(\mathcal{V}x)(t),\ t \in I, \tag{3.2}$$

and

$$(\mathcal{V}x)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{m'(s)v(t, s, (\mathcal{A}x)(s))}{(m(t) - m(s))^{1-\beta}} ds,\ t \in I,\ 0 < \beta < 1. \tag{3.3}$$

For ease of presentation, we divide the proof into a sequence of steps.

Step 1: $\mathcal{T}$ *maps the space $C(I)$ into itself.*

In view of conditions (a_1) and (a_2), it suffices to show that $\mathcal{V}$ maps $C(I)$ into itself. Fix $\varepsilon > 0$, take t_1, $t_2 \in I$ with $|t_2 - t_1| \leq \varepsilon$, and assume without loss of generality that $t_2 \geq t_1$. Then, we have

$$\begin{aligned}
&|(\mathcal{V}x)(t_2) - (\mathcal{V}x)(t_1)| \\
&= \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1) - m(s))^{1-\beta}} ds \right| \\
&\leq \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1) - m(s))^{1-\beta}} ds \right| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)|v(t_2, s, (\mathcal{A}x)(s)) - v(t_1, s, (\mathcal{A}x)(s))|}{(m(t_2) - m(s))^{1-\beta}}\ ds
\end{aligned}$$

$$+ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} \frac{m'(s)|v(t_1, s, (\mathcal{A}x)(s))|}{(m(t_2) - m(s))^{1-\alpha}} ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} m'(s)|v(t_1, s, (\mathcal{A}x)(s))|[(m(t_2) - m(s))^{\beta-1}$$
$$- (m(t_1) - m(s))^{\beta-1}]ds. \tag{3.4}$$

Now, let

$$\omega_c(v, \varepsilon) = \sup\{|v(t_2, s, y) - v(t_1, s, y)| : s,\ t_1,\ t_2 \in I,$$
$$s \le t_1,\ s \le t_2,\ |t_2 - t_1| \le \varepsilon,\ y \in [-c, c]\}.$$

Using the fact that $m(t_2) - m(0) \ge m(t_1) - m(0)$, from (3.4) we obtain

$$|(\mathcal{V}x)(t_2) - (\mathcal{V}x)(t_1)|$$
$$\le \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)\omega_{\phi(\|x\|)}(v, \varepsilon)}{(m(t_2) - m(s))^{1-\beta}} ds + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} \frac{m'(s)\Phi(\phi(\|x\|))}{(m(t_2) - m(s))^{1-\beta}} ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^{t_1} m'(s)\Phi(\phi(\|x\|))[(m(t_1) - m(s))^{\beta-1} - (m(t_2) - m(s))^{\beta-1}]ds$$
$$\le \frac{(m(t_2) - m(0))^\beta}{\Gamma(\beta+1)} \omega_{\phi(\|x\|)}(v, \varepsilon) + \frac{(m(t_2) - m(t_1))^\beta}{\Gamma(\beta+1)} \Phi(\phi(\|x\|))$$
$$+ \frac{\Phi(\phi(\|x\|))}{\Gamma(\beta+1)} [(m(t_1) - m(0))^\beta - (m(t_2) - m(0))^\beta + (m(t_2) - m(t_1))^\beta]$$
$$\le \frac{(m(t_2) - m(0))^\beta}{\Gamma(\beta+1)} \omega_{\phi(\|x\|)}(v, \varepsilon)$$
$$+ \frac{\Phi(\phi(\|x\|))}{\Gamma(\beta+1)} [(m(t_1) - m(0))^\beta - (m(t_2) - m(0))^\beta + 2(m(t_2) - m(t_1))^\beta]$$
$$\le \frac{(m(t_2) - m(0))^\beta}{\Gamma(\beta+1)} \omega_{\phi(\|x\|)}(v, \varepsilon) + \frac{2(m(t_2) - m(t_1))^\beta}{\Gamma(\beta+1)} \Phi(\phi(\|x\|))$$
$$\le \frac{(m(1) - m(0))^\beta}{\Gamma(\beta+1)} \omega_{\phi(\|x\|)}(v, \varepsilon) + \frac{2[\omega(m, \varepsilon)]^\beta}{\Gamma(\beta+1)} \Phi(\phi(\|x\|)).$$

Thus,

$$\omega(\mathcal{V}x, \varepsilon) \le \frac{1}{\Gamma(\beta+1)} [(m(1) - m(0))^\beta \omega_{\phi(\|x\|)}(v, \varepsilon) + 2[\omega(m, \varepsilon)]^\beta \Phi(\phi(\|x\|))]. \tag{3.5}$$

If $\varepsilon \to 0$, we have $\omega(m, \varepsilon) \to 0$ and $\omega_{\phi(\|x\|)}(v, \varepsilon) \to 0$ due to the uniform continuity of the function v on $I \times I \times [-\phi(\|x\|), \phi(\|x\|)]$. Therefore, the function $\mathcal{V}x$ is continuous on the interval I.

Step 2: *$\mathcal{T}$ maps the ball B_{r_0} into itself.*

For $t \in I$, from (a_2) and (a_7) we have

$$|(\mathcal{T}x)(t)| \le \left| h(t) + \frac{f(t, x(t))}{\Gamma(\beta)} \int_0^t \frac{m'(s)v(t, s, (\mathcal{A}x)(s))}{(m(t) - m(s))^{1-\beta}} ds \right|$$
$$\le |h(t)| + \frac{|f(t, x(t)) - f(t, 0)| + |f(t, 0)|}{\Gamma(\beta)} \int_0^t \frac{m'(s)|v(t, s, (\mathcal{A}x)(s))|}{(m(t) - m(s))^{1-\beta}} ds$$

$$\begin{aligned}&\le \|h\| + \frac{a\|x\| + f^*}{\Gamma(\beta)}\Phi(\phi(\|x\|))\int_0^t \frac{m'(s)}{(m(t)-m(s))^{1-\beta}}ds\\&= \|h\| + \frac{(a\|x\| + f^*)(m(t)-m(0))^\beta}{\Gamma(\beta+1)}\Phi(\phi(\|x\|))\end{aligned}$$

and so

$$\|\mathcal{T}x\| \le \|h\| + \frac{(a\|x\| + f^*)(m(1)-m(0))^\beta}{\Gamma(\beta+1)}\Phi(\phi(\|x\|)). \tag{3.6}$$

If $\|x\| \le r_0$, then by (a_7), inequality (3.6) yields

$$\|\mathcal{T}x\| \le \|h\| + \frac{(ar_0 + f^*)(m(1)-m(0))^\beta}{\Gamma(\beta+1)}\Phi(\phi(r_0)).$$

Therefore, the operator $\mathcal{T}$ maps B_{r_0} into itself.

Step 3: *$\mathcal{T}$ maps the set $B_{r_0}^+ = \{x \in B_{r_0} : x(t) \ge 0,\ t \in I\}$ into itself.*

Notice that the set $B_{r_0}^+$ is nonempty, bounded, closed, and convex. Therefore, by our assumptions, we see that $\mathcal{T}$ maps $B_{r_0}^+$ into itself.

Step 4: *$\mathcal{T}$ is continuous on $B_{r_0}^+$.*

Fix $\varepsilon > 0$ and take x, $y \in B_{r_0}^+$ with $\|x-y\| \le \varepsilon$. Then, for $t \in I$, we have

$$\begin{aligned}&|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)|\\
&\le \left|\frac{f(t,x(t))}{\Gamma(\beta)}\int_0^t \frac{m'(s)v(t,s,(\mathcal{A}x)(s))}{(m(t)-m(s))^{1-\beta}}ds - \frac{f(t,y(t))}{\Gamma(\beta)}\int_0^t \frac{m'(s)v(t,s,(\mathcal{A}y)(s))}{(m(t)-m(s))^{1-\beta}}ds\right|\\
&\le \frac{1}{\Gamma(\beta)}\left|f(t,x(t))\int_0^t \frac{m'(s)v(t,s,(\mathcal{A}x)(s))}{(m(t)-m(s))^{1-\beta}}ds\right.\\
&\quad \left. - f(t,y(t))\int_0^t \frac{m'(s)v(t,s,(\mathcal{A}x)(s))}{(m(t)-m(s))^{1-\beta}}ds\right|\\
&\quad + \frac{1}{\Gamma(\beta)}\left|f(t,y(t))\int_0^t \frac{m'(s)v(t,s,(\mathcal{A}x)(s))}{(m(t)-m(s))^{1-\beta}}ds\right.\\
&\quad \left. - f(t,y(t))\int_0^t \frac{m'(s)v(t,s,(\mathcal{A}y)(s))}{(m(t)-m(s))^{1-\beta}}ds\right|\\
&\le \frac{|f(t,x(t)) - f(t,y(t))|}{\Gamma(\beta)}\int_0^t \frac{m'(s)|v(t,s,(\mathcal{A}x)(s))|}{(m(t)-m(s))^{1-\beta}}ds\\
&\quad + \frac{|f(t,y(t))|}{\Gamma(\beta)}\int_0^t \frac{m'(s)|v(t,s,(\mathcal{A}x)(s)) - v(t,s,(\mathcal{A}y)(s))|}{(m(t)-m(s))^{1-\beta}}ds\\
&\le \frac{a|x(t)-y(t)|}{\Gamma(\beta)}\int_0^t \frac{m'(s)\Phi(\phi(\|x\|))}{(m(t)-m(s))^{1-\beta}}ds\\
&\quad + \frac{|f(t,y(t)) + f(t,0)| + |f(t,0)|}{\Gamma(\beta)}\int_0^t \frac{m'(s)\alpha_v(\varepsilon)}{(m(t)-m(s))^{1-\beta}}ds\\
&\le \frac{a|x(t)-y(t)|}{\Gamma(\beta+1)}(m(t)-m(0))^\beta\Phi(\phi(\|x\|)) + \frac{a|y(t)| + |f(t,0)|}{\Gamma(\beta+1)}(m(t)-m(0))^\beta\alpha_v(\varepsilon)\end{aligned}$$

by (a_2), where

$$\alpha_v(\varepsilon) = \sup\{|v(t,s,u_2) - v(t,s,u_1)| : t,s \in I,\ u_1,u_2 \in [0,\phi(r_0)],\ \|u_2 - u_1\| \le \varepsilon\}.$$

Therefore,

$$\|\mathcal{T}x-\mathcal{T}y\| \leq \frac{a\|x-y\|}{\Gamma(\beta+1)}(m(1)-m(0))^{\beta}\Phi(\phi(\|r_0\|)) + \frac{ar_0+f^*}{\Gamma(\beta+1)}(m(1)-m(0))^{\beta}\alpha_v(\varepsilon). \tag{3.7}$$

As $\varepsilon \to 0$, we have that $\alpha_v(\varepsilon) \to 0$ since v is uniformly continuous on the set $I \times I \times [0, \phi(r_0)]$. It then follows from (3.7) that $\mathcal{T}$ is continuous on $B_{r_0}^+$.

Step 5: *Estimate $\mathcal{T}$ with respect to the monotonic term d.*

We take $\emptyset \neq X \subset B_{r_0}^+$ and fix an arbitrary $x \in X$ and t_1, $t_2 \in I$ with $t_1 \leq t_2$. Then, in view of our assumptions, we obtain

$$\begin{aligned}
d(\mathcal{T}x) &= |(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)| - [(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)] \\
&\leq |h(t_2) - h(t_1)| - [h(t_2) - h(t_1)] + |(Fx)(t_2)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_1)| \\
&\quad - [(Fx)(t_2)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_1)] \\
&\leq |(Fx)(t_2)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_2)| \\
&\quad + |(Fx)(t_1)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_1)| \\
&\quad - [(Fx)(t_2)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_2)] \\
&\quad - [(Fx)(t_1)(\mathcal{V}x)(t_2) - (Fx)(t_1)(\mathcal{V}x)(t_1)] \\
&\leq \frac{d(Fx)}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \\
&\quad + \frac{(Fx)(t_1)}{\Gamma(\beta)} \left\{ \left| \int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2)-m(s))^{1-\beta}} ds - \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1)-m(s))^{1-\beta}} ds \right| \right. \\
&\quad \left. - \left[\int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1) - m(s))^{1-\beta}} ds \right] \right\}.
\end{aligned} \tag{3.8}$$

Next, we will show that

$$\int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \geq \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1) - m(s))^{1-\beta}} ds.$$

We have

$$\begin{aligned}
&\int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1) - m(s))^{1-\beta}} ds \\
&= \int_0^{t_2} \frac{m'(s)v(t_2, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \int_0^{t_2} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \\
&\quad + \int_0^{t_2} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \\
&\quad + \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds - \int_0^{t_1} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_1) - m(s))^{1-\beta}} ds \\
&\geq \int_{t_1}^{t_2} \frac{m'(s)v(t_1, s, (\mathcal{A}x)(s))}{(m(t_2) - m(s))^{1-\beta}} ds \\
&\quad + \int_0^{t_1} m'(s)v(t_1, s, (\mathcal{A}x)(s))[(m(t_2) - m(s))^{\beta-1} - (m(t_1) - m(s))^{\beta-1}] ds
\end{aligned}$$

$$\begin{aligned}
&\geq \int_{t_1}^{t_2} \frac{m'(s)v(t_1,t_1,(\mathcal{A}x)(t_1))}{(m(t_2)-m(s))^{1-\beta}}ds \\
&\quad + \int_0^{t_1} m'(s)v(t_1,t_1,(\mathcal{A}x)(t_1))[(m(t_2)-m(s))^{\beta-1}-(m(t_1)-m(s))^{\beta-1}]ds \\
&= v(t_1,t_1,(\mathcal{A}x)(t_1))\left[\int_0^{t_2} \frac{m'(s)}{(m(t_2)-m(s))^{1-\beta}}ds - \int_0^{t_1} \frac{m'(s)}{(m(t_1)-m(s))^{1-\beta}}ds\right] \\
&= v(t_1,t_1,(\mathcal{A}x)(t_1))\frac{(m(t_2)-m(0))^{\beta}-(m(t_1)-m(0))^{\beta}}{\beta} \\
&\geq 0,
\end{aligned}$$

where, in addition to our assumptions, we used the fact that $(m(t_2)-m(s))^{\beta} \geq (m(t_1)-m(s))^{\beta}$ for $0 \leq s < t_1$. Therefore, (3.8) yields

$$\begin{aligned}
d(\mathcal{T}x) &\leq \frac{d(Fx)}{\Gamma(\beta)} \int_0^{t_2} \frac{m'(s)v(t_2,s,(\mathcal{A}x)(s))}{(m(t_2)-m(s))^{1-\beta}}ds \\
&\leq \frac{\Phi(\phi(r_0))}{\Gamma(\beta+1)}(m(t_2)-m(0))^{\beta}d(Fx) \\
&\leq \frac{a\Phi(\phi(r_0))}{\Gamma(\beta+1)}(m(1)-m(0))^{\beta}d(x),
\end{aligned}$$

and consequently

$$d(\mathcal{T}X) \leq \frac{a\Phi(\phi(r_0))}{\Gamma(\beta+1)}(m(1)-m(0))^{\beta}d(X). \tag{3.9}$$

Step 6: *An estimate of $\mathcal{T}$ with respect to ω_0.*

Fix $\varepsilon > 0$, take $x \in X$ and t_1, $t_2 \in I$ with $|t_2 - t_1| \leq \varepsilon$, and assume without loss of generality that $t_1 \leq t_2$. Then again using our assumptions, we obtain

$$\begin{aligned}
\omega(\mathcal{T}x,\varepsilon) &= |(\mathcal{T}x)(t_2)-(\mathcal{T}x)(t_1)| \\
&\leq |h(t_2)-h(t_1)| + |(Fx)(t_2)(\mathcal{V}x)(t_2)-(Fx)(t_2)(\mathcal{V}x)(t_1)| \\
&\quad + |(Fx)(t_2)(\mathcal{V}x)(t_1)-(Fx)(t_1)(\mathcal{V}x)(t_1)| \\
&\leq \omega(h,\varepsilon) + |(Fx)(t_2)||(\mathcal{V}x)(t_2)-(\mathcal{V}x)(t_1)| \\
&\quad + |(Fx)(t_2)-(Fx)(t_1)||(\mathcal{V}x)(t_1)| \\
&\leq \omega(h,\varepsilon) + \frac{ar_0+f^*}{\Gamma(\beta+1)}[(m(1)-m(0))^{\beta}\omega_{\phi(r_0)}(v,\varepsilon)+2[\omega(m,\varepsilon)]^{\beta}\Phi(\phi(r_0))] \\
&\quad + \frac{a\omega(x,\varepsilon)+\delta_f(\varepsilon)}{\Gamma(\beta+1)}\Phi(\phi(r_0))(m(1)-m(0))^{\beta},
\end{aligned}$$

where

$$\delta_f(\varepsilon) = \sup\left\{|f(t_2,y)-f(t_1,y)| : t_1,\ t_2 \in I,\ y \in [0,r_0],\ |t_2-t_1| \leq \varepsilon\right\}.$$

Therefore,

$$\begin{aligned}
\omega(\mathcal{T}X,\varepsilon) \leq \omega(h,\varepsilon) &+ \frac{ar_0+f^*}{\Gamma(\beta+1)}[(m(1)-m(0))^{\beta}\omega_{\phi(r_0)}(v,\varepsilon)+2[\omega(m,\varepsilon)]^{\beta}\Phi(\phi(r_0))] \\
&+ \frac{(m(1)-m(0))^{\beta}\Phi(\phi(r_0))}{\Gamma(\beta+1)}\delta_f(\varepsilon) + \frac{a\Phi(\phi(r_0))(m(1)-m(0))^{\beta}}{\Gamma(\beta+1)}\omega(X,\varepsilon).
\end{aligned}$$

The last inequality implies

$$\omega_0(\mathcal{T}X) \leq \frac{a\Phi(\phi(r_0))}{\Gamma(\beta+1)}(m(1)-m(0))^\beta\omega_0(X). \tag{3.10}$$

Step 7: *$\mathcal{T}$ is contraction with respect to μ.*

The definition of the measure of noncompactness μ and inequalities (3.9) and (3.10) yield

$$\mu(\mathcal{T}X) \leq \frac{a\Phi(\phi(r_0))}{\Gamma(\beta+1)}(m(1)-m(0))^\beta\mu(X).$$

Since $a\Phi(\phi(r_0))(m(1)-m(0))^\beta < \Gamma(\beta+1)$, $\mathcal{T}$ is a contraction operator with respect to μ.

Step 8: *Application of the Darbo fixed point theorem.*

In view of the previous steps, we can apply Theorem 2.1 to obtain that $\mathcal{T}$ has at least one fixed point, or equivalently, Eq.(1.1) has at least one nondecreasing solution in B_{r_0}. This completes the proof of the theorem. □

4. Examples

First, we present some interesting examples of operators $\mathcal{A}$ satisfying assumption (a_5) of Theorem 3.1.

Example 4.1. Consider the operator $\mathcal{A}$ defined on $C(I)$ by

$$(\mathcal{A}x)(t) = \max_{0\leq\tau\leq t}|x(\tau)|, \text{ for } t\in I.$$

In [7], it is proved that $\mathcal{A}$ maps $C(I)$ into itself and that $\mathcal{A}$ is a continuous operator. Moreover, for $x\in C(I)$,

$$\|\mathcal{A}x\| = \sup\{|(\mathcal{A}x)(t)| : t\in I\} = \sup\{\max_{0\leq\tau\leq t}|x(\tau)| : t\in I\} \leq \sup\{|x(t)| : t\in I\} = \|x\|.$$

Therefore, in this case, the function ϕ appearing in assumption (a_5) is given by $\phi(t)=t$.

Notice that it is easily seen that for any nonnegative function $x\in C(I)$, the function $\mathcal{A}x$ is nondecreasing and nonnegative on I.

Example 4.2. Consider the operator $\mathcal{A}$ defined on $C(I)$ by

$$(\mathcal{A}x)(t) = \int_0^t x(s)\,ds, \text{ for } t\in I.$$

It is clear that $\mathcal{A}$ maps $C(I)$ into itself and it is easily seen that $\mathcal{A}$ is a continuous operator. Moreover, for $x\in C(I)$, we have

$$\begin{aligned}\|\mathcal{A}x\| &= \sup\{|(\mathcal{A}x)(t)| : t\in I\}\\ &= \sup\left\{\left|\int_0^t x(s)\,ds\right| : t\in I\right\}\\ &\leq \sup\{\int_0^t |x(s)|\,ds : t\in I\} = \|x\|.\end{aligned}$$

It is also clear that for any nonnegative function $x \in C(I)$, the function $\mathcal{A}x$ is nondecreasing and nonnegative on I. Therefore, $\mathcal{A}$ satisfies (a_5) with ϕ the identity mapping on $\mathbb{R}_+$.

Notice that if the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy condition (a_5), then $\mathcal{A}_1 + \mathcal{A}_2$ and $\lambda\mathcal{A}_1$ also satisfy it. This algebraic property gives us the possibility of constructing a great variety of operators that satisfy (a_5).

Next, we present a numerical example to illustrate our results.

Example 4.3. Consider the Uryshon-Volterra quadratic integral equation of fractional order having the form

$$x(t) = t^2 + \frac{x(t)}{\Gamma(1/2)} \int_0^t \frac{(t+s)\int\limits_0^s x(u)\,du}{(1+s)\alpha\sqrt{\ln\left(\frac{1+t}{1+s}\right)}} ds, \; t \in I, \tag{4.1}$$

where α is a positive parameter. Eq.(4.1) is a particular case of Eq.(1.1) with $h(t) = t^2$, $f(t,x) = x$, $\beta = 1/2$, $m(s) = \ln(1+s)$, $v(t,s,x) = \frac{(t+s)x}{\alpha}$, and $(\mathcal{A}x)(t) = \int\limits_0^t x(s)\,ds$. Clearly, condition (a_1) is satisfied and $\|h\| = 1$. It is also easy to see that (a_2), (a_3), and (a_6) are satisfied with $a = 1$ and $f^* = \max\limits_{t \in I} f(t,0) = 0$. To see that the operator $\mathcal{A}$ satisfies (a_5), we refer to Example 4.2.

The function $v(t,s,x) = \frac{(t+s)x}{\alpha}$ is clearly nondecreasing with respect to each variable, continuous on $I \times I \times \mathbb{R}$ and it maps $I \times I \times \mathbb{R}_+$ to $\mathbb{R}_+$. Moreover,

$$|v(t,s,x)| = \frac{(t+s)|x|}{\alpha} \le \frac{2}{\alpha}|x|$$

for any t, $s \in I$ and $x \in \mathbb{R}$, so (a_4) is satisfied with $\Phi(t) = \frac{2}{\alpha}t$.

Finally, the inequality in condition (a_7) takes the form

$$\Gamma(3/2) + r\sqrt{\ln 2}\frac{2}{\alpha}r \le r\Gamma(3/2).$$

Notice that the quadratic equation

$$\frac{2\sqrt{\ln 2}}{\alpha}r^2 - r\Gamma(3/2) + \Gamma(3/2) = 0$$

has as its solutions

$$r = \frac{\Gamma(3/2) \pm \sqrt{[\Gamma(3/2)]^2 - \frac{8\sqrt{\ln 2}}{\alpha}\Gamma(3/2)}}{\frac{4\sqrt{\ln 2}}{\alpha}}.$$

These solutions are real and distinct provided

$$[\Gamma(3/2)]^2 - \frac{8\sqrt{\ln 2}}{\alpha}\Gamma(3/2) > 0,$$

or, equivalently, if

$$\alpha > \frac{8\sqrt{\ln 2}}{\Gamma(3/2)}.$$

Therefore, if $\alpha > \frac{8\sqrt{\ln 2}}{\Gamma(3/2)}$, in condition (a_7) we can take

$$r_0 = \frac{\Gamma(3/2) - \sqrt{[\Gamma(3/2)]^2 - \frac{8\sqrt{\ln 2}}{\alpha}\Gamma(3/2)}}{\frac{4\sqrt{\ln 2}}{\alpha}}.$$

Moreover, in our case,

$$\begin{aligned} a\Phi(\phi(r_0))(m(1)-m(0))^\beta &= \frac{2}{\alpha} r_0 (\ln 2 - \ln 1)^{1/2} \\ &= \frac{2}{\alpha}\frac{\Gamma(3/2) - \sqrt{[\Gamma(3/2)]^2 - \frac{8\sqrt{\ln 2}}{\alpha}\Gamma(3/2)}}{\frac{4\sqrt{\ln 2}}{\alpha}}\sqrt{\ln 2} \\ &= \frac{1}{2}\left(\Gamma(3/2) - \sqrt{[\Gamma(3/2)]^2 - \frac{8\sqrt{\ln 2}}{\alpha}\Gamma(3/2)}\right) \\ &< \Gamma(3/2). \end{aligned}$$

This proves that (a_7) is satisfied. Therefore, by Theorem 3.1, Eq.(4.1) has at least one continuous and nondecreasing solution $x(t)$ with $\|x\| \le r_0$.

Remark 4.1. If we replace $\int_0^t x(s)\, ds$ by $\max_{0\le\tau\le s} |x(\tau)|$ in Eq.(4.1), then the same argument (see Example 4.1) shows that the Uryshon-Volterra type integral equation

$$x(t) = t^2 + \frac{x(t)}{\Gamma(1/2)}\int_0^t \frac{(t+s)\max_{0\le\tau\le s}|x(\tau)|}{(1+s)\alpha\sqrt{\ln\left(\frac{1+t}{1+s}\right)}}ds,\ t\in I,\ \alpha>0,$$

has at least one continuous and nondecreasing solution $x(t)$ if $\alpha > \frac{8\sqrt{\ln 2}}{\Gamma(3/2)}$ and $\|x\| \le r_0$, where

$$r_0 = \frac{\Gamma(3/2) - \sqrt{[\Gamma(3/2)]^2 - \frac{8\sqrt{\ln 2}}{\alpha}\Gamma(3/2)}}{\frac{4\sqrt{\ln 2}}{\alpha}}.$$

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