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Conditional Duration Model and the Unobserved Market Heterogeneity of Traders: An Infinite Mixture of Non-Exponentials

Modelo de duración condicionada y heterogeneidad inobservada de los agentes. Una mezcla infinita de distribuciones no exponenciales

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Abstract

This paper extends the conditional duration model proposed by Luca & Zuccolotto (2003) proposing an infinite mixture of distributions based on non-exponentials that account for the unobserved market heterogeneity of traders. The model we propose takes into account the fact that reaction times follow a gamma distribution and that the intensity parameter follows the reciprocal of an inverse Gaussian distribution. This extension allows us to capture, not only various density shapes of durations, but also non-monotonic shapes of hazard functions. The model also allows us to test the unobserved heterogeneity of traders. This mixture model is easy to fit and characterises the behaviour of the conditional durations reasonably well.

***Key words*:** Autoregressive conditional duration model, Exponential distribution, Gamma distribution, Heterogeneity, Reciprocal inverse gaussian distribution.

Resumen

Este trabajo extiende el modelo de duración condicionada propuesto por Luca & Zuccolotto (2003) introduciendo una mezcla infinita de distribuciones no exponenciales que permite incorporar la heterogeneidad inobservada en el mercado por los agentes. El modelo propuesto tiene en cuenta el hecho de que el tiempo de respuesta sigue una distribución gamma y que el parámetro que mide la intensidad sigue una distribución recíproca inversa

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Gaussiana. Esta modelización permite no sólo capturar distintas formas de la distribución de la duración sino que también captura funciones de azar no monótonas. El modelo propuesto es fácil de ajustar a datos de duración proporcionando resultados razonables y competitivos con otros modelos utilizados en la literatura.

***Palabras clave*:** modelo de duración autorregresivo condicional, distribución exponencial, distribución Gamma, heterogeneidad, distribución recíproca inversa gaussiana.

1. Introduction

The most popular econometric model for the intraday trading process has traditionally been the autoregressive conditional duration (ACD) model (Engle & Russell 1998) and its extensions: logarithmic ACD (Bauwens & Giot 2000), stochastic conditional duration (Bauwens & Veredas 2004) and stochastic volatility duration. To cope with higher-order dynamics in the duration process (Ghysels, Gouriéroux & Jasiak 2004), a nonlinear version based on self-exciting threshold autoregressive processes (Zhang, Russell & Tsay 2001), or a family of ACD models that encompasses most specifications proposed in the literature (Fernandes & Grammig 2006), among others.

However, the recent focus of several studies of the financial duration of transactions has taken into account the heterogeneity of agents. This heterogeneity plays an important role in determining the size of price movements, the amount being exchanged, and the frequency at which orders are presented and executed, among other factors (Luca & Gallo 2009). For example, financial market microstructure theories divide traders into informed and non-informed traders. The assumption of interaction among agents, i.e, the coexistence of informed traders who possess private information and liquidity traders whose information set is publicly available (O'Hara (1995) and Ghysels (2000)) suggests that financial durations may obey different probability laws. In addition, there are many reasons to believe that arrival rates for informed and uninformed agents exhibit temporal dependence: each having its own distinct pattern.

Since durations reflect the heterogeneity of traders, modelling their baseline distribution requires us to take into account random effects that account for unobserved heterogeneity, rather than fixed effects, which can suffer from the incidental parameter problem. The distribution of the duration is assumed to be derived from a mixture of distributions given that heterogeneity could reflect differing rates of information arrival (i.e., Clark 1973, Luca & Gallo 2009 and references therein).

The basic idea of enabling the inclusion of the unobservable heterogeneity of individuals in models, by means of diverse probability laws, is not new in microeconometric literature (Lancaster 1990). For example, this approach is taken in shared-frailty models, which are the survival-data analogue to regression models with random effects.[1]

[1] A frailty is a latent random effect that enters multiplicatively on the hazard function.

However, in a financial context, durations could also reflect the heterogeneity of traders. Therefore, modelling the baseline distribution requires a mixture of distributions. For example, Luca & Zuccolotto (2003), Luca & Gallo (2004) and Luca & Gallo (2009) have proposed a link between statistical and financial aspects within a set of distributional assumptions based on financial market microstructure theories. These are related to a wide variety, or heterogeneity, of agents and also of trading conditions (for example, different degrees of information possessed by various traders and differing attitudes toward risk or budget constraints, among others).

These authors have proposed a mixture of distributions based on exponentials in the context of ACD models, such as finite mixtures (i.e., the traders are assumed to be divided into a finite number of groups),[2] or infinite mixtures (i.e., when every trader is considered to have an individual behaviour).

However, unlike Luca & Zuccolotto (2003), our paper contributes to the financial duration literature by considering another baseline distribution for ACD models. More specifically, we propose an infinite mixture of non-exponential distributions for reaction times, thus we draw attentran to a more complex unobserved heterogeneity based on the intensity of duration we do this by using a reciprocal inverse Gaussian distribution, which is related to the inverse Gaussian distribution.

The reciprocal inverse Gaussian distribution, just like the connected inverse Gaussian distribution, has a flexible shape and location in the positive support. This shape is allowed to vary according to the position of the data.

There are several motivations for doing so. First, the inverse Gaussian distribution (also known as the Wald distribution) has many applications in studying lifetime and number of event occurrences (Lancaster 2003, Jørgensen 1982, Zhang, Russell & Tsay 1983, Seshadri 1993, Chhikara & Folks 1989, Abraham & Balakrishnan 1998, Balakrishnan & Nevzorov 2003, among others).[3] When modelling the unobserved heterogeneity of individuals, the inverse Gaussian distribution is also employed in frailty duration models (Hougard 1984, Hanagal & Dabade 2013). However, the reciprocal of the inverse Gaussian distribution (which is not commonly used by statisticians and has been little explored in the literature) allows us to introduce more flexibility and tractability into the modelling of scale parameters and to obtain closed form expressions for the infinite mixture.

Second, following Grammig & Maurer (2000), it is well known that a good candidate distribution should allow for both greater flexibility in modelling and non-monotonic hazard shapes (i.e., QML estimation may provide very inaccurate estimates if the baseline hazard function is non-monotonic). This argument sug-

[2] For example, Luca & Gallo (2009) suggest using a mixture of two distributions with time-varying weights (Mixture ACD model). These authors show that the limitations of the standard base model, and its inadequate modelling of the behaviour in the tail of the distribution, are resolved by their model.

[3] This is a member of the natural exponential family of distributions and can be considered an alternative to the exponential, log-normal, Frechet and Weibull distributions, among others. However, it also provides flexibility in modelling when early occurrences of failure are dominant in a lifetime distribution and the failure rate is expected to be non-monotonic.

gests that another way to increase the flexibility of ACD models is to use a mixture of distributions.

Finally, parameter estimates are very sensitive to the choice of mixing distribution and hence it is important to consider one that takes this non-monotonicity in the baseline hazard into consideration. The proposed approach facilitates the natural parameterisation of a point process in terms of a conditional mean duration for the ACD model[4] and provides estimates with a high degree of fit in terms of the log likelihood function for the family of exponentials.

The rest of this paper is organised as follows: in Section 2, we describe two mixture models based on the exponential and gamma-reciprocal inverse Gaussian ACD model. An empirical example is developed in Section 3, and some conclusions are drawn in Section 4.

2. The New Mixture ACD Model

Let $x_i = t_i - t_{i-1}$ be the duration between two consecutive periods, where t_i is the time for period i, and the expected conditional duration for the ith trade is expressed as $\psi_i = E(x_i \mid x_{i-1}, \ldots, x_1; \theta_1)$, where $x_i = \psi_i \, \varepsilon_i$. Therefore, standardised or excess durations are described by $x_i/\psi_i \equiv \varepsilon_i \sim \text{iid}\, D(\theta_2)$, where D is a general distribution defined within the interval $(0, \infty)$ with $E(\varepsilon_i) = 1$, and where θ_1 and θ_2 are vectors of unknown parameters. ψ_i is called the conditional duration and can be expressed as a linear function of past durations and lagged conditional durations. Hence, the ACD (p, q) model can be written as:

$$\psi_i = \omega + \sum_{j=1}^{q} \alpha_j \, x_{i-j} + \sum_{j=1}^{p} \beta_j \, \psi_{i-j}, \quad i = 1, 2, \ldots, N,$$

where $\omega > 0$, $\alpha_j \geq 0$ and $\beta_j \geq 0$ for all j. Although not necessary, these sign restrictions are convenient to ensure the positivity of ψ_i in the estimation.

In this paper, we focus particularly on $D(\theta_2)$. Any distribution defined as a positive support can be specified for D to estimate ACD models. Simple distributional assumptions for the conditional excess durations have been employed, such as exponential and Weibull distributions (Engle & Russell 1998). However, these pdfs are far from capturing the most salient features of the errors, namely their variability. Therefore, alternative hypotheses have been considered, such as the Generalised gamma distribution (Lunde 1999), or the Burr distribution (Grammig & Maurer 2000) (both nest Weibull and exponential as special cases), and other standardised financial duration distributions such as Birnbaum-Saunders (Bhatti 2010). There are distributions which accommodate certain stylised facts such as over-dispersion (standard deviation greater than the mean), slowly-decreasing autocorrelations (Bauwens, Giot, Gramming & Veredas 2004) or

[4]In the basic formulation, ACD considers that all heterogeneity is captured by the conditional expectation term, which is linear in lagged durations and exhibits persistence decay at an exponential rate.

a finite and infinite mixture of distributions to model the behaviour in the tail of the distribution (Luca & Zuccolotto 2003, Luca & Gallo 2004, Luca & Gallo 2009).

Regarding the consistency of the estimation, (Engle & Russell 1998) show that consistent and asymptotically Normal estimates of vector parameters are obtained by maximising by QML. This is the case even if the distribution of the standardised duration, $D(\theta_2)$, is not exponential. Drost & Werker (2004) show that consistent estimates are obtained when the QML estimation method is based on the standard gamma family (including the exponential).

In this paper, we examine exponential and gamma distributions with reciprocal inverse Gaussian heterogeneity. Unlike Luca & Zuccolotto (2003), who studied an exponential inverse gamma distribution, we take into account the non-monotonic hazard function (i.e., that the intensity function conditional on past durations could be constant, increasing or decreasing with respect to duration (like Grammig & Maurer 2000).

2.1. The mixture model

A mixture of distributions is usually employed for modelling situations with characteristics that differ from those that would be anticipated under a simple component distribution. This is what occurs with the exponential distribution, where the variance is determined by the mean. For this reason, general families of distributions, such as mixtures, are often taken as alternative models that offer greater flexibility. Apart from this flexibility, a mixture model can be thought of as a market which is heterogeneous in which the mixing distribution represents a measure of this heterogeneity.

Let the pdf of the gamma distribution be $f(x) = \frac{1}{\theta^{\sigma}\Gamma(\sigma)}x^{\sigma-1}e^{-x/\theta}$, with a scale parameter $\theta > 0$ and shape parameter $\sigma > 0$. A new class of probability distributions with a domain in $\mathbb{R}^+$ is now introduced by mixing the θ parameter with the reciprocal of the inverse Gaussian distribution. This family can be considered as an alternative to the exponential-inverse Gaussian distribution described in Bhattacharya & Kumar (1986) and Frangos & Karlis (2004), and also to the gamma-generalised inverse Gaussian distribution proposed in Gómez-Déniz, Calderín & Sarabia (2013). Distribution mixtures have often been used in statistical research, especially in the construction of duration models (see Luca & Zuccolotto 2003, Luca & Gallo 2004, Luca & Gallo 2009) because they make it possible to model heterogeneity.

It is straightforward to show that

$$g(z) = \frac{\gamma}{\sqrt{2\pi z}} \exp\left[-\frac{(\gamma z + \delta)^2}{2z}\right], \quad z > 0,\ \gamma > 0,\ \delta > 0 \tag{1}$$

is the probability density function of the reciprocal of a variable distributed according to the inverse Gaussian distribution with parameters $\gamma > 0$ and $\delta > 0$. That is, if Y, $Y > 0$, follows an inverse Gaussian distribution with parameters $\gamma > 0$ and $\delta > 0$, then the random variable $Z = 1/Y$ follows the distribution given in (1). See, for instance, Jørgensen, Seshadri & Whitmore (1991).[5] Henceforth, when a positive random variable Z follows the pdf given in (1) we write $Z \sim \mathcal{RIG}(\gamma, \delta)$ and $X \sim \mathcal{G}(\sigma, \theta)$ to denote that the random variable X follows a gamma distribution. Additionally, the reciprocal inverse Gaussian distribution can be derived, as the inverse Gaussian distribution, as a particular case of the generalised inverse Gaussian distribution (see Mohtashami & Mohtashami 2011, for details).

We begin with the definition of the gamma-reciprocal inverse Gaussian distribution.

Definition 1. A random variable X follows a gamma-reciprocal inverse Gaussian distribution if it admits the following stochastic representation:

$$\begin{aligned} X \mid \sigma, \theta &\sim \mathcal{G}(\sigma, \theta) \quad (2) \\ \theta &\sim \mathcal{RIG}(\delta, \gamma), \quad \theta \in \Theta = (0, \infty), \quad (3) \end{aligned}$$

where $\sigma, \delta, \gamma > 0$. Henceforth, this distribution is denoted by $X \sim \mathcal{GRIG}(\sigma, \delta, \gamma)$.

The next result gives us closed expressions for the pdf of the new distribution.

Theorem 1. *Let $\varepsilon_i \sim \mathcal{GRIG}(\sigma, \delta, \gamma)$, i.e. ε_i follow the representation given by expressions (2)-(3). Then its pdf is given by:*

$$f(\varepsilon_i) = \sqrt{\frac{2\gamma}{\pi}} \frac{\gamma^\sigma e^{\gamma\,\delta}}{\Gamma(\sigma)} x^{\sigma-1} \left[\phi(\varepsilon_i, \delta)\right]^{1/2-\sigma} K_{\sigma-1/2}\left(\gamma\phi(\varepsilon_i, \delta)\right), \quad (4)$$

where $\phi(\varepsilon_i, \delta) = \sqrt{\delta^2 + 2\varepsilon_i}$ and $K_\nu(z)$ represents the modified Bessel function of the second kind (see Jørgensen 1982).

***Proof*.** The pdf can be obtained directly by using the well-known compounding formula

$$f(\varepsilon_i) = \int_0^\infty f(\varepsilon_i \mid \theta) g(\theta)\, d\theta, \quad (5)$$

and by arranging the parameters. □

The mean and variance of the new distribution can also be obtained by compounding and are given by

$$\begin{aligned} E(\varepsilon_i) &= \lambda = \frac{\sigma(1+\gamma\delta)}{\gamma^2}, \quad (6) \\ var(\varepsilon_i) &= \frac{\sigma\left((\sigma+1)\left(\left(\gamma^2\delta^2+3\right)+3\gamma\delta\right)-\sigma(\gamma\delta+1)^2\right)}{\gamma^4}. \end{aligned}$$

The next result gives the closed expression for the cumulative distribution function (cdf) of this new distribution, in which the parameter σ is assumed to be integer and known.

[5] Note that there are several different parameterizations of the inverse Gaussian distribution (see, for example Zhang et al. 1983).

Theorem 2. *Let $\sigma > 0$ be an integer and known and $\varepsilon_i \sim \mathcal{GRIG}(\delta, \gamma)$, i.e. ε_i follows the representation given by expressions (2)-(3). Then its cdf is given by:*

$$F(\varepsilon_i) = 1 - \sqrt{\frac{2}{\pi}}\gamma^{3/2} e^{\gamma\,\delta} \sum_{j=0}^{\sigma-1} \frac{(\gamma x)^j}{j!} \left(\phi(\varepsilon_i, \delta)\right)^{1/2-j} K_{j-1/2}\left(\gamma\phi(\varepsilon_i, \delta)\right). \tag{7}$$

***Proof*.** The cdf can be computed by using:

$$F(\varepsilon_i) = \int_0^{\varepsilon_i} \int_0^{\infty} f(t_i \mid \theta) f(\theta) dt\, d\theta.$$

Now, by applying Fubini's theorem and taking into account the expression, for integer values of σ,

$$\frac{1}{\Gamma(\sigma)} \int_0^{\varepsilon_i} \frac{1}{\theta^{\sigma}} t^{\sigma-1} e^{-t/\theta} dt = 1 - \sum_{j=0}^{\sigma-1} \frac{(\varepsilon_i/\theta)^j e^{-\varepsilon_i/\theta}}{j!},$$

is satisfied (see for instance (Castillo, Hadi, Balakrishnan & Sarabia 2005), p. 82), then the desired result is obtained after some algebra. □

2.2. Two Simple Sub-Models

Two simple sub-models obtained from the model above are now studied in some detail.

2.2.1. The Exponential-Reciprocal Inverse Gaussian

It is known that the exponential distribution is a particular case of the gamma distribution and obtained by taking $\sigma = 1$ in (2). In this case, after the compounding process with the reciprocal inverse Gaussian distribution we obtain the exponential-inverse Gaussian distribution mixture, the pdf of which is obtained from (4). It should be taken into account that $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} \exp(-z)$ and thus, after some simple algebra, we obtain

$$f(\varepsilon_i) = \frac{\gamma e^{\gamma\,\delta}}{\phi(\varepsilon_i, \delta)} \exp\left[-\gamma\phi(\varepsilon_i, \delta)\right], \tag{8}$$

for $\varepsilon_i > 0$, $\delta > 0$ and $\gamma > 0$, which are the heterogeneity parameters.

It is straightforward to see that

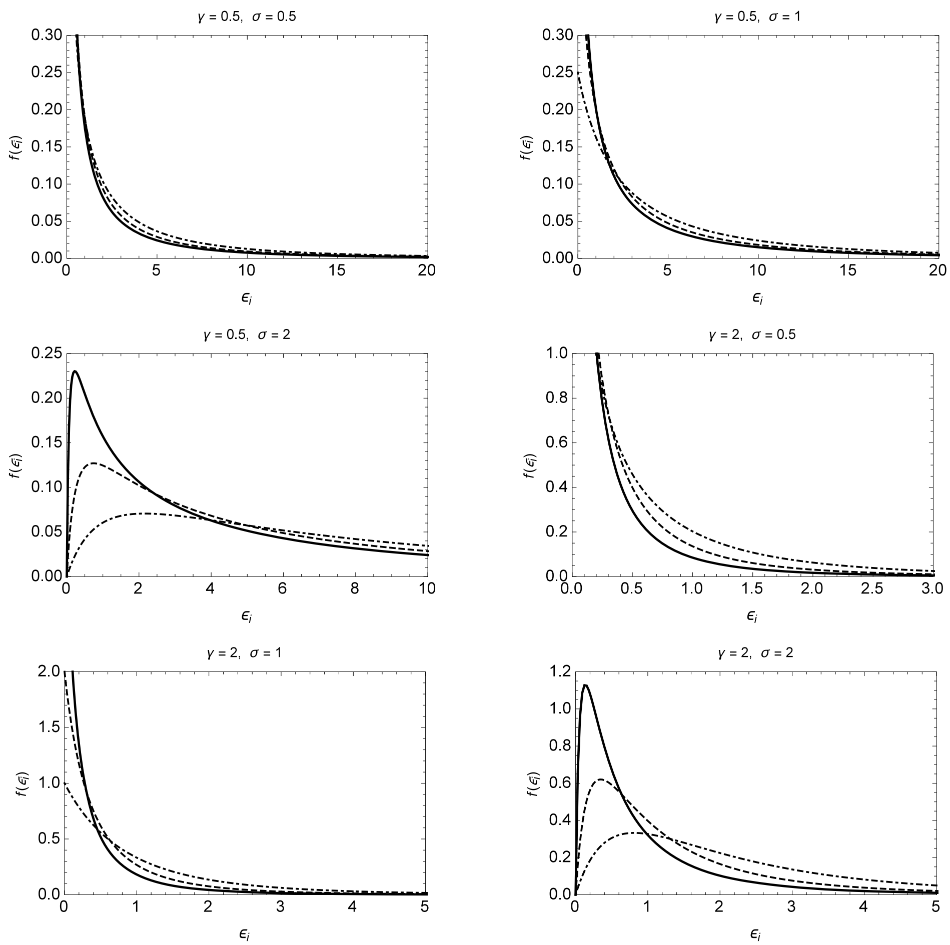
$$\frac{d^2}{dx^2}\left(\log f(\varepsilon_i)\right) = \frac{2 + \gamma\phi(\varepsilon_i, \delta)}{[\phi(\varepsilon_i, \delta)]^4} > 0,$$

and so the distribution is log-convex. Therefore, the cumulative distribution function, $F(\varepsilon_i)$, and the survival function, $1 - F(\varepsilon_i)$ are also log-convex and the hazard rate function is nonincreasing.

To further investigate the properties of this mixture model, we examine the behaviour of some parameters of interest.

The mode is always at the origin of the support of the Distribution. Some pdf curves are shown in Figure 1.

FIGURE 1: Some examples of the probability density function of the gamma-reciprocal inverse Gaussian mixture distribution for selected parameter values, $\delta = 0.5$ (thick), $\delta = 1$ (dashed) and $\delta = 2$ (dotdashed).



The mean and the variance are given by

$$
\begin{aligned}
E(\varepsilon_i) &= \kappa = \frac{1+\gamma\,\delta}{\gamma^2}, \qquad (9)\\
var(\varepsilon_i) &= \frac{\gamma^2\delta^2+4\gamma\delta+5}{\gamma^4}.
\end{aligned}
$$

The survival function of the exponential-reciprocal inverse Gaussian mixture distribution is obtained from (7). This gives:

$$\bar{F}(\varepsilon_i) = \exp\left\{\gamma\left[\delta - \phi(\varepsilon_i, \delta)\right]\right\}. \tag{10}$$

By applying (8) in conjunction with (10) we obtain the hazard rate function given by:

$$h(\varepsilon_i) = \frac{\gamma}{\phi(\varepsilon_i, \delta)},$$

which is clearly decreasing, as expected from a mixture of the exponential distribution.

For many insurance and financial risks, the right-tail risk, representing low-frequency and large-loss events, is usually measured in terms of the right-tail index (Frangos & Karlis 2004), given by:

$$\zeta(\varepsilon_i) = \frac{1}{E(\varepsilon_i)} \int_0^\infty \sqrt{\bar{F}(t)}\, dt - 1.$$

In some computations, for the exponential-reciprocal inverse Gaussian mixture distribution, this value is given by

$$\zeta(\varepsilon_i) = 1 + \frac{2}{1 + \gamma\delta},$$

and, therefore a value larger than 1, which is that of the exponential distribution.

Moreover, in recent years there has been increasing interest (especially in actuarial and financial settings) in computing quantiles of probability of the distribution of a particular risk. Two measures of risk that are of particular importance in this framework are the value of risk (VaR) and the tail of the value of risk (TVaR); see Furman & Zitikis (2008) for details. For the distribution we are considering, these measures are obtained in simple forms, and are given by:

$$\begin{aligned}
\mathrm{VaR}(q) &= \frac{1}{2}\left[\left(\delta - \frac{1}{\gamma}\log q\right)^2 - \delta^2\right], \\
\mathrm{TVaR}(q) &= \mu + \frac{q \log q}{\gamma^2}\left(1 + \gamma\delta - \frac{1}{2}\log q\right),
\end{aligned}$$

for $0 < q < 1$.

Finally, the mean residual life, which is defined as $m(t) = E(X - t \mid x > t)$ gives

$$m(t) = \frac{1}{\gamma^2}\left[1 + \gamma\phi(t)\right].$$

Because $\frac{d}{dt}m(t) > 0$ for all t, the resulting mean residual life is an increasing function on t.

2.2.2. The Gamma$(2, \theta)$-Reciprocal Inverse Gaussian Distribution

One of the disadvantages of employing the exponential-reciprocal inverse Gaussian distribution is that, with this mixture, the hazard rate function decreases. This problem can be overcome by considering the case in which $\sigma = 2$, which corresponds to the $\mathcal{G}(2, \theta)$-reciprocal inverse Gaussian mixture distribution, the pdf of which is given by:

$$f(\varepsilon_i) = \frac{\varepsilon_i \gamma e^{\delta\gamma}}{\phi^2(\varepsilon_i, \delta)} \left[\gamma + \frac{1}{\phi(\varepsilon_i, \delta)}\right] \exp\left[-\gamma\phi(\varepsilon_i, \delta)\right]. \tag{11}$$

The next result shows that the above distribution is unimodal.

Theorem 3. *The density given in (11) is unimodal with a modal value satisfying the following equation*

$$\Psi(\varepsilon_i) = 2\varepsilon_i^2\gamma^2 - \delta^2\left[1 + \gamma\phi(\varepsilon_i, \delta)\right] + x\left[1 + \gamma(\gamma\delta^2 + \phi(\varepsilon_i, \delta))\right] = 0. \tag{12}$$

***Proof*.** By computing the derivative of (11) and equating it to zero, we obtain, after some simple algebra, the equation given in (12). Now, taking into account that $\Psi(\infty) = \infty$, $\Psi(0) = -\delta^2(1 + \gamma\delta) < 0$ and the fact that

$$\Psi'(\varepsilon_i) = 1 + \frac{3\varepsilon_i\gamma}{\phi(\varepsilon_i, \delta)} + \gamma^2(4x + \delta^2) > 0$$

the theorem is proved. □

The mean and the variance are given by

$$\begin{aligned} E(\varepsilon_i) &= \vartheta = \frac{2(1 + \gamma\delta)}{\gamma^2}, \qquad (13) \\ var(\varepsilon_i) &= \frac{2}{\gamma^4}\left(3(\gamma\delta(\gamma\delta + 3) + 3) - 2\delta^2\left(\gamma + \frac{1}{\delta}\right)^2\right). \end{aligned}$$

Figure 1 shows some examples of the pdf of the gamma$(2, \theta)$-reciprocal inverse Gaussian mixture distribution for $\gamma = 0.5$ and $\gamma = 2$.

Again, the survival function is obtained from (7). This gives

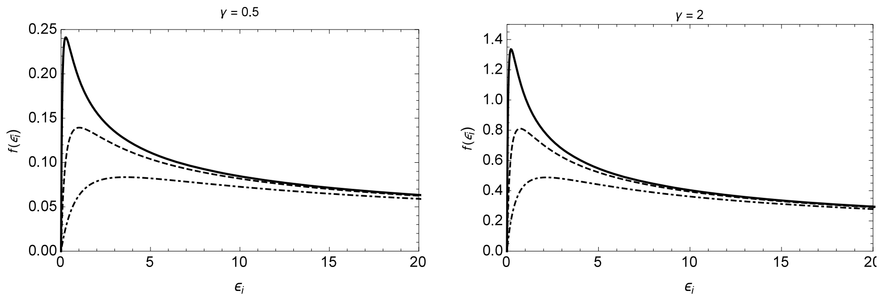
$$\bar{F}(\varepsilon_i) = \gamma\left[\frac{1}{\gamma} + \frac{\varepsilon_i}{\phi(\varepsilon_i, \delta)}\right] \exp\left\{\gamma\left[\delta - \phi(\varepsilon_i, \delta)\right]\right\}. \tag{14}$$

From (11) and (14) can we obtain the hazard rate function, given by:

$$h(\varepsilon_i) = \frac{\gamma\varepsilon_i}{\phi} \frac{1 + \gamma\phi(\varepsilon_i, \delta)}{\phi(\varepsilon_i, \delta)[\gamma\ \varepsilon_i + \phi(\varepsilon_i, \delta)]}.$$

This function is non-monotonic with respect to duration, as is to be expected from a mixture of exponential distributions. Figure 2 shows some curves for this hazard rate function for various parameter values.

FIGURE 2: Some examples of the hazard rate function of the gamma$(2,\theta)$-reciprocal inverse Gaussian mixture distribution for various parameter values, $\delta = 0.5$ (thick), $\delta = 1$ (dashed) and $\delta = 2$ (dotdashed).



2.3. Likelihood of the Conditional Duration

With the usual change of variable proposed in duration models, we obtain the conditional pdf of x_i given ψ_i. From this, the log-likelihood function is given by

$$\begin{aligned}\log f(x_i \mid \psi_i) \quad = \quad & \log\sigma - 2\log(\Gamma(\sigma)) + \log\delta - \log M - \log\psi_i - \frac{1}{2}\log 2 \\ & + \left(\frac{1}{4} - \frac{\sigma}{2}\right)\log\left(\delta^2 + \frac{2\varepsilon_i\phi}{\psi_i}\right) + (\sigma - 1)\log\left(\frac{\varepsilon_i\phi}{\psi_i}\right) \\ & + \left(\sigma - \frac{1}{2}\right)\left(2\log\gamma + \log\left[\phi(x_i\lambda/\psi_i, \delta)\right] - \log 2\right) + \gamma\delta \\ & + \log\left(\frac{1}{\gamma\delta} + 1\right) + \log\left(\chi(\underline{\varphi}) + \chi(\overline{\varphi}) + 2\sum_{j=0}^{M-1}\chi(\underline{\varphi} + hj)\right),\end{aligned}$$

where λ is given in (6). See Appendix A for details of the other elements of this expression.

For $\sigma = 1$, when applying the infinite exponential mixture, the conditional pdf of x_i given ψ_i, is

$$f(x_i \mid \psi_i) = \frac{\gamma\kappa}{\psi_i}\ \frac{\exp\left\{\gamma\left[\delta - \phi(\varepsilon_i\kappa/\psi_i, \delta)\right]\right\}}{\phi(\varepsilon_i\kappa/\psi_i, \delta)},$$

where κ is given in (9).

The log-likelihood function is then

$$\begin{aligned}\log f(x_i \mid \psi_i) \quad = \quad & \log(1 + \gamma\delta) - \log\gamma - \log\psi_i \\ + \quad & \gamma\left[\phi(x_i\kappa/\psi_i, \delta)\right] - \frac{1}{2}\log\left[\phi(x_i\kappa/\psi_i, \delta)\right].\end{aligned}$$

As an alternative to the maximum likelihood method of estimation, a maximum expectation algorithm could be straightforwardly developed, to facilitate the estimation procedure. In this respect, see Frangos & Karlis (2004) and Gómez-Déniz et al. (2013) who addressed the estimation by using the exponential-inverse Gaussian distribution and the gamma-inverse Gaussian distribution, respectively.

Since we do not provide the estimated parameters or the Fisher's information matrix in closed-form expression the next subsection includes a bootstrapping technique. This allows for measures of accuracy (defined in terms of bias and standard deviation) to be assigned toof the estimated parameters of the two distributions, the exponential-reciprocal inverse Gaussian and the gamma$(2, \theta)$-reciprocal inverse Gaussian distribution.

2.4. Simulation Experiment

in this section we present some simulation results that were obtained by the bootstrap experiment. The proposed of therm is to study the behavior of the maximum likelihood estimators. In this work we have used the Mathematica package to create random variates directly from the pdf's (8) and (11).

We have considered three different sets of model parameters (see Table 1) for the case of the exponential-reciprocal inverse Gaussian distribution and two sets of model parameters in the case of the gamma$(2, \theta)$-reciprocal inverse Gaussian distribution (see Table 2). In all the cases we have used $n = 500$, 1000 and 200. We have proposed the average estimates and square root of the mean squared errors based on 1000 replications. The results are reported in Tables 1 and 2. As the sample size increases, the biases and the mean squared errors decrease, which verifies the consistency properties of maximum likelihood estimates.

Table 1: Average estimates (first row) and the square root of the mean squared errors (second row) for the exponential-reciprocal inverse Gaussian distribution.

n	$\delta = 0.5$	$\gamma = 1$	$\delta = 3$	$\gamma = 2$	$\delta = 0.75$	$\gamma = 2$
500	0.66260	1.07758	0.73602	1.97440	2.34040	1.6209
	0.11380	0.07977	0.12875	0.18829	0.55175	0.28380
1000	0.50187	0.95921	2.91440	1.95555	0.71118	2.03135
	0.06443	0.04774	0.75168	0.39719	0.07792	0.13000
2000	0.50685	1.02411	3.36927	2.20952	0.80300	2.04216
	0.04532	0.03371	0.54636	0.29348	0.08065	0.11565

Table 2: Average estimates (first row) and the square root of the mean squared errors (second row) for the gamma$(2, \theta)$-reciprocal inverse Gaussian distribution.

n	$\delta = 0.5$	$\gamma = 1$	$\delta = 1$	$\gamma = 0.5$
500	0.53553	1.03282	0.79510	0.45954
	0.07671	0.05190	0.09765	0.01928
1000	0.45411	0.96222	0.99347	0.53990
	0.03952	0.03621	0.08912	0.01931
2000	0.51444	0.99679	0.94204	0.48733
	0.03000	0.02406	0.06026	0.01189

3. An Empirical Example

As an illustration of the application of our specification to duration models, we estimated a simple ACD model using data obtained from the transaction durations of IBM stock on five consecutive trading days from November 1 to November 7, 1990 (Tsay 2002). Focusing on positive transaction durations, we obtained 3534 observations. In addition, the data were adjusted by removing the deterministic component. Thus, a total of 3534 positive adjusted durations were used.

To compare several distribution results, we examined exponential, Weibull, Burr, infinite mixture of (Luca & Zuccolotto 2003) and exponential-inverse Gaussian ACD models, together with the density function and the logarithm of likelihood for various duration models (see Table 3). Let Θ be a vector of unknown parameters to be estimated. Then, the density functions and the logarithm of each of the duration models used in this paper are as follows:

$$\text{Exponential: } \ell(\Theta, x_i) = \sum_{i=1}^{N} \left[-\log \psi_i - \frac{x_i}{\psi_i} \right],$$

$$\text{Weibull: } \ell(\Theta, x_i) = \sum_{i=1}^{N} \left[\log \gamma - \log x_i + \gamma \log \left(\frac{x_i}{\phi_i} \right) - \left(\frac{x_i}{\phi_i} \right)^{\gamma} \right],$$

where $\phi_i = \psi_i \left[\Gamma\left(1 + 1/\gamma\right) \right]^{-1}$.

$$\text{Generalised gamma: } \ell(\Theta, x_i) = \sum_{i=1}^{N} \Big[\log \gamma - \log \Gamma(\delta) + (\delta\gamma - 1) \log x_i - \delta\gamma(\log \phi + \log \psi_i) - \left(\frac{x_i}{\phi \psi_i} \right)^{\gamma} \Big],$$

where $\phi = \Gamma(\delta)/\Gamma(\delta + 1/\gamma)$,

$$\text{Burr: } \ell(\Theta, x_i) = \sum_{i=1}^{N} \Big[\log \kappa - \kappa \log \epsilon_i + (\kappa - 1) \log x_i - \left(\frac{1}{\sigma^2} + 1 \right) \log(1 + \sigma^2 \epsilon_i^{-\kappa} x_i^{\kappa}) \Big],$$

where $\epsilon_i = \frac{\psi_i (\sigma^2)^{1+1/\kappa} \Gamma(1/\sigma^2 + 1)}{\Gamma\left(1 + \frac{1}{\kappa}\right) \Gamma\left(\frac{1}{\sigma^2} - \frac{1}{\kappa}\right)}$.

For the infinite exponential mixture of Luca & Zuccolotto (2003),

$$\ell(\Theta, x_i) = \sum_{i=1}^{N} \Big[(\theta_1 + 1) \log \theta_1 - \log \psi_i + \log(\theta_1 + 1) - (\theta_1 + 2) \log \left(\theta_1 + \frac{x_i}{\psi_i} \right) \Big],$$

Table 3 shows the estimated parameters, with the asymptotic t-statistics corrected to take into consideration the presence of unobserved heteroskedasticity, and, also, the value of the logarithm of maximum likelihood.

Table 3: Quasi maximum likelihood estimates, statistics and misspecification tests of the different ACD(1,1) models.

	Exponential	Weibull	Generalised Gamma	Burr	Infinite exponentials	Exponential-reciprocal inverse Gaussian	Gamma-reciprocal inverse Gaussian
δ			4.0689			1.7748	0.401
			(5.66)			(15.47)	(21.32)
γ		0.8788	0.4038			1.6082	
		(78.48)	(10.14)			(15.47)	
κ				0.9812			
				(60.21)			
σ^2				0.1885			
				(5.81)			
σ							0.5767
							(188.24)
θ_1					3.7437		
					(8.81)		
ω	0.1803	0.1687	0.1411	0.1119	0.1177	0.1470	0.2727
	(2.35)	(2.33)	(3.01)	(2.82)	(2.71)	(2.87)	(2.78)
α_1	0.065	0.0639	0.0627	0.0499	0.0573	0.0637	0.1165
	(4.93)	(5.39)	(5.75)	(5.70)	(5.54)	(6.48)	(6.28)
β_1	0.8811	0.8852	0.8968	0.8966	0.9081	0.8933	0.8907
	(27.9)	(30.61)	(47.59)	(42.69)	(43.77)	(44.34)	(41.19)
$\ell_{\max}$	−7688.09	−7583.66	−7633.65	−7615.51	−7613.70	−7610.02	−6992.73
AIC	4.3526	4.3224	4.2947	4.3127	4.3111	4.3096	3.9602
BIC	4.3579	4.3294	4.3034	4.3214	4.3181	4.3183	3.9690
$E(x_i \mid \psi_i)$	1.0014	1.0058	0.9894	1.2457	0.9943	0.9961	0.9667
$\sigma_{x_i \mid \psi_i}$	1.2276	1.2336	1.2157	1.2629	1.2095	1.2234	1.1883
$\text{MSE}_{x_i \mid \psi_i}$	13.701	13.658	13.860	13.373	16.368	16.393	16.415
$\text{MAE}_{x_i \mid \psi_i}$	2.2906	2.2862	2.3025	2.2624	2.7582	2.7593	2.7928
Excess-Dispersion	4.7728	4.8365	4.5032	5.2195	4.7384	4.6346	4.0722
$Q(1)$	0.0680	0.0850	0.1566	0.146	0.376	0.1125	0.0998
	[0.79]	[0.77]	[0.69]	[0.70]	[0.54]	[0.74]	[0.75]
$Q(5)$	2.8626	2.8591	2.7562	2.9510	3.043	2.8385	2.6580
	[0.72]	[0.72]	[0.74]	[0.70]	[0.69]	[0.72]	[0.75]
$Q(10)$	5.1054	5.1193	5.1449	5.147	4.784	5.0560	5.3740
	[0.88]	[0.88]	[0.88]	[0.88]	[0.90]	[0.89]	[0.86]
$Q(20)$	11.2538	11.0604	10.6352	10.742	10.658	10.479	10.787
	[0.94]	[0.94]	[0.95]	[0.95]	[0.95]	[0.96]	[0.95]

Note: The t-values are shown in parentheses and the p-values in brackets.

Because the tests for ACD models involve basic residual examinations that test for the functional form of the conditional mean duration or test for the distribution of the error term, Table 3 shows the following statistics. They are based on standardised durations: mean, standard deviation, mean square error ($\text{MSE}_{x_i|\psi_i}$), mean absolute error ($\text{MAE}_{x_i|\psi_i}$), and excess-dispersion. Thus,

$$
\begin{aligned}
E(x_i \mid \psi_i) &= \frac{1}{N}\sum_{i=1}^{N}\frac{x_i}{\psi_i}, \\
\sigma_{x_i|\psi_i} &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i}{\psi_i} - E(x_i \mid \psi_i)\right)^2}, \\
\text{MSE}_{x_i|\psi_i} &= \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \psi_i\right)^2,
\end{aligned}
$$

$$\text{MAE}_{x_i|\psi_i} = \frac{1}{N}\sum_{i=1}^{N} \mid x_i - \psi_i \mid,$$

$$\text{Excess-dispersion} = \sqrt{N}\frac{\sigma^2_{x_i|\psi_i} - 1}{\sigma_{(x_i|\psi_i - 1)^2}}.$$

We also include the Akaike and Schwarz Bayesian information criteria (AIC and BIC, respectively) and the Ljung-Box statistic ($Q(k)$) for $k = 1$, $k = 5$, $k = 10$ and $k = 20$ lags for autocorrelation in the standardised residuals.

In general, these results indicate that all the parameters, in each of the models estimated, are statistically significant at any conventional level of significance, but also that the standardised residual does not present serial dependence. For example, the heterogeneity parameters for the gamma-reciprocal inverse Gaussian are positive ($\gamma, \delta > 0$), but they are also positive in the infinite exponential of (Luca & Zuccolotto 2003)'s model ($\theta_1 > 0$). Henceforth, misspecification tests based on the autocorrelation of standardised residuals indicate the non-presence of serial dependence. All of the models examined present good behaviour in their standardised residuals, and wherefore, the models cannot be compared by means of this portmanteau test. Additionally, in terms of the log-likelihood measures and of AIC and BIC information criteria, the gamma-reciprocal inverse Gaussian distribution achieves the best fit. Moreover, with respect to statistical error measures, the gamma-reciprocal inverse Gaussian has the lowest excess dispersion. Finally, the durations show overdispersion phenomena in all models, as is shown by the fact that the mean value is lower than the variance.

Finally, we used the Vuong closeness test for model selection: a likelihood-ratio-type test based on the Kullback-Leibler information criterion. This statistic makes probabilistic statements about two models and the limiting distribution is standard Normal. Table 4 shows the Vuong statistic results for our non-nested models. Model 0 is shown in the rows and Model 1 in the columns. As most of the results are both negative and large, Model 1 is favoured, and, Therefore, so we conclude that the gamma-reciprocal inverse Gaussian model outperforms the other models at all conventional significance levels. The only exception is the Weibull model as 0. In this case, the results are inconclusive.

Table 4: Vuong statistic test for nonnested models

	Exponential	Weibull	Generalised Gamma	Burr	Infinite exponentials	Exponential-reciprocal inverse Gaussian	Gamma-reciprocal inverse Gaussian
Exponential	-	−3.40	−3.92	−4.35	−6.15	−4.48	−18.35
		[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
Weibull		-	−0.39	−0.12	−0.17	−0.16	−0.16
			[0.35]	[0.45]	[0.43]	[0.43]	[0.43]
Generalized Gamma			-	6.08	5.03	7.54	−24.71
				[0.00]	[0.00]	[0.00]	[0.00]
Burr				-	−2.48	−3.22	−24.26
					[0.01]	[0.00]	[0.00]
Infinite exponentials					-	0.47	−23.74
						[0.32]	[0.00]
Exponential-reciprocal inverse Gaussian						-	−25.20
							[0.00]
Gamma-reciprocal inverse Gaussian							-

Note: The p-values are shown in brackets.

4. Conclusions

In this paper, we have proposed a baseline ACD model based on a mixture of gamma and reciprocal inverse Gaussian distributions to take into account the more complex unobserved heterogeneity arising from the variety of agents and trading conditions in financial markets.

The closed form solution obtained for the mixture distribution means that our proposed model is easy to fit. In this respect, the statistical measures used, the autocorrelation tests performed on standardised residuals and the Vuong closeness test conducted for non-nested models show that the gamma-reciprocal inverse Gaussian ACD model performs better than those without heterogeneity, such as the exponential, Weibull, and Burr models, but also with regard to other infinite exponential mixtures. Therefore, we conclude that our model characterises the behaviour of conditional durations reasonably well.

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Appendix

The integral representation of the Bessel function of the second kind is given by:

$$K_\nu(z) = \frac{z^\nu \sqrt{\pi}}{2^\nu \Gamma\left(\nu + 1/2\right)} \int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2}\, dt,$$

which, after changing the variable $t = 1/y$, can be rewritten as

$$K_\nu(z) = \frac{z^\nu \sqrt{\pi}}{2^\nu \Gamma\left(\nu + 1/2\right)} \int_0^1 \frac{(1-y^2)^{\nu-1/2}}{y^{2\nu+1}} \exp\left(-\frac{z}{y}\right)\, dy.$$

The latter integral can now be approximated by using the composite trapezoidal rule:

$$\int_0^1 \frac{(1-y^2)^{\nu-1/2}}{y^{2\nu+1}} \exp\left(-\frac{z}{y}\right) dy \approx \frac{\overline{y}-\underline{y}}{2M}\left[\chi^2(\underline{y}) + \chi^2(\overline{y}) + 2\sum_{j=0}^{M} \chi^2(\underline{y}+jh)\right],$$

where $\underline{y}$ and $\overline{y}$ are appropriate values that are near to 0 and 1, respectively,

$$\chi^2(y) = \frac{(1-y^2)^{\nu-1/2}}{y^{2\nu+1}} \exp\left(-\frac{z}{y}\right)$$

M is the number of grid points considered and $h = 1/M$.