# Some advances in open problems of isogeometric analysis

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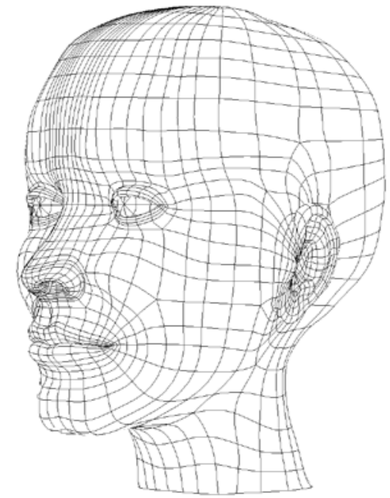
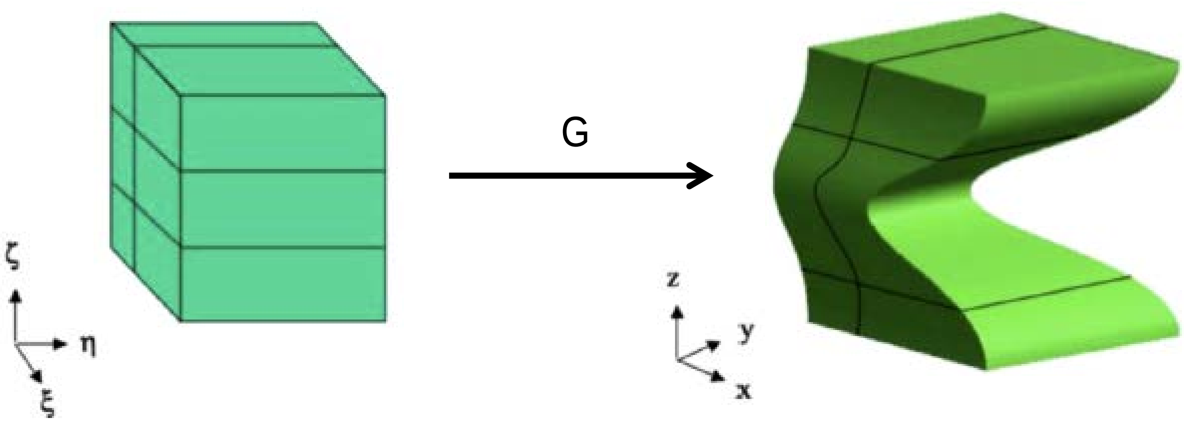
# Isogeometric analysis: integrating FEA and CAD

**Isogeometric analysis** (IGA) has arisen as an attempt to unify the field of CAD and classical finite element method



**The main idea:** using for analysis the same functions that are used in CAD representation of the computational domain, preserving thus the "exact" geometry

*Global parameterization of the physical domain*



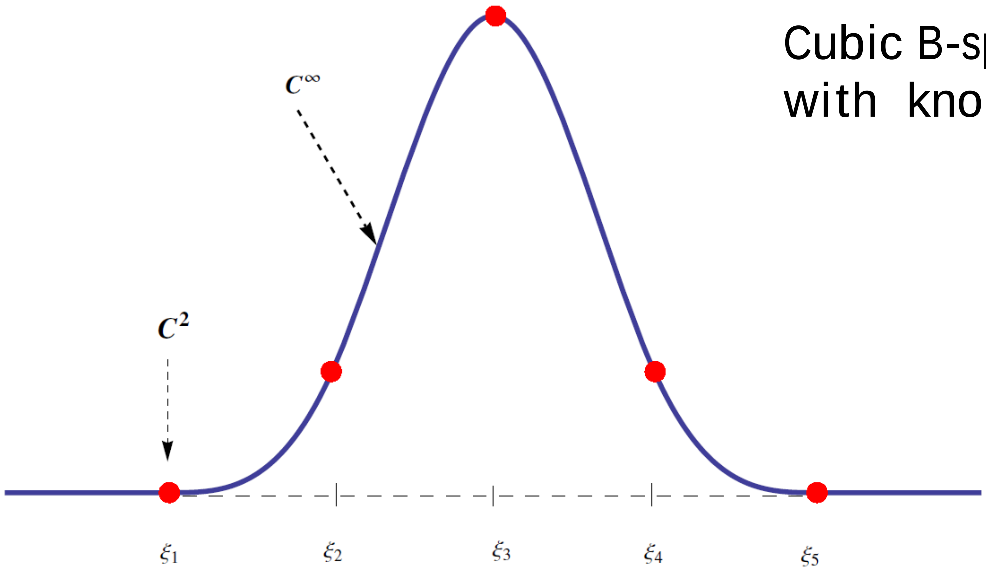

Basis functions: cubic splines

Higher smoothness:
$C^2$ instead of $C^0$ of FEM

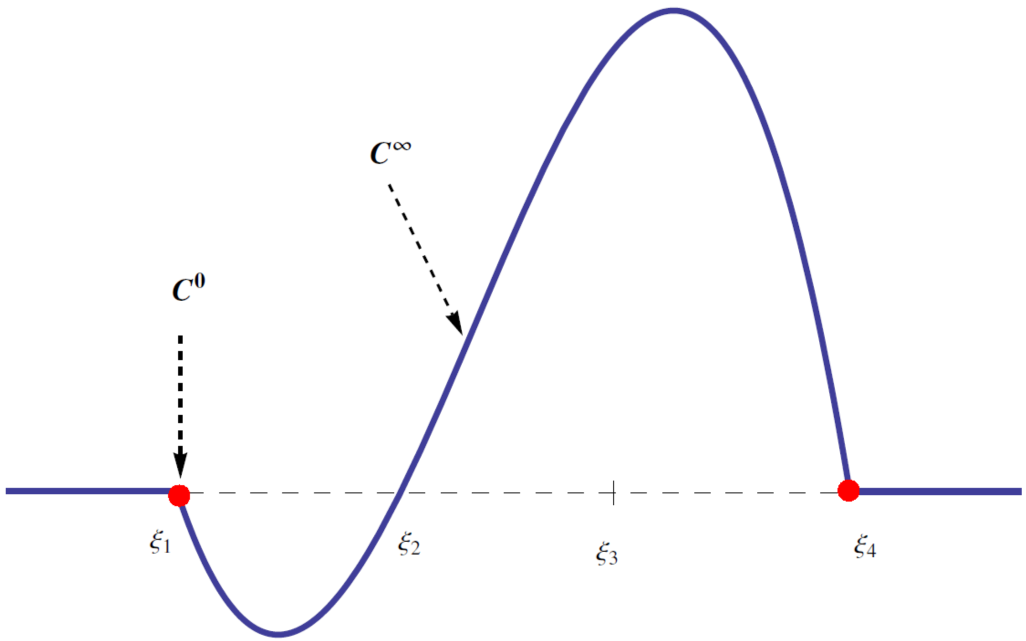
# Isogeometric analysis: integrating FEA and CAD

## B-spline basis functions

Cubic B-spline
with knot vector $\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$




Langrange basis function of order 3



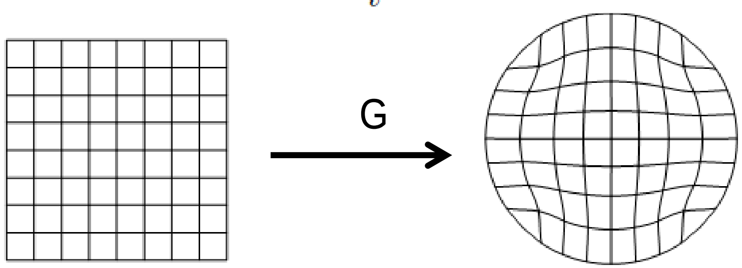
# Isogeometric analysis: integrating FEA and CAD

## Isoparametric approach in IGA

(T. Hughes et al. 2005) uses **only one global geometry function**

$$G : \Omega_0 = [0,1]^2 \to \Omega$$

$G(\xi) = \sum_i R_i(\xi)\mathbf{d}_i$ with tensor-product NURBS basis functions $R_i$

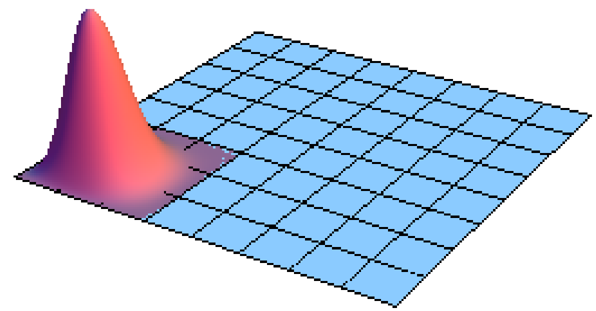



$\xi \in \Omega_0$

$(\xi = (\xi, \eta))$

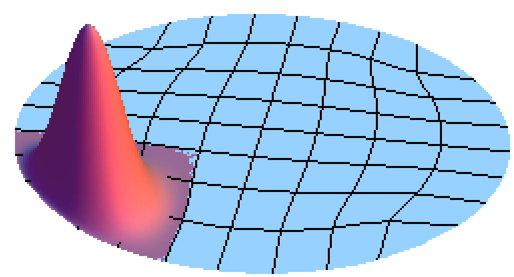
$x(\xi) \in \Omega$

$R_i$



The basis functions of $V_h \subset V$ are $\phi_i = R_i \circ G^{-1}$

$\phi_i$

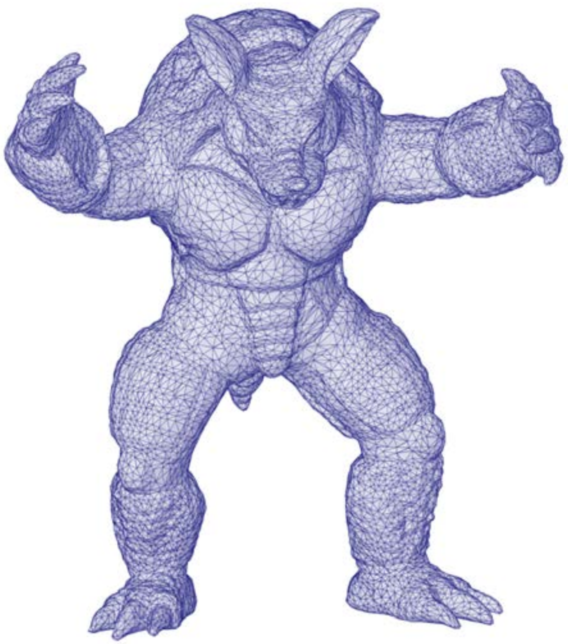


# Isogeometric analysis: integrating FEA and CAD

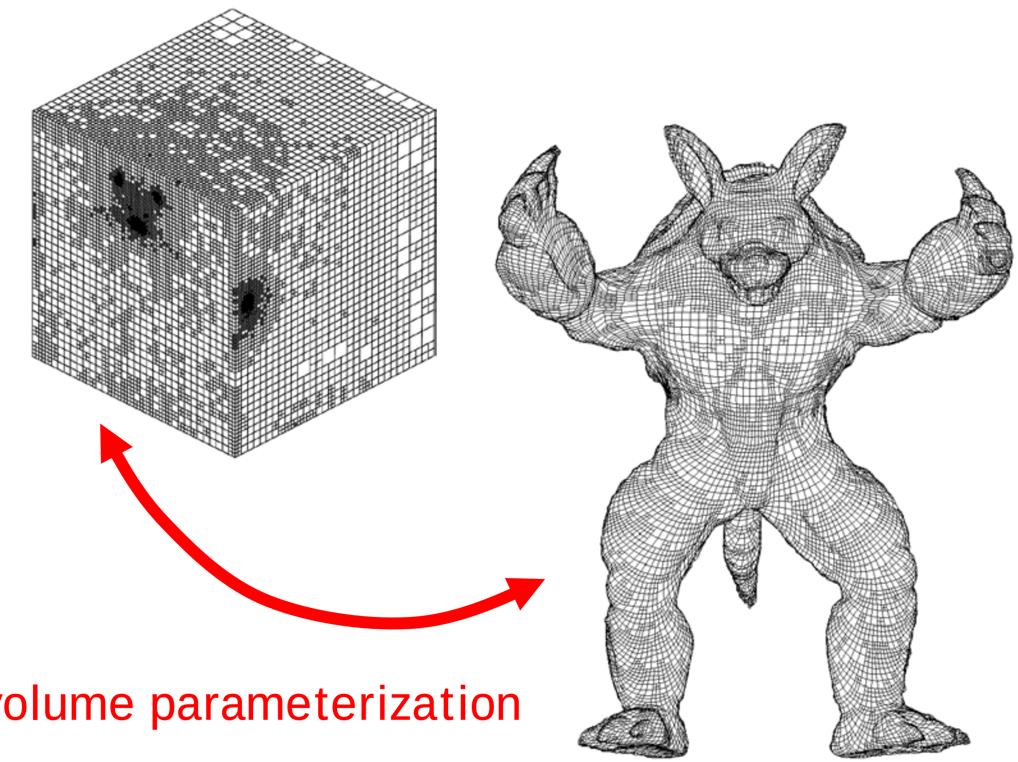
## Open problems

- **Volumetric parameterization of computational domain from its surface representation.**

CAD provides only surface representation of the geometry. For application of IGA it is necessary to have robust and effective method to obtain analysis-suitable volume parameterization

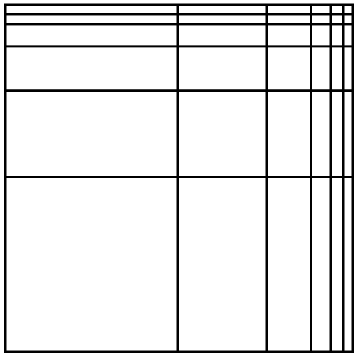


Input surface

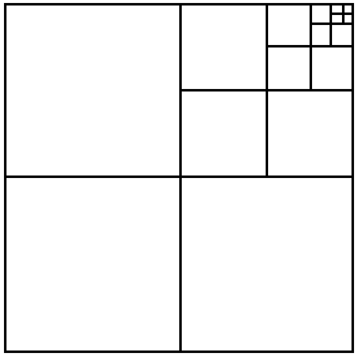


## ❑ Local refinement.

Tensor product structure does not allow local enrichment of approximation space: knot insertion propagates thru the domain. A strategy for defining spline spaces over meshes with T-juntions (T-meshes) is needed.



global refinement



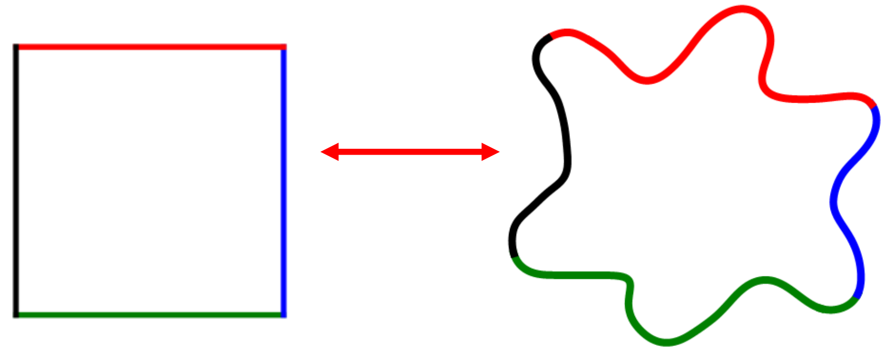
local refinement

# Our results

- A method for spline parameterization of 2D and 3D geometries
- A new strategy for constructing cubic spline spaces over quadtree (2D) and octree (3D) T-meshes

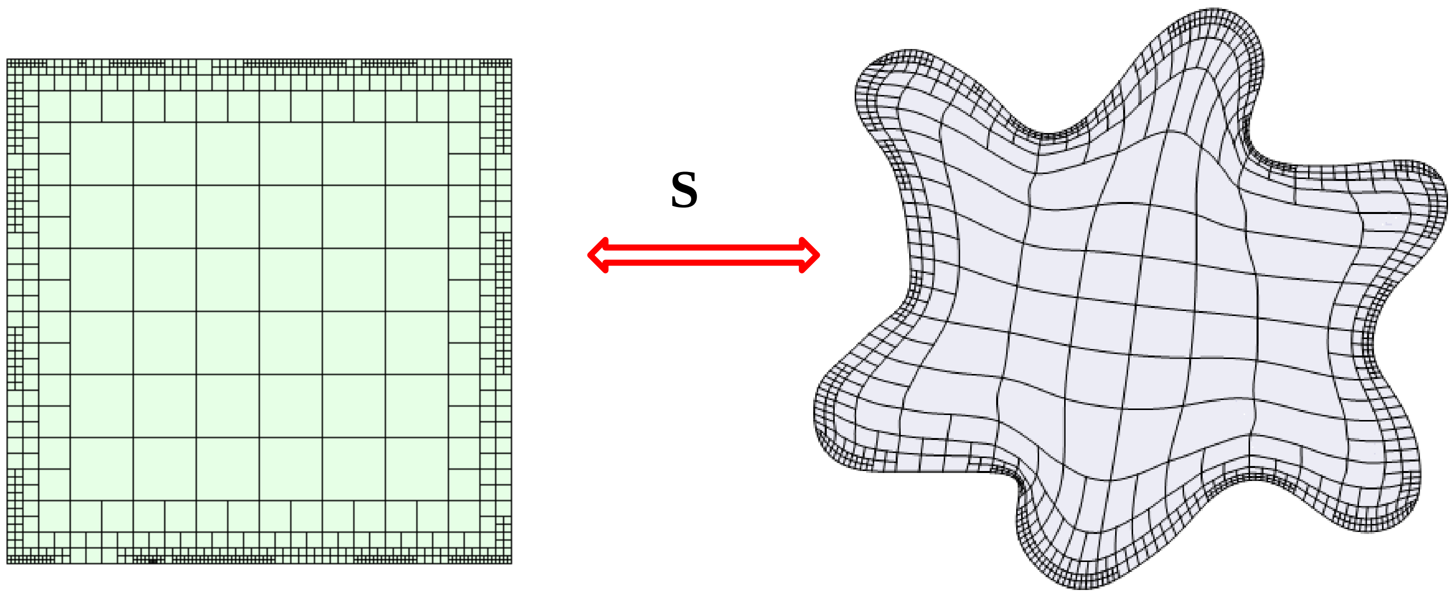
# Our results:
# spline parameterization of 2D geometries

**Goal**: construct a global transformation from parametric to physical domain from boundary representation of the geometry



**Good quality parameterization:**

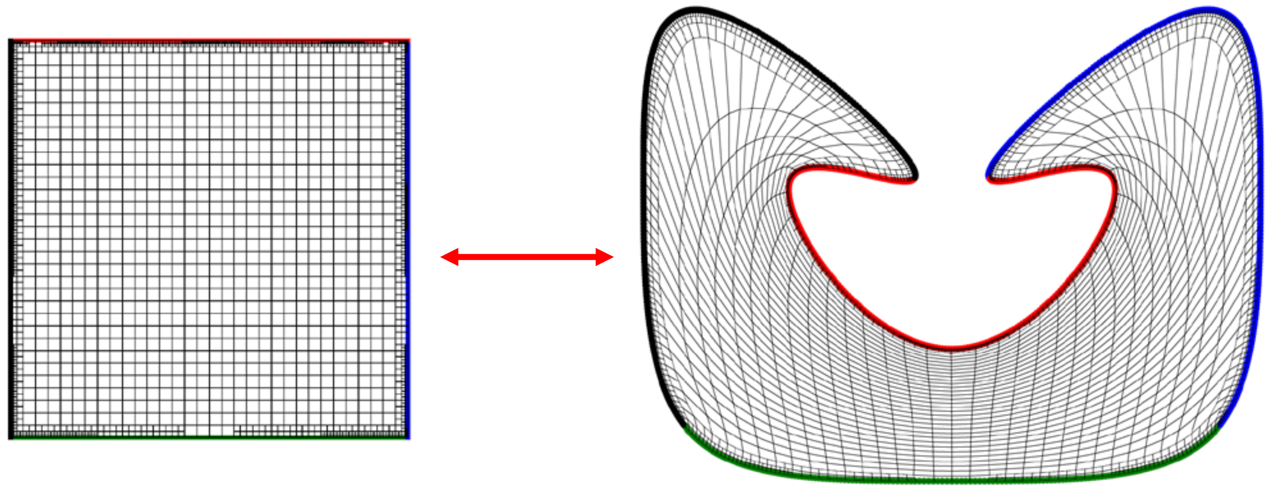
- Strictly positive Jacobian
- Good orthogonality and uniformity of isoparametric curves



# Spline parameterization of 2D geometries

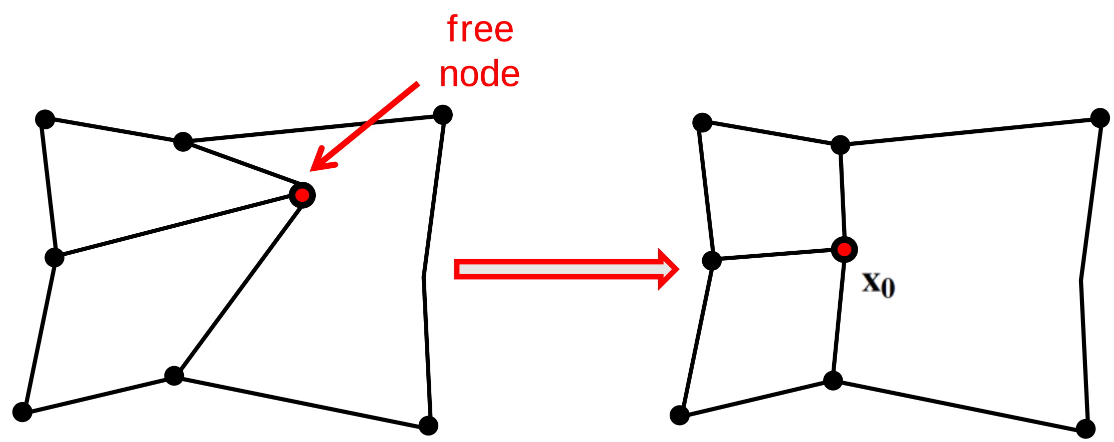
## The key: untangling and smoothing procedure for T-mesh

Parametric T-mesh is deformed isomorphically into the physical T-mesh



**Local optimization:** determine a new position of the free node to improve mesh quality

Minimize the objective function $K(\mathbf{x})$ to find the optimal position $\mathbf{x}_0$ of the free node




*local mesh*    *optimized local mesh*

# Simultaneous Untangling and Smoothing of T-meshes

Triangle decomposition of the T-mesh cells

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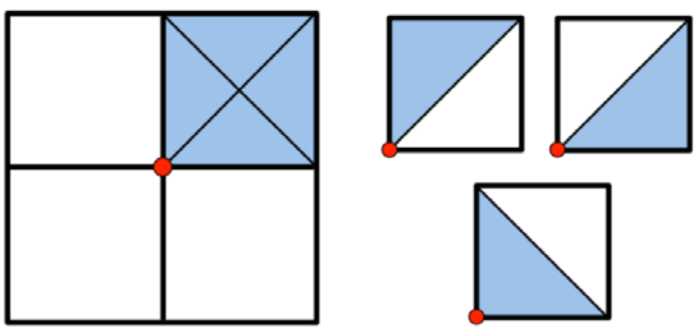
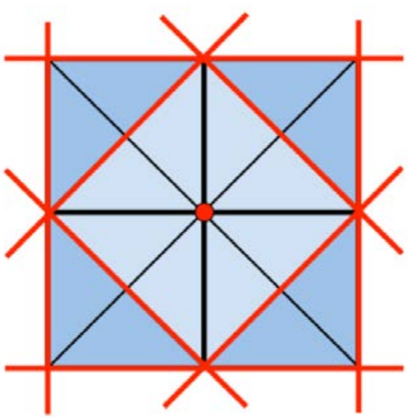
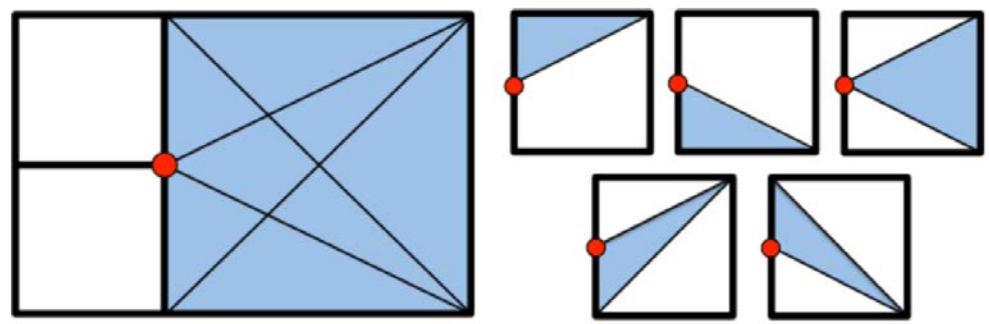
## Case 1: Free node is a regular node

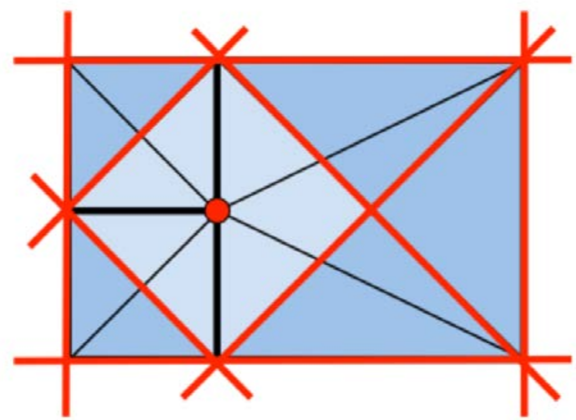




*Barriers and feasible region for a regular node*

## Case 2: Free node is a hanging node





*Barriers and feasible region for a hanging node*

Local submesh is decomposed in triangles.

Shape quality measure for a triangle: ***mean ratio***

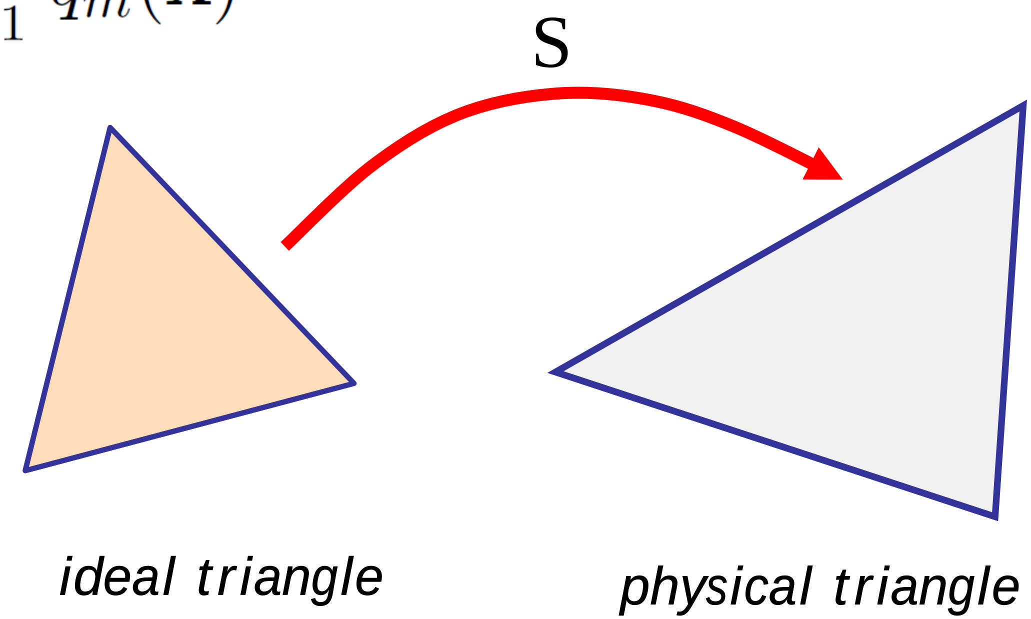
$$q = \frac{2\det(S)}{\|S\|^2}$$

Objective function:

$$K(\mathbf{x}) = \sum_{m=1}^{M} \frac{1}{q_m(\mathbf{x})}$$

$$K(\mathbf{x}) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2\det(S_m)}$$

**M**: number of elements of the local submesh



# Simultaneous Untangling and Smoothing of T-meshes

T-mesh transformation along the SUS process: Video

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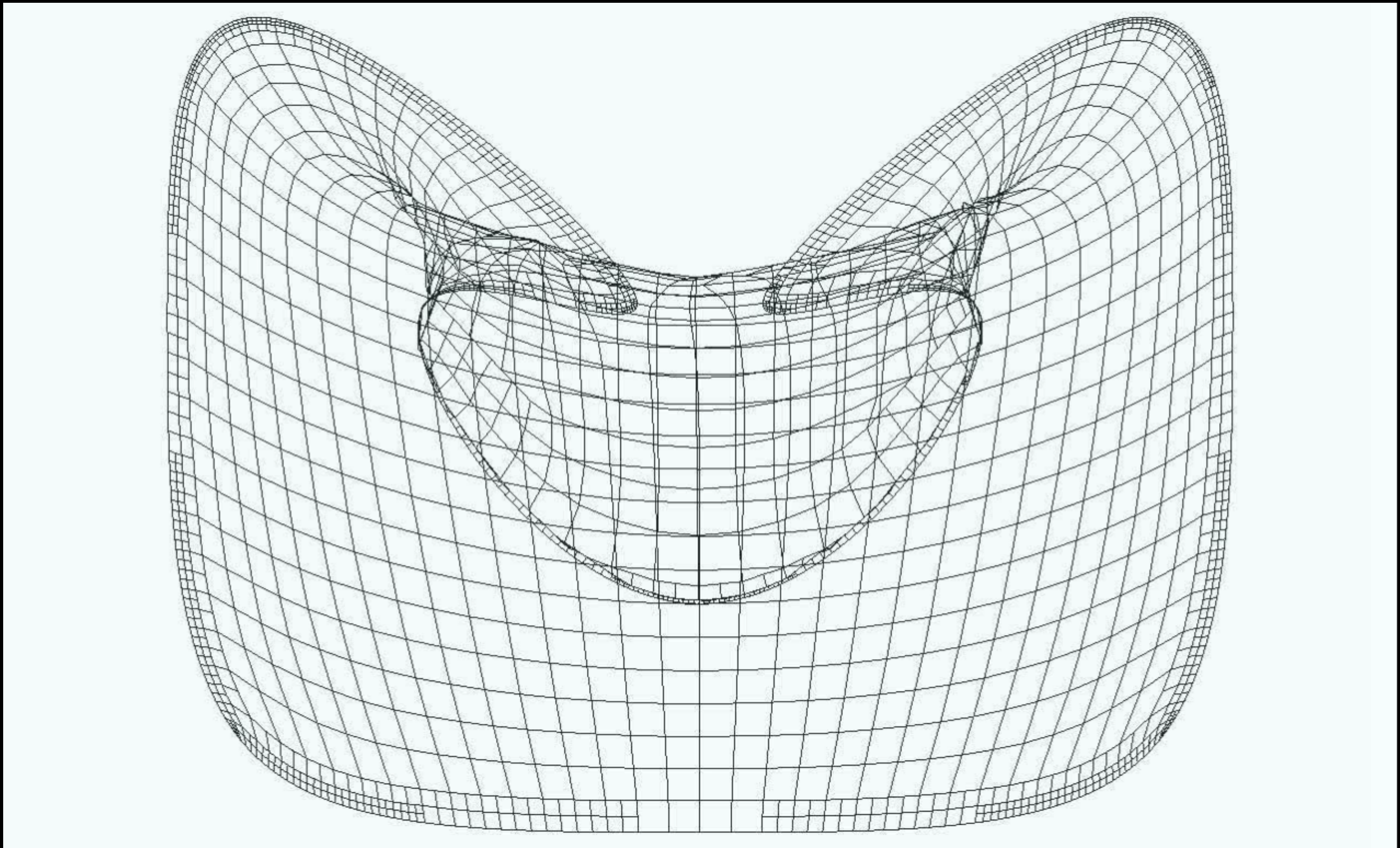
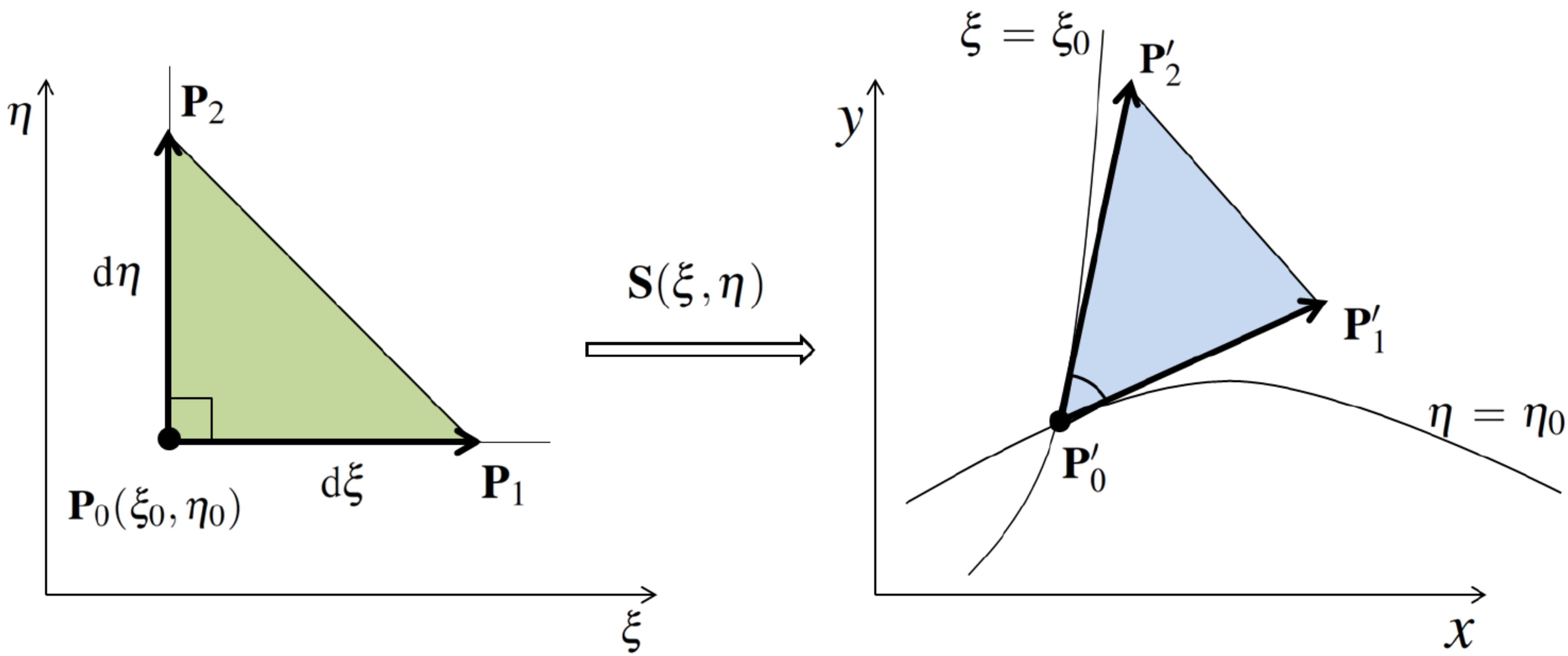


# Mean Ratio Jacobian $J_r(\xi)$ of Parametric Transformation

A quality metric of the mapping at any point $P_0$

$$-1 \leq J_r(\boldsymbol{\xi}) = \frac{2\det(\boldsymbol{J})}{\|\boldsymbol{J}\|^2} \leq 1$$

where $\boldsymbol{J}$ is the jacobian matrix of the spline mapping $\mathbf{S}$



# Spline parameterization of 2D geometries

Puzzle piece (Mean Ratio Jacobian)

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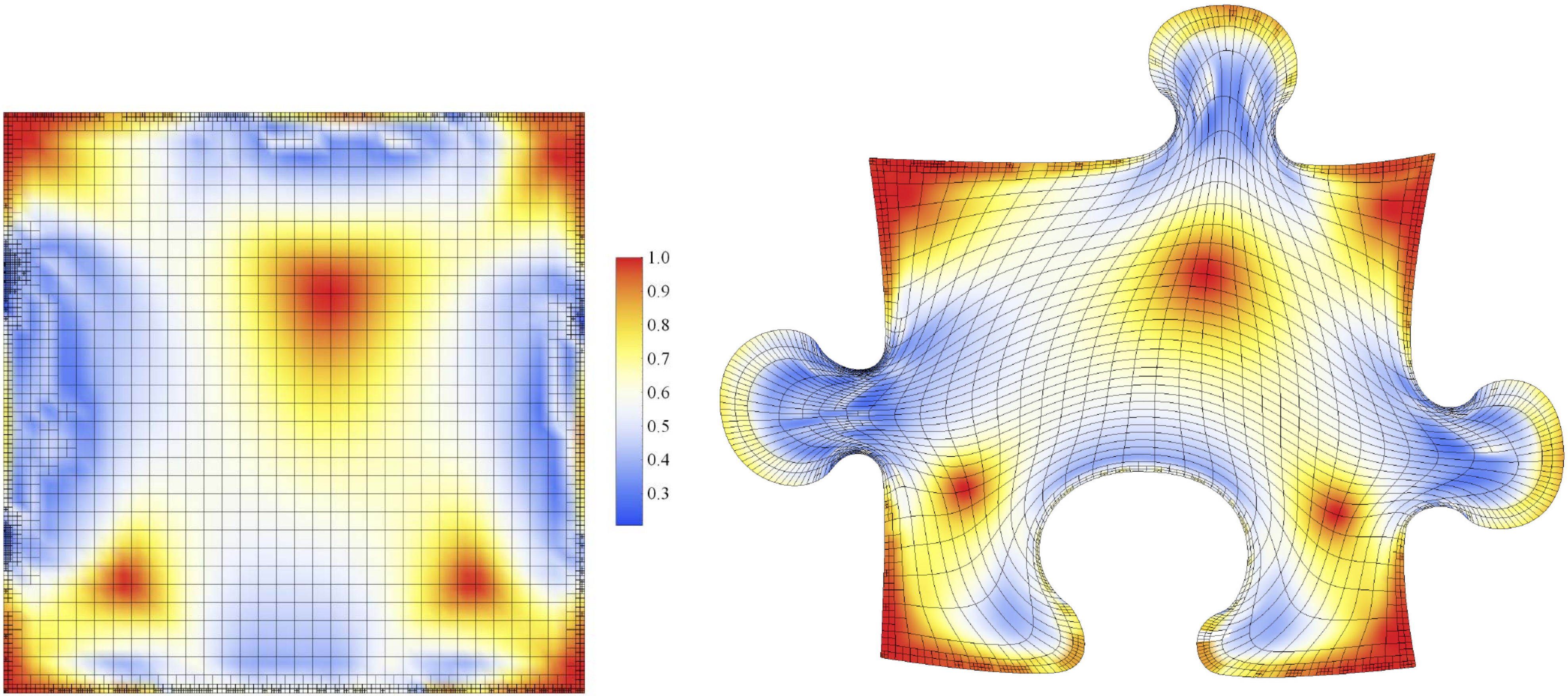
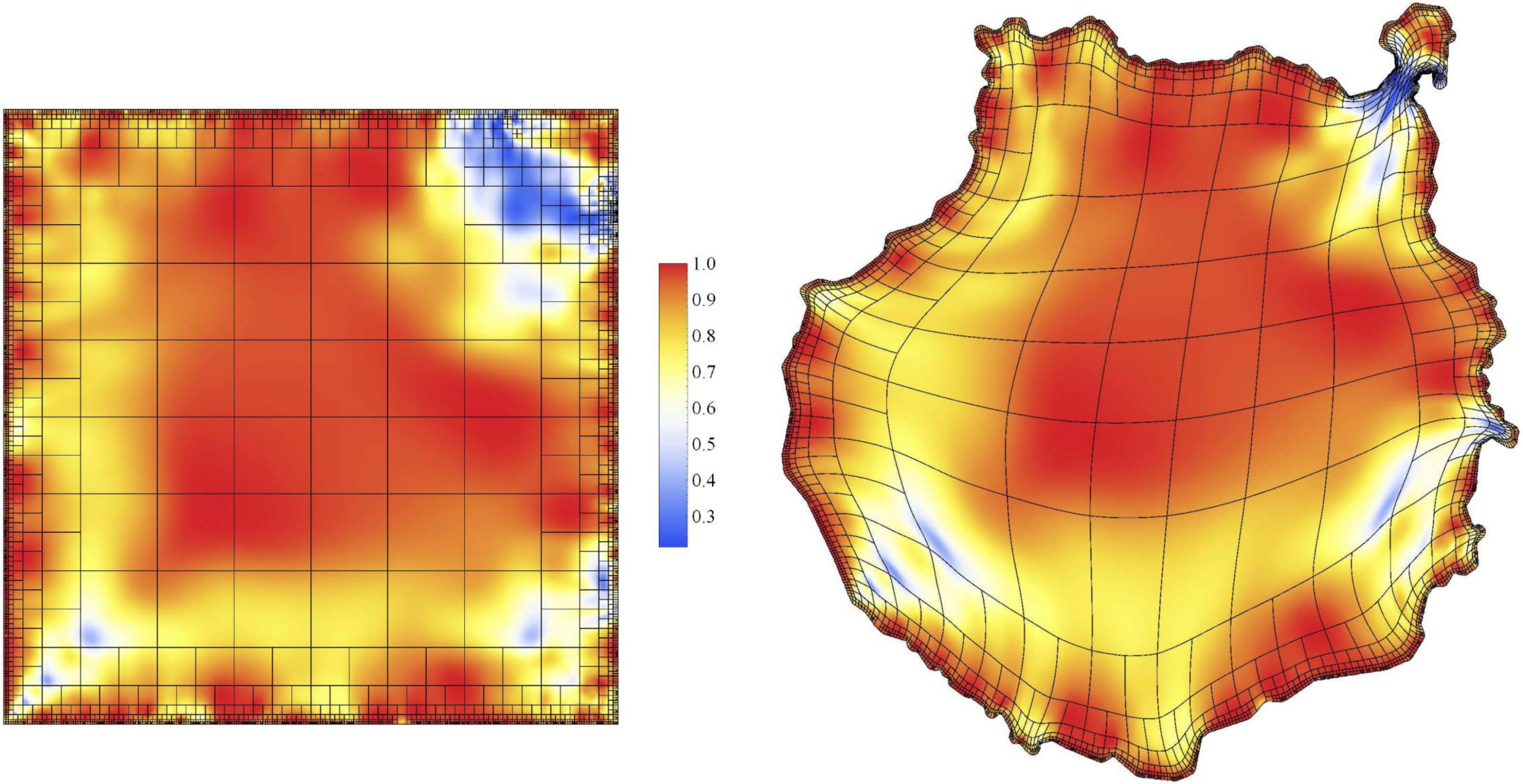


# Good quality parameterization for application of IGA

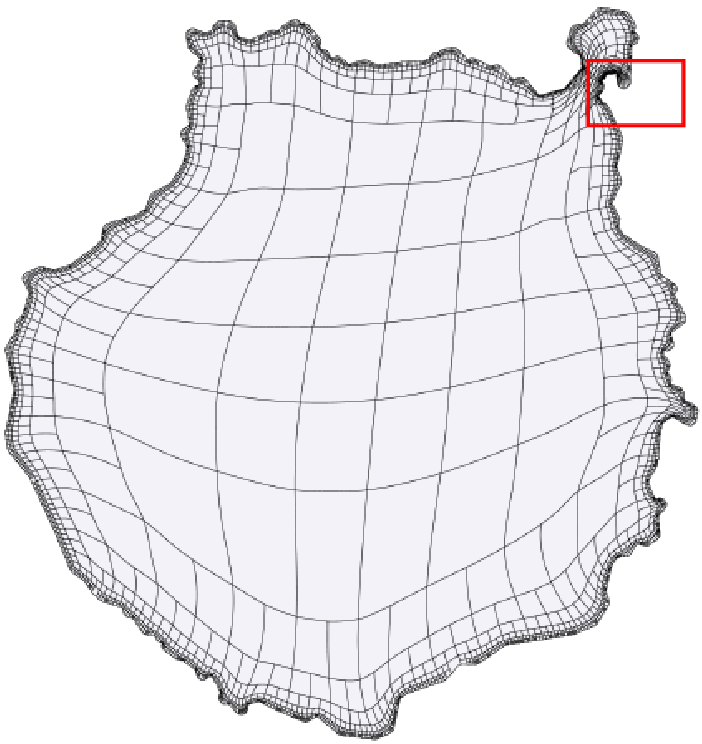
Mean ratio Jacobian as a quality metric



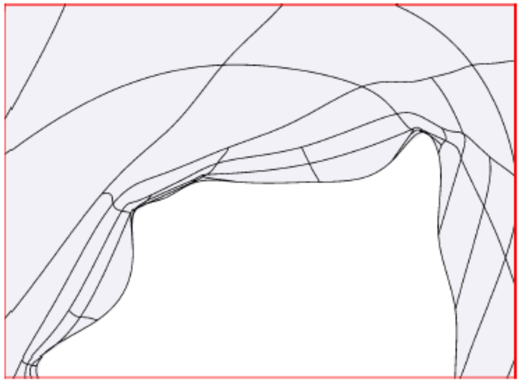
# Spline parameterization of 2D geometries

## Adaptive refinement to improve parameterization quality

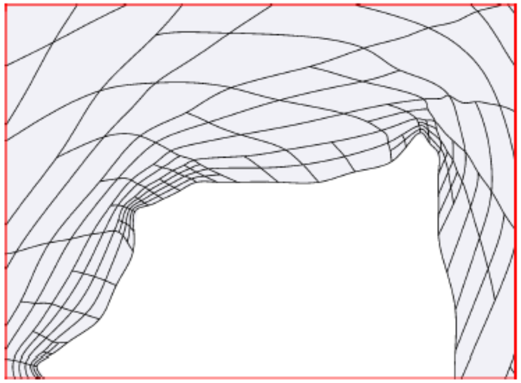
Low quality zone is refined and optimized again



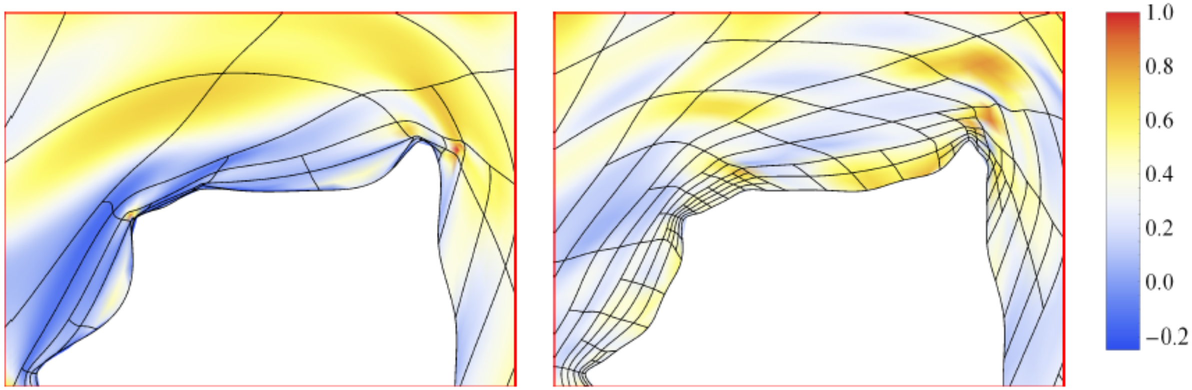
*Isla de Gran Canaria*



*inicial T-spline*



*improved T-spline*




*Mean ratio Jacobian*

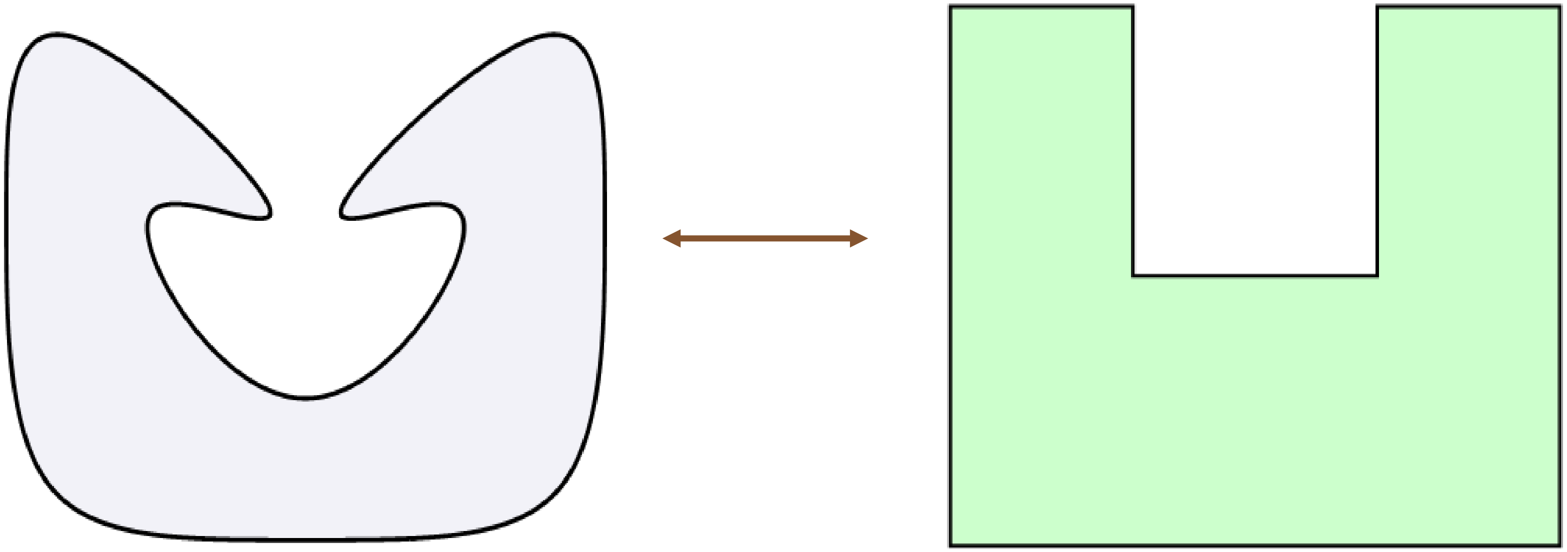
*Mean ratio Jacobian*

# Spline parameterization of 2D geometries

Limitations and future research

- **What can we parameterize with a square? Something similar to a square and a little bit more.**

We need more complex polycube-type parametric domain that fits better the geometry
[ Li (2007), Liu (2014)]



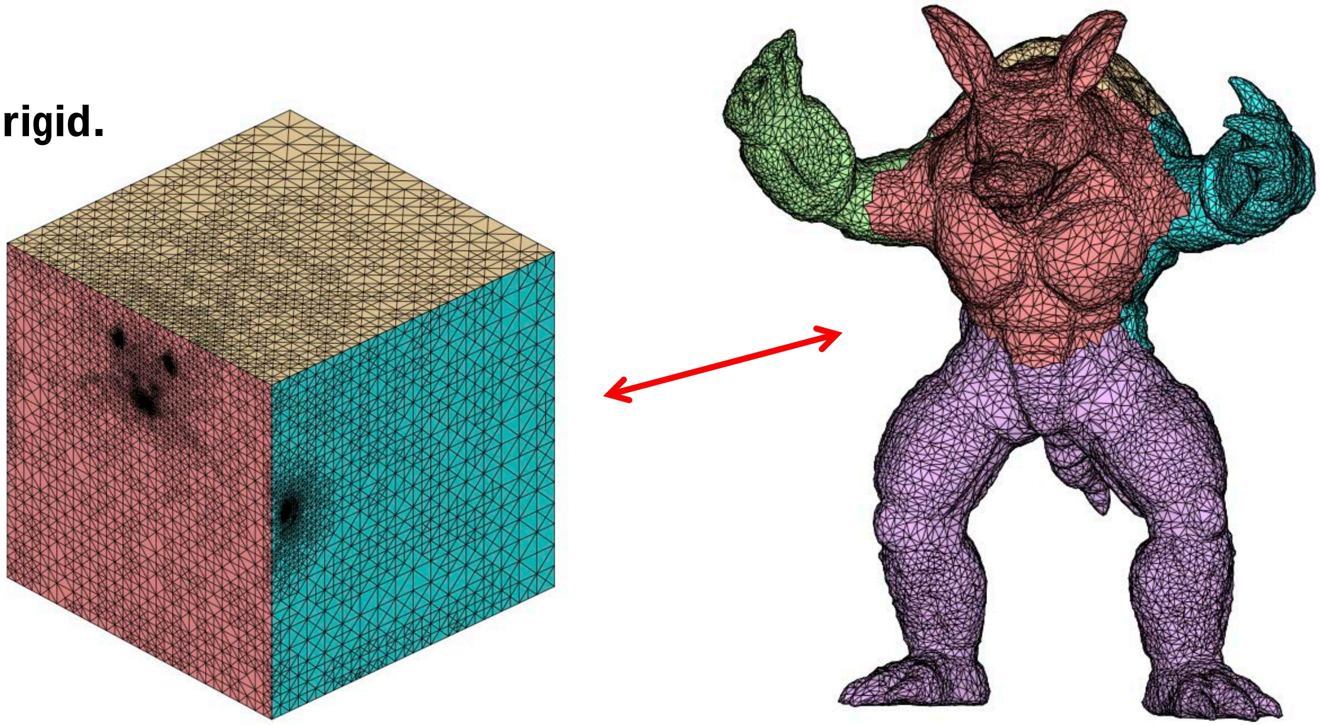
# Spline parameterization of 2D and 3D geometries

## Limitations and future research

**All the above mentioned problems for 2D domain are aggravated considerably for 3D object:**

- An appropiate selection of the edges may not exist
- More complex untangling and optimization procedure, the number of options increases a lot (O. Ushakova: Nondegeneracy tests for hexahedral cells, CMAME 2011)
- Jacobian close do cero along the edges of the object

**Tensor product structure is too rigid.**



# Our results:
# Polynomial spline spaces over hierarchical T-meshes

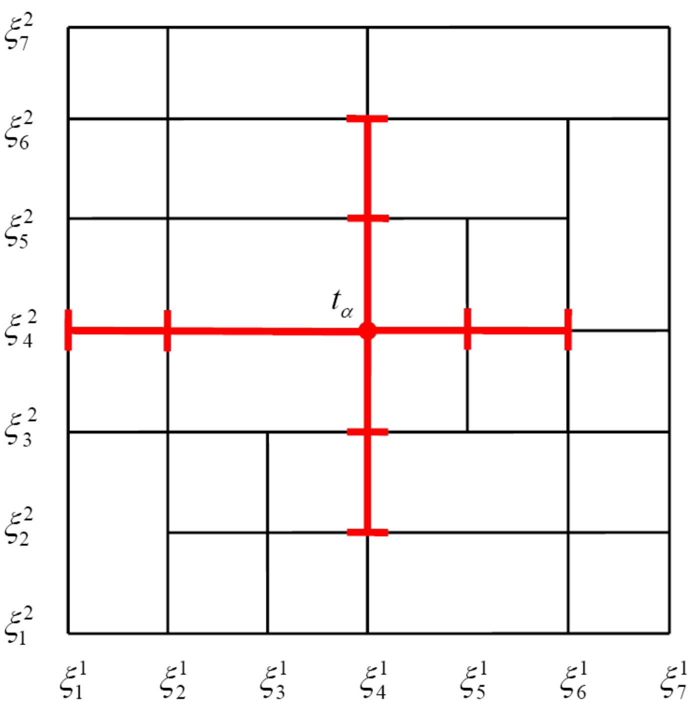
➢ A new strategy for constructing cubic spline space over arbitrary quadtree (2D) and octree (3D) T-meshes

- For a given T-mesh, it allows to obtain a set of cubic spline functions that span a space with nice properties: $C^2$ continuous, nested spaces, linear independence
- Simple rules for inferring local knot vectors to define spline blending functions
- Straightforward implementation in 2D and 3D

# Spline spaces over T-meshes

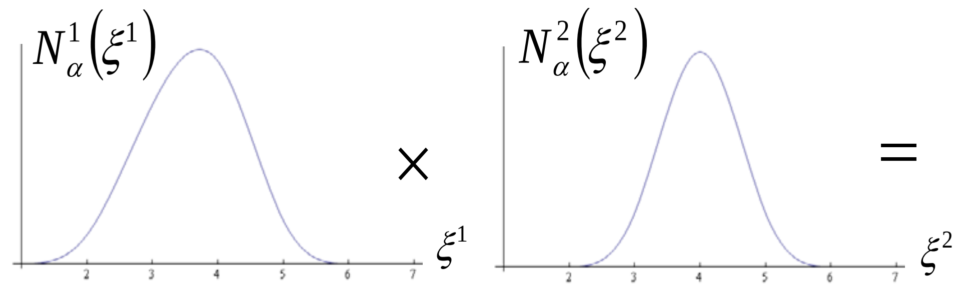
T-splines, Sederberg (2003)

***T-mesh and anchor $t_a$***



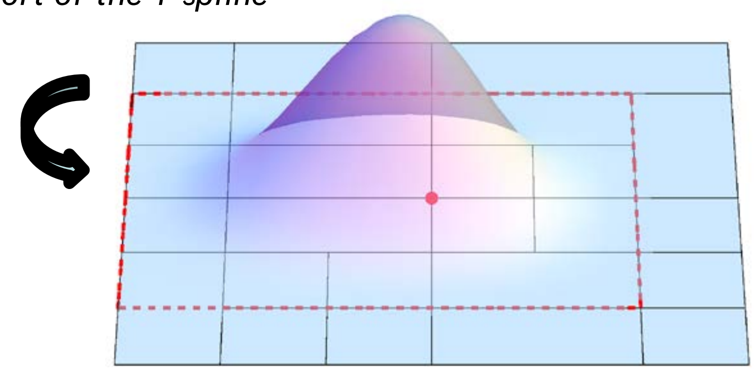

***Knots associated to anchor $t_a$:***

$$\Xi_\alpha^1 = \{\xi_1^1, \xi_2^1, \xi_4^1, \xi_5^1, \xi_6^1\} \qquad \Xi_\alpha^2 = \{\xi_2^2, \xi_3^2, \xi_4^2, \xi_5^2, \xi_6^2\}$$




$$B_\alpha(\xi^1, \xi^2) = N_\alpha^1(\xi^1) N_\alpha^2(\xi^2)$$

*support of the T-spline*



***Bivariate Cubic T-spline Basis Function***

*Normalized rational T-splines*

$$R_\alpha(\xi^1, \xi^2) = \frac{B_\alpha(\xi^1, \xi^2)}{\sum_{\beta \in A} B_\beta(\xi^1, \xi^2)}$$

# Spline spaces over T-meshes

T-splines, Sederberg (2003). Approximation capability of rational basis functions.

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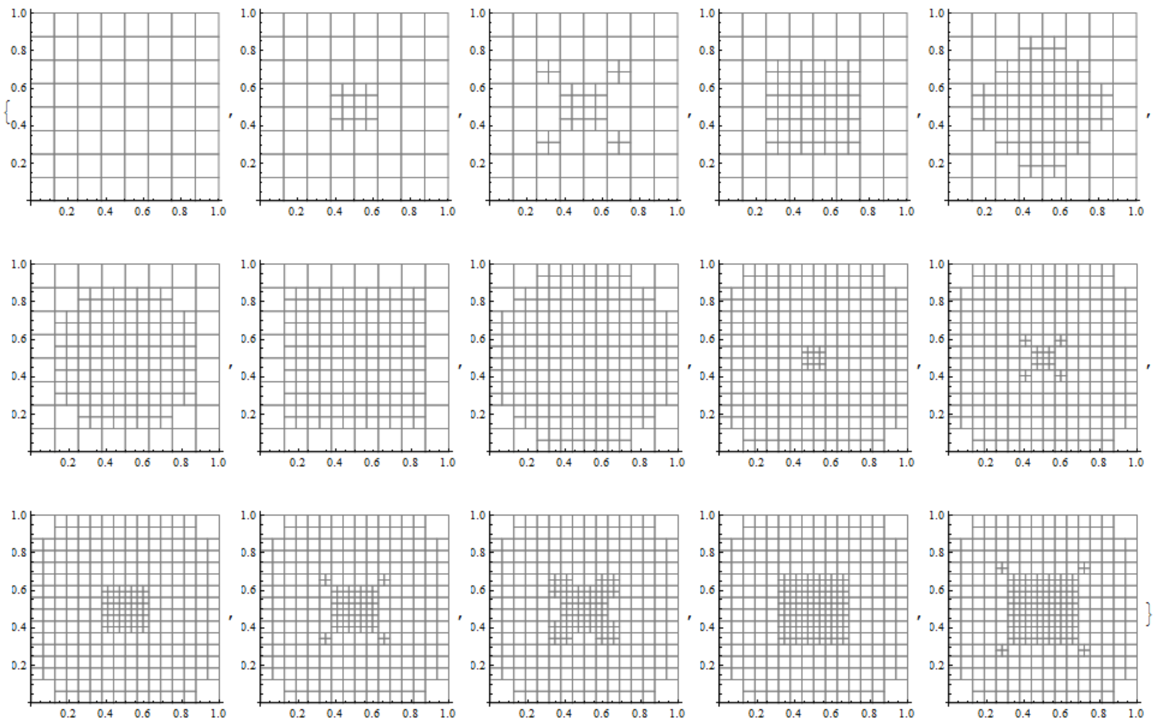
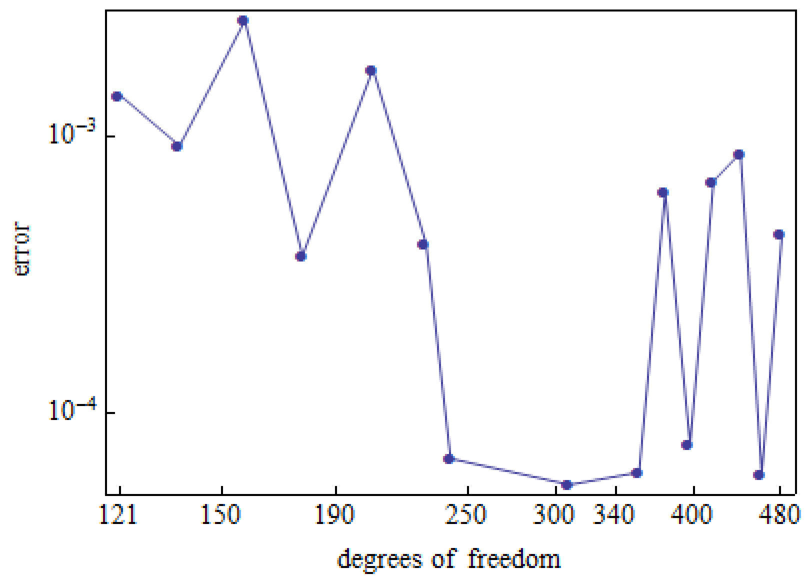
**Adaptive refinement** for the approximation of smooth gaussian-type function $u(x,y) = e^{-(x^2+y^2)}$ via interpolation with rational T-splines



Error convergence en $L^2$ norm



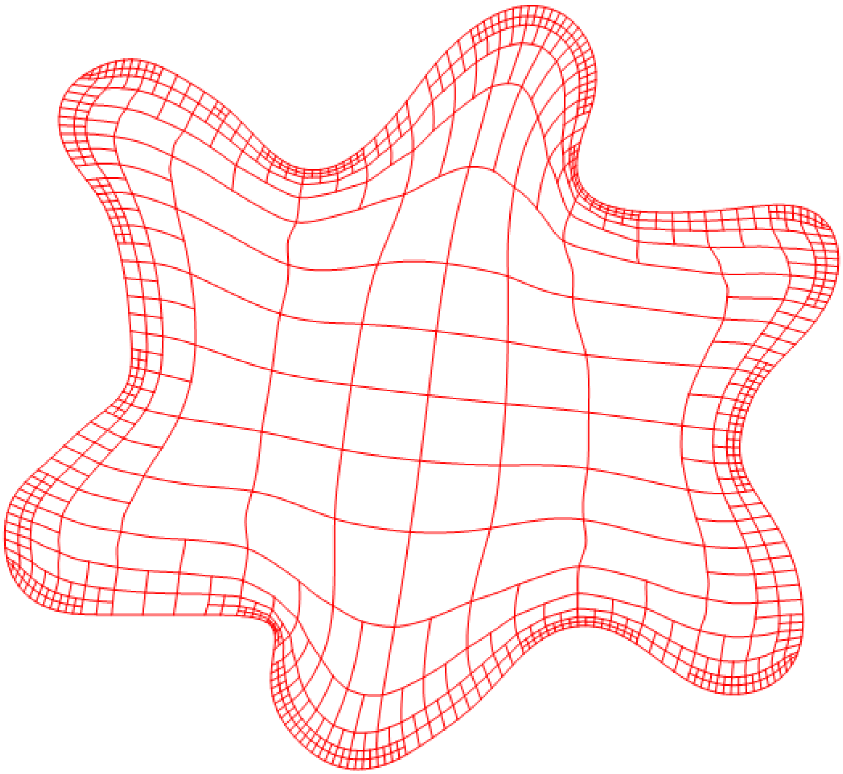


**It is time to drop the "R" from NURBS, EWC, 2014**

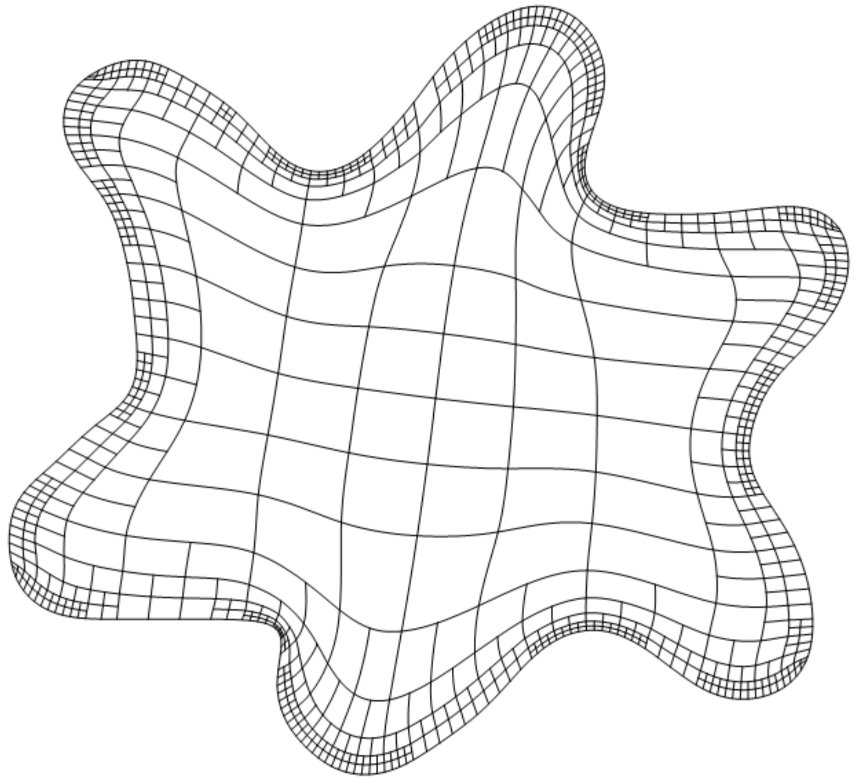
Les A. Piegl • Wayne Tiller • Khairan Rajab

# Spline spaces over T-meshes

T-splines, Sederberg (2003). Aproximation capability?



T-spline parameterization, Sederberg
(rational blending functions )

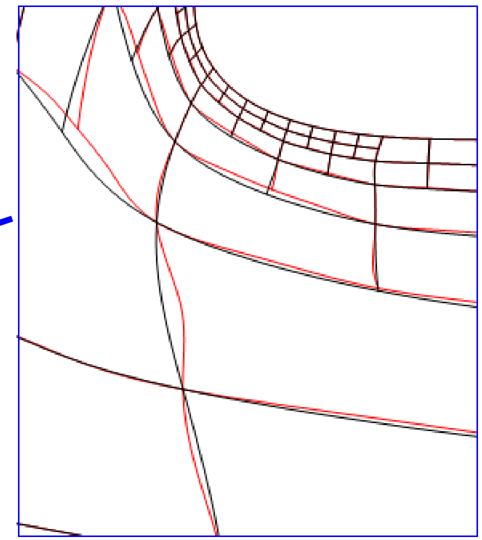


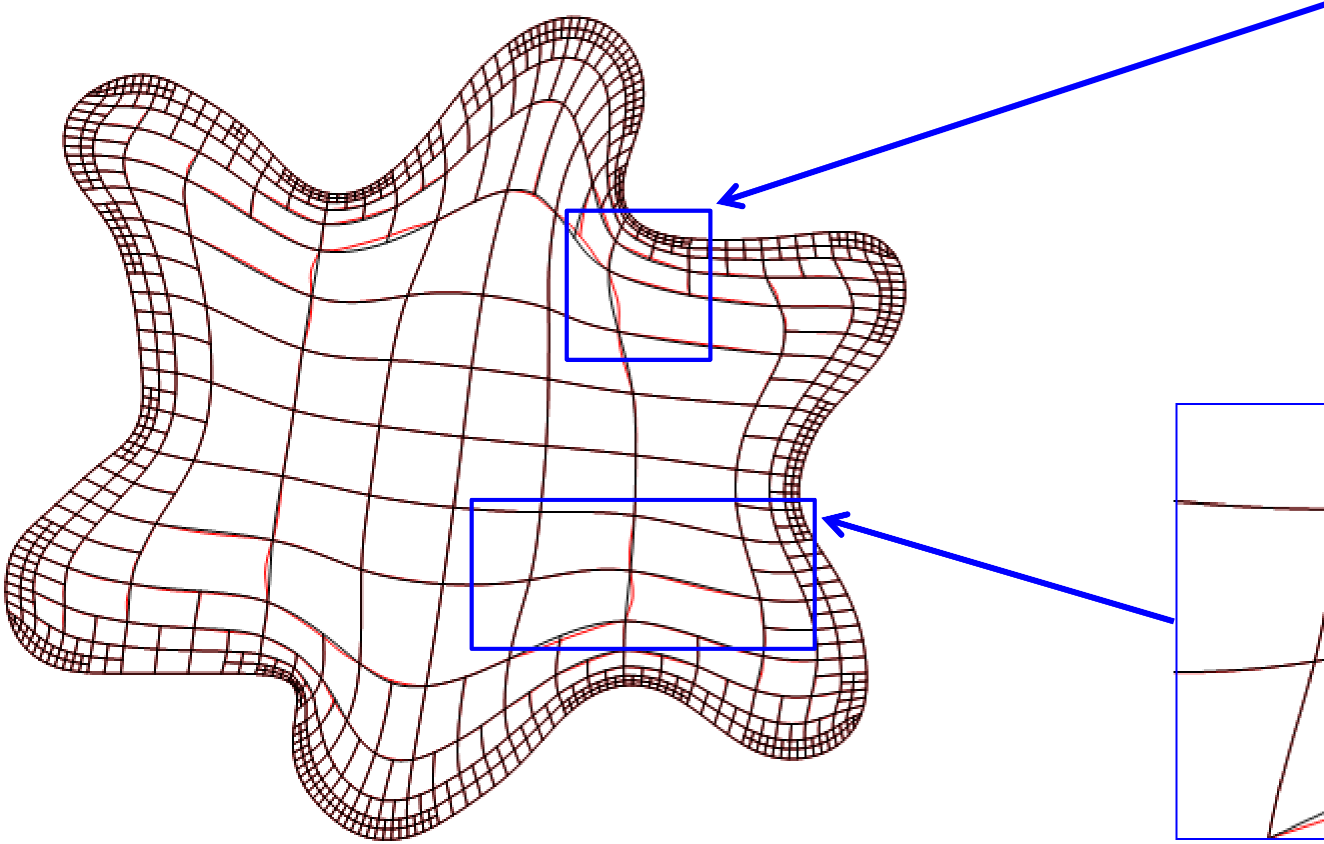
Spline parameterization with the new strategy
(polynomial blending functions)

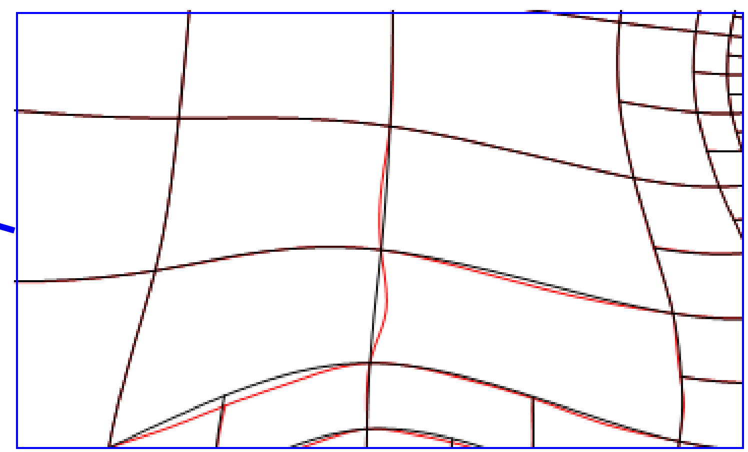
# Good quality parameterization for application of IGA

## Comparison of rational T-spline parameterization with the new strategy

rational T-spline parameterization  presents more oscillation





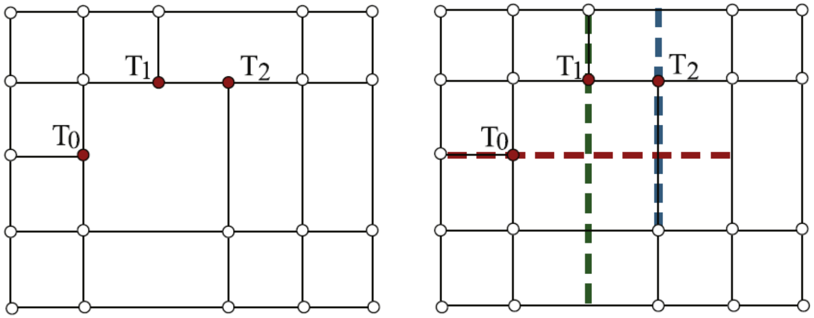


# Spline spaces over T-meshes

## Available strategies

**Analysis suitable T-spline**, Scott (2011)

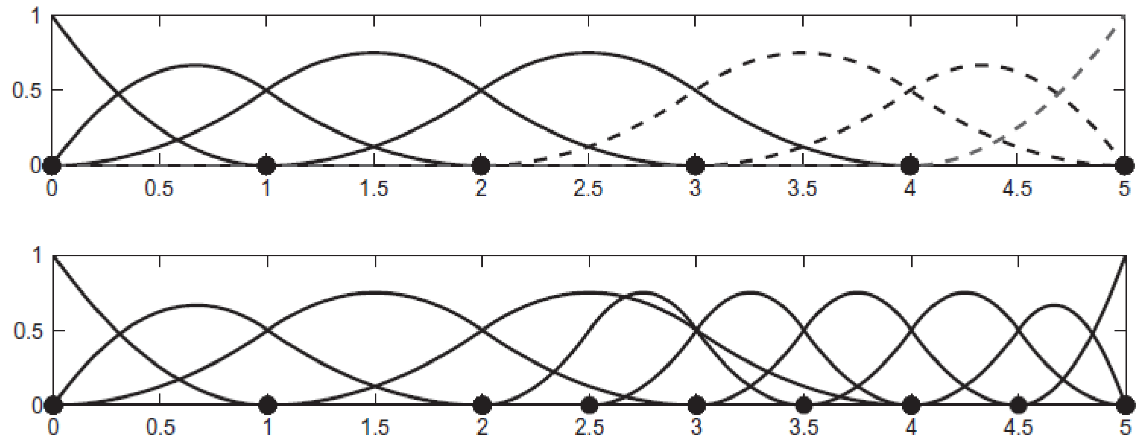
- nested spaces
- linear independence
- optimal support overlapping
- topoligical restriction for the T-mesh
  difficult to implement **=> Does not exist in 3D yet!**




(a) Extensions that cross.

**Hierachical refinement for IGA,** Vuong (2011), Schillinger (2012), Bornemann(2012)

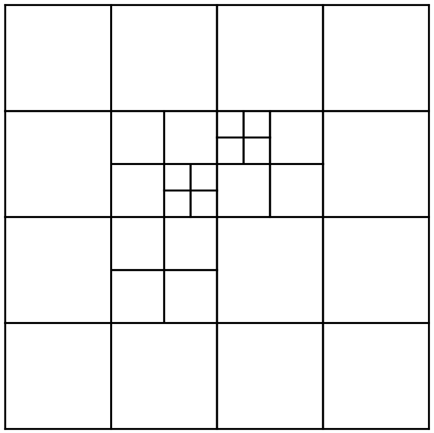
- nested spaces
- linear independence
- relatively easy implementation
- excesive support overlapping
- imposibility to define spline
  space over an arbitrary mesh



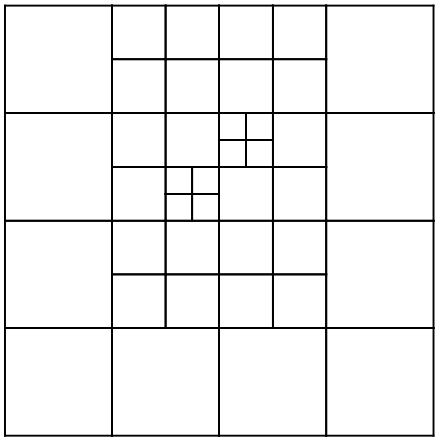
## Our objective

- simple and easy implementation
- nice properties: linear independence and nested spaces

We work with 0-balanced quadtree and octree T-meshes



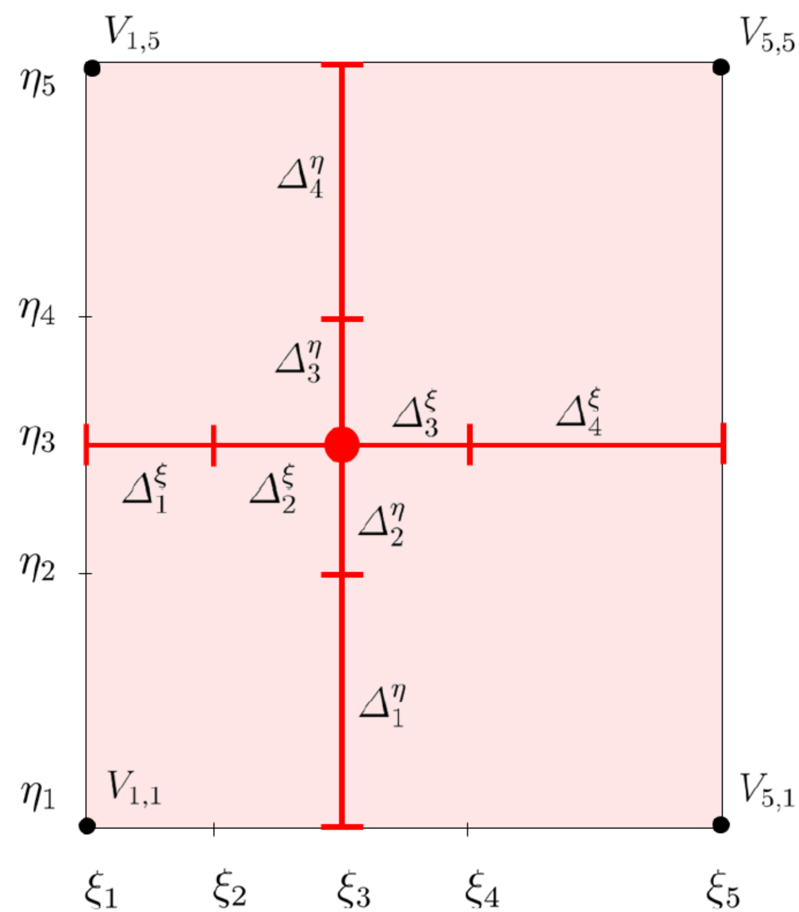
*unbalanced T-mesh*



*0-balanced T-mesh*

**Steps to define spline basis functions over a given T-mesh:**

- Mesh pretreatment: 0-balancing
- Inferring local knot vectors
- Modification of local knot vectors




**Condition 1:** Local knot vectors verify:

$$\Delta_1^j \geqslant \Delta_2^j = \Delta_3^j \leqslant \Delta_4^j$$

**Condition 2:** The frame of the function support should be situated over the mesh skeleton:

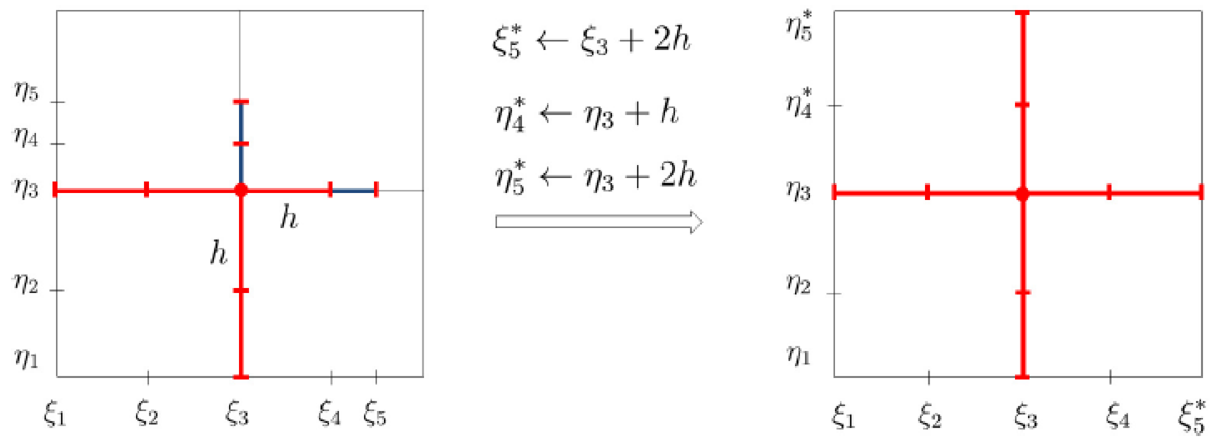
$$\mathrm{frm}(\mathrm{supp}\, N_\alpha) \in \mathrm{skt}(T)$$

# Spline spaces over T-meshes

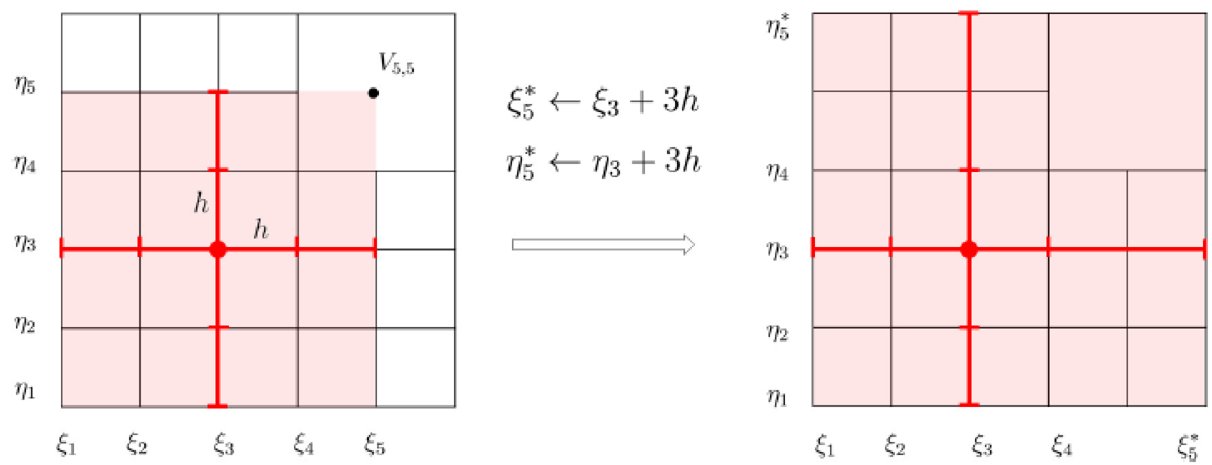
## Definition of polynomial spline functions. Extension rules in 2D

Support modification
to fulfill the imposed conditions

**Simple extension rules:**
**duplicate some knot intervals**




(a) Condition 1 is not satisfied because $\Delta_4^\xi < \Delta_3^\xi$ and $\Delta_3^\eta < \Delta_2^\eta$.



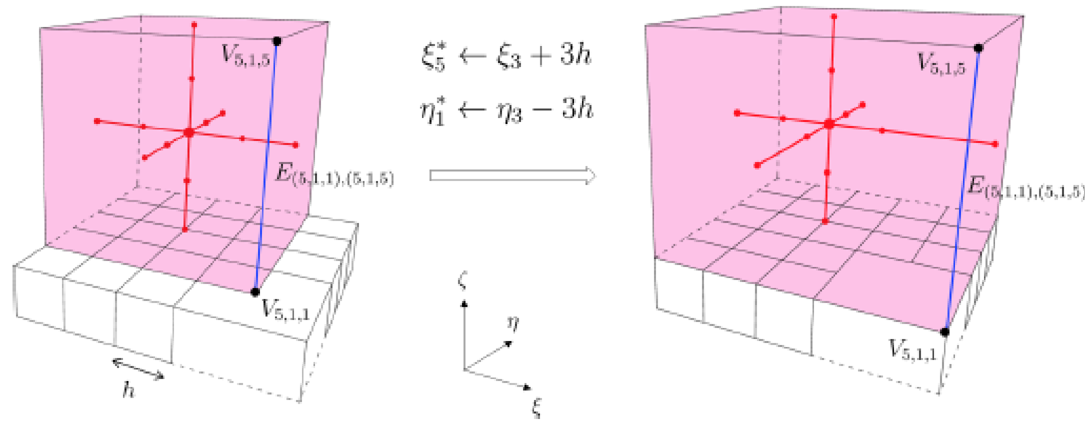

(b) Condition 2 is not satisfied because $V_{5,5} \notin skt(T)$.

# Spline spaces over T-meshes

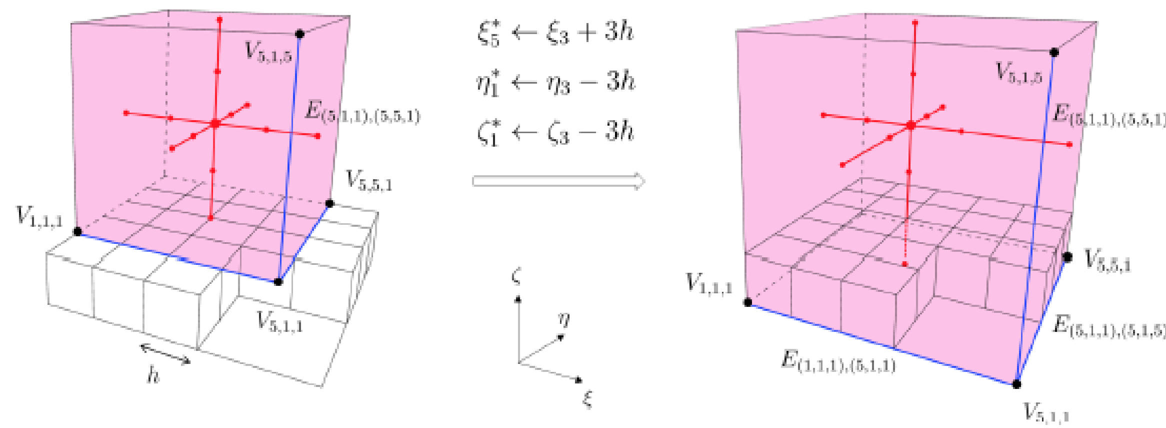
## Definition of polynomial spline functions. Extension rules in 3D

**Support modification**
**to fulfill the imposed conditions**

**Simple extension rules in 3D as well !**




(a) Only one edge violating Condition 2. The node $V_{5,1,1}$ is sited in the center of the face of size $2h$. The support is extended in two directions.




(b) Three edges violating Condition 2. The node $V_{5,1,1}$ is sited in the center of the cell of size $2h$. The support is extended in three directions.

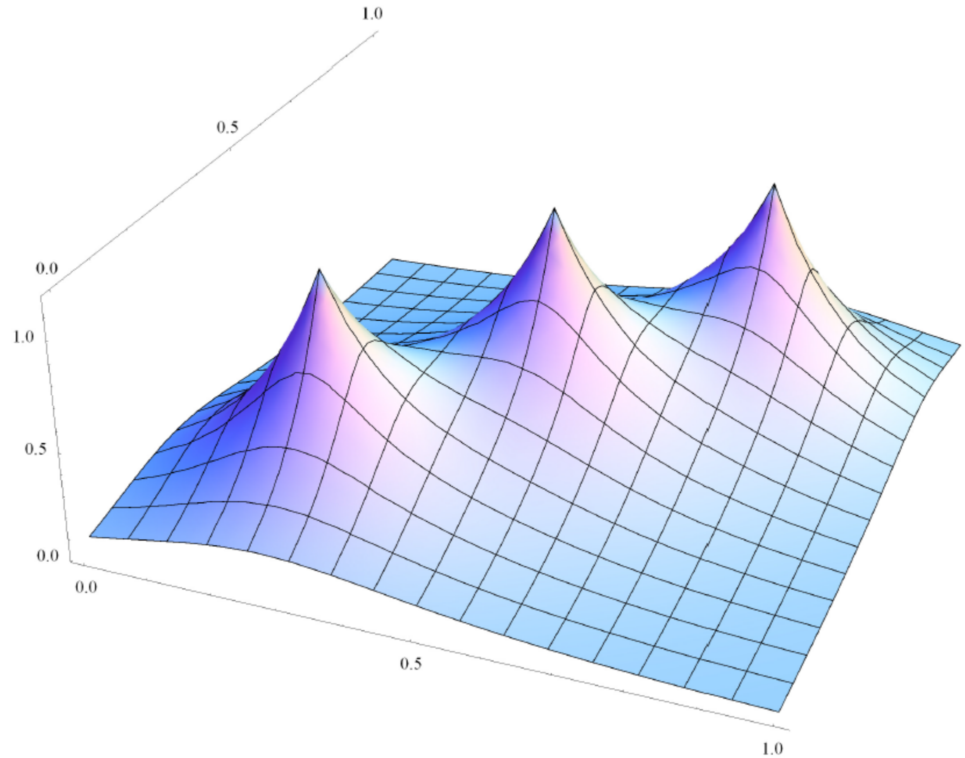
# Isogeometric analysis on 2D domain

## Poisson problem with Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

Analytical solution:

$$\begin{aligned} u(x,y) = \exp\left(-7\sqrt{(x-0.5)^2+(y-0.5)^2}\right) + \\ + \exp\left(-7\sqrt{(x-0.25)^2+(y-0.25)^2}\right) + \\ + \exp\left(-7\sqrt{(x-0.75)^2+(y-0.75)^2}\right). \end{aligned}$$

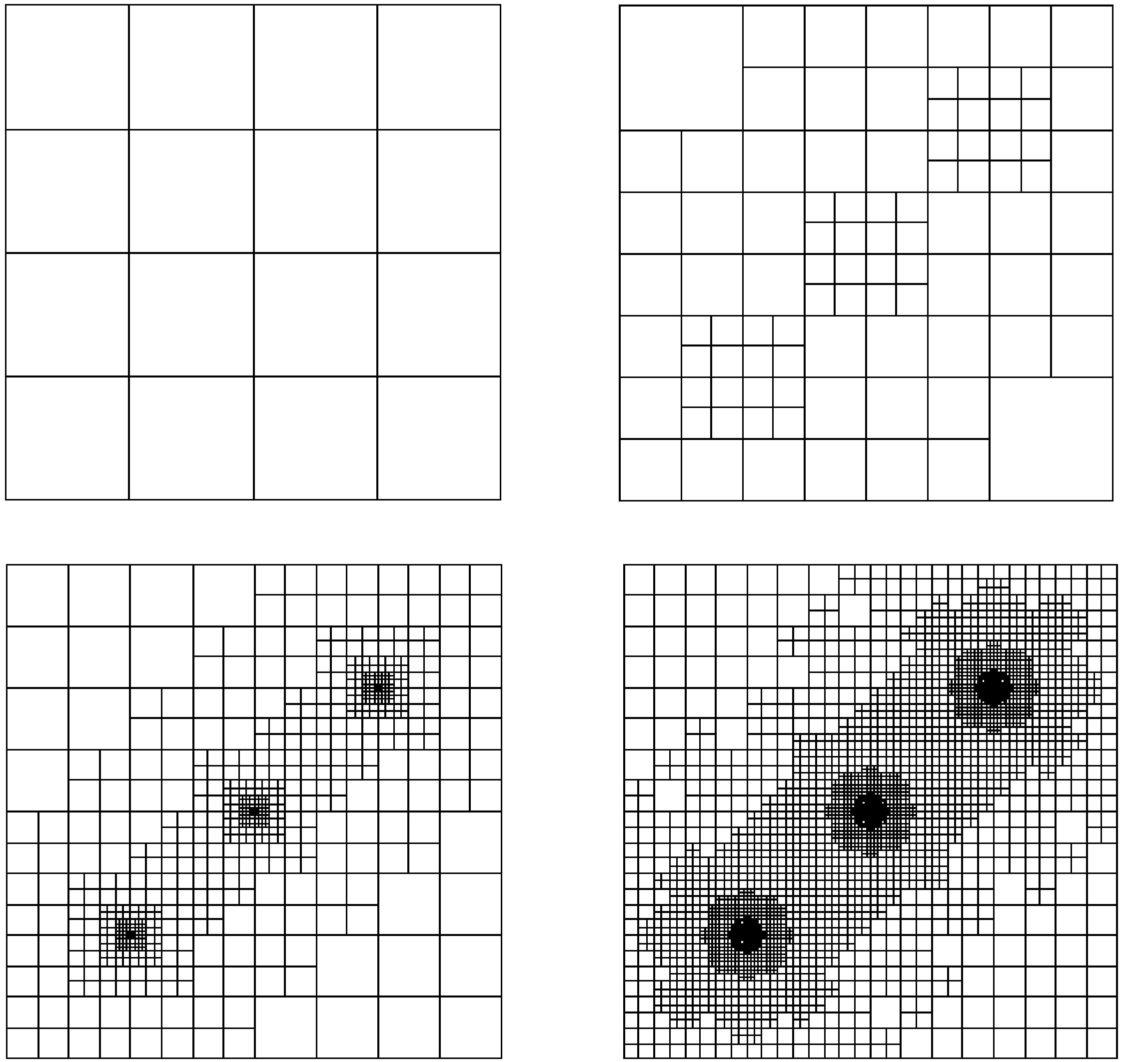



Residual-type error indicator:

$$\eta(\Omega_e)^2 = \int_{\Omega_e} h^2 \left(f + \Delta u_h\right)^2 \, d\Omega$$

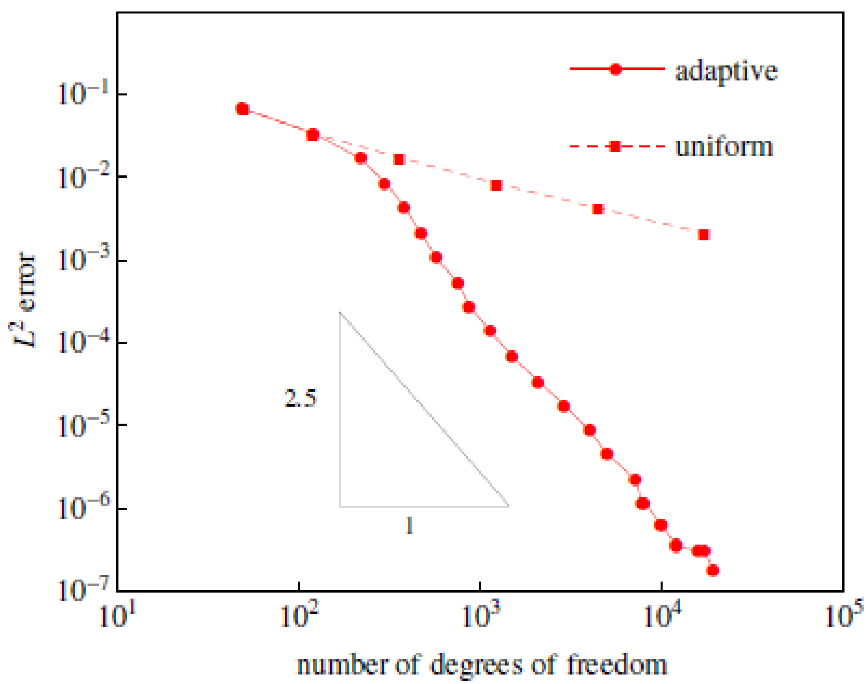
# Adaptive refinement for 2D Poisson problem

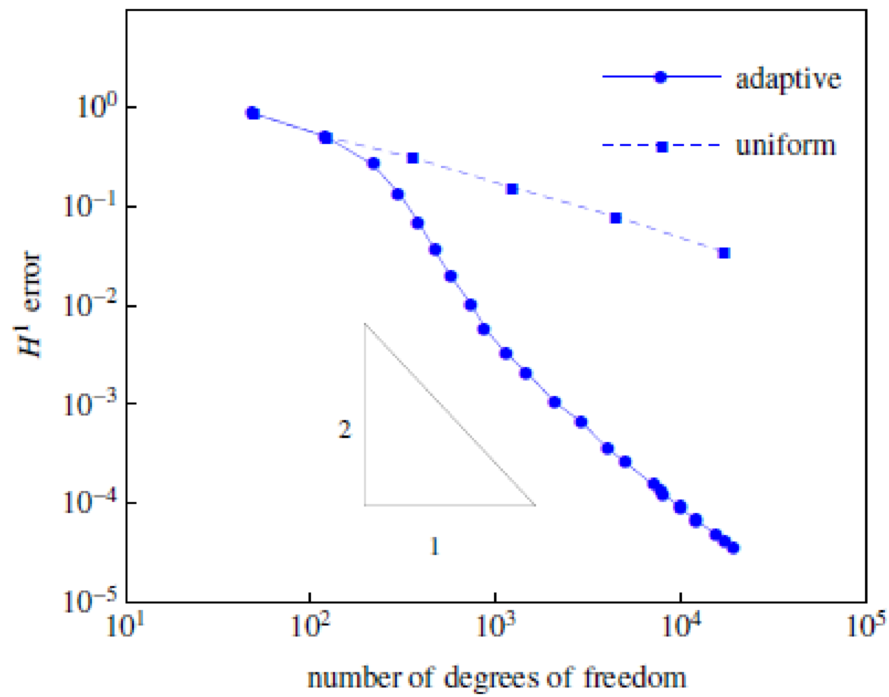
Error indicator : $\eta(\Omega_e)^2 = \int_{\Omega_e} h^2 (f + \Delta u_h)^2 \, d\Omega$



# Adaptive refinement for 2D Poisson problem

Monotonous convergence: $L^2$-norm and $H^1$-seminorm error



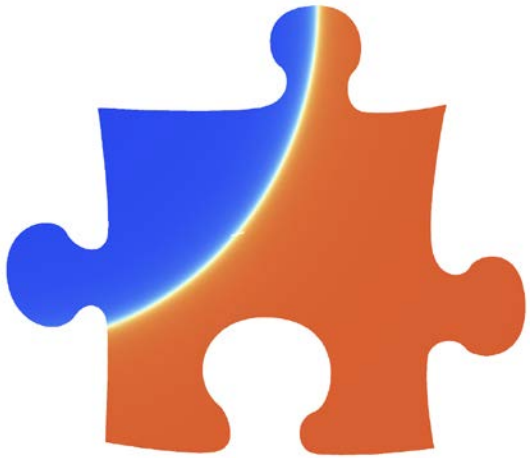



# Isogeometric analysis on 2D complex domain

## Poisson problem with Dirichlet boundary condition on a puzzle piece

SIANI

$$
\begin{aligned}
-\triangle u &= f && \text{in } \Omega, \\
u &= g && \text{on } \partial\Omega.
\end{aligned}
$$



- Analytical solution: Steep wave front given by

$$u(r) = \arctan(\alpha(r - r_0)), \qquad \text{where } r = \sqrt{(x - x_c)^2 + (y - y_c)^2}\,,$$

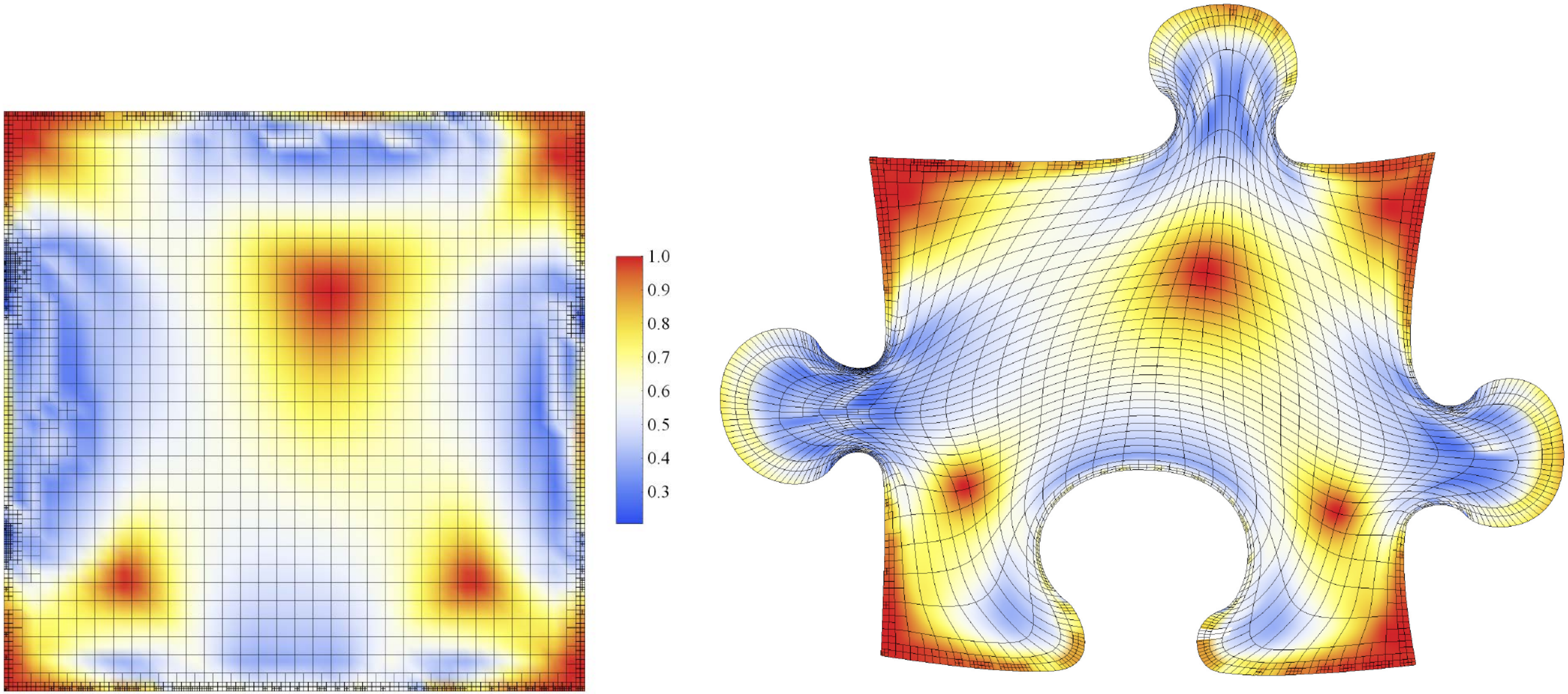
$$(x_c, y_c) = (0, 0),\ \alpha = 200 \text{ and } r_0 = 0.6$$

- Adaptive strategy: Residual-type error indicator:

$$\eta(\Omega_e)^2 = \int_{\Omega_e} h^2 \left(f + \Delta u_h\right)^2 \, \mathrm{d}\Omega$$

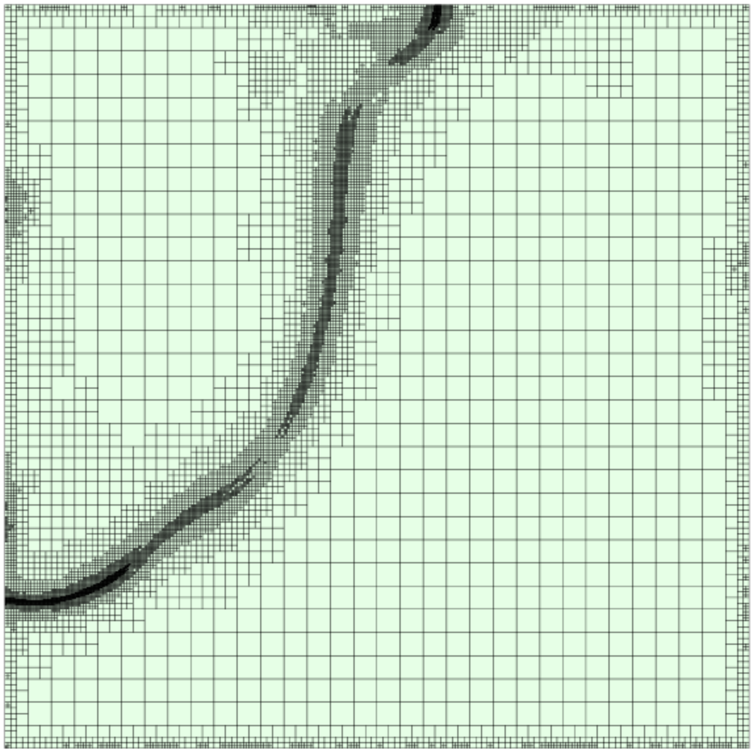
# Isogeometric analysis for 2D Poisson problem

Parameterization of computational domain

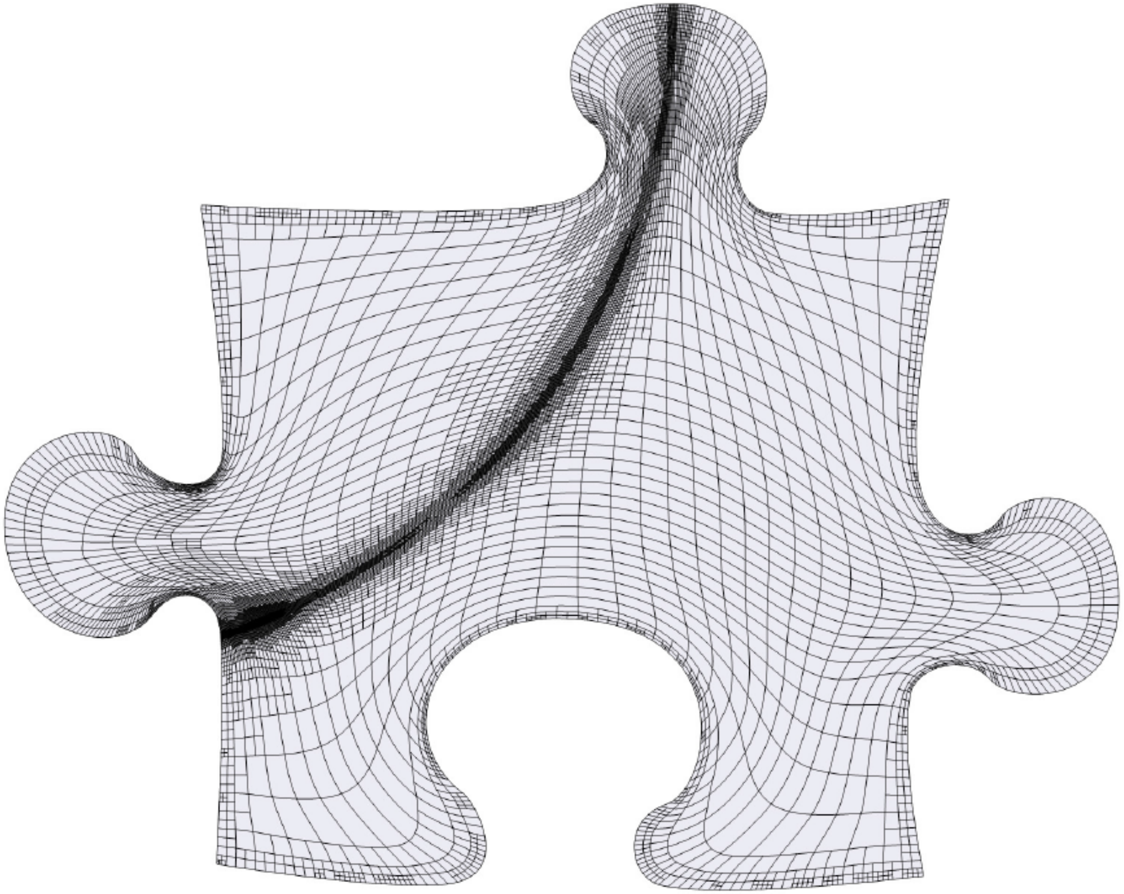


# Adaptive refinement for 2D Poisson problem

Adaptive T-mesh after 13 refinement steps



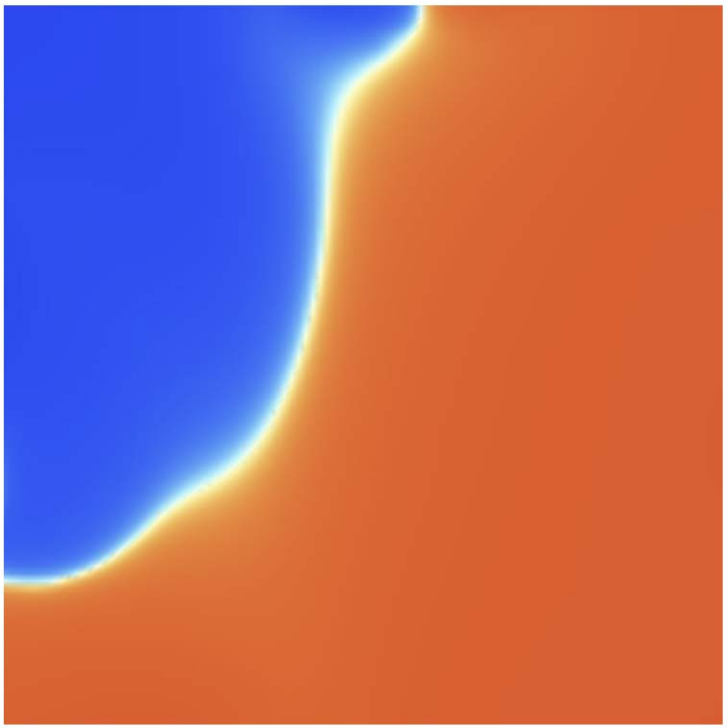
final refinement in the parametric domain



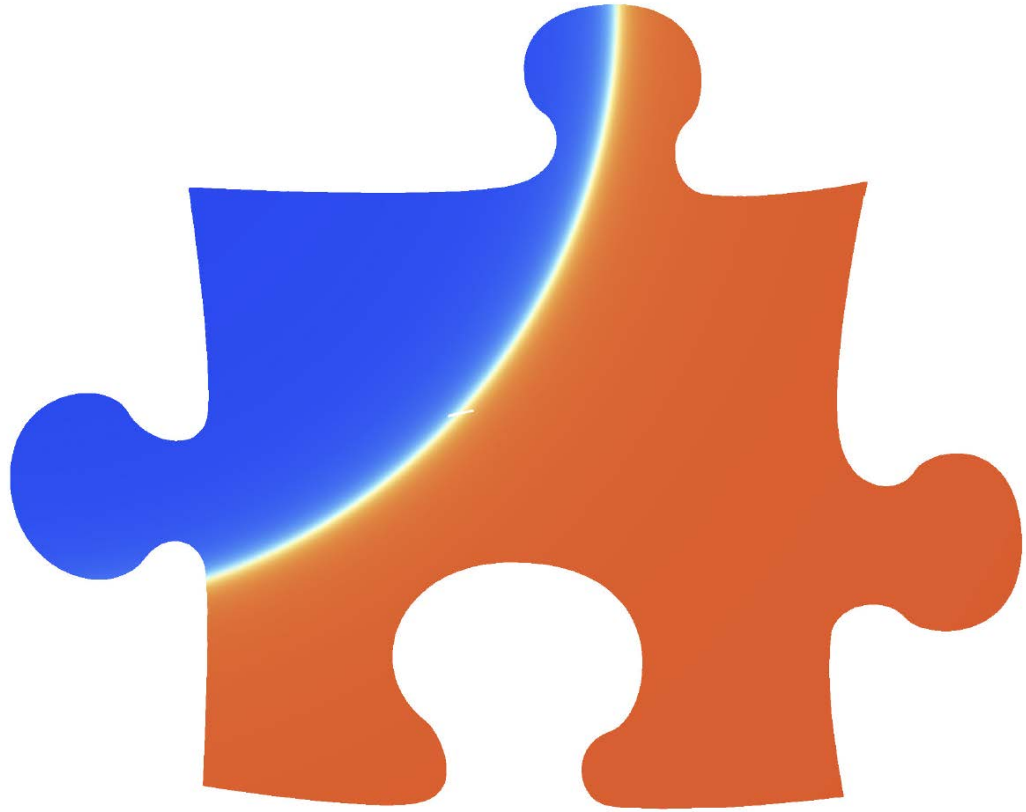
final refinement in the physical domain

# Adaptive refinement for 2D Poisson problem

Numerical solution



numerical solution in the parametric domain



numerical solution in the physical domain

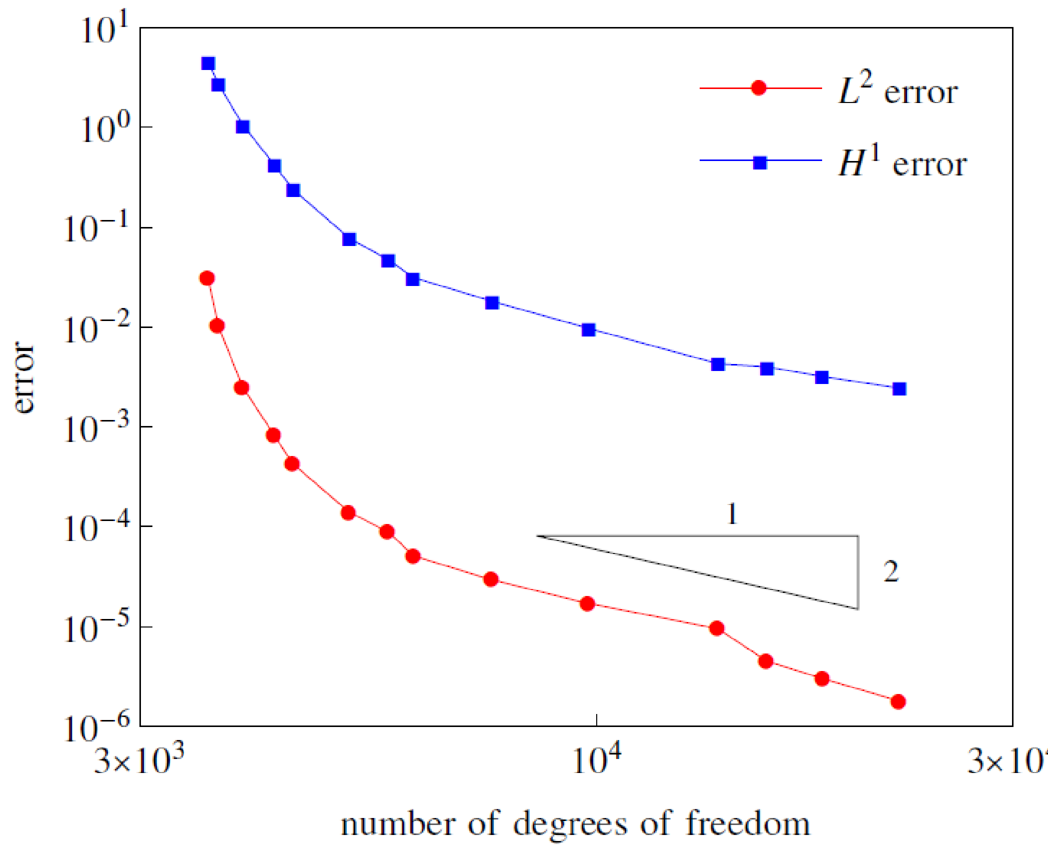
# Adaptive refinement for 2D Poisson problem

## Convergence behavior

Convergence of $L^2$-norm and $H^1$-seminorm error

expected order for 2D problem:

$L^2$-norm: 2
$H^1$-seminorm: 3/2



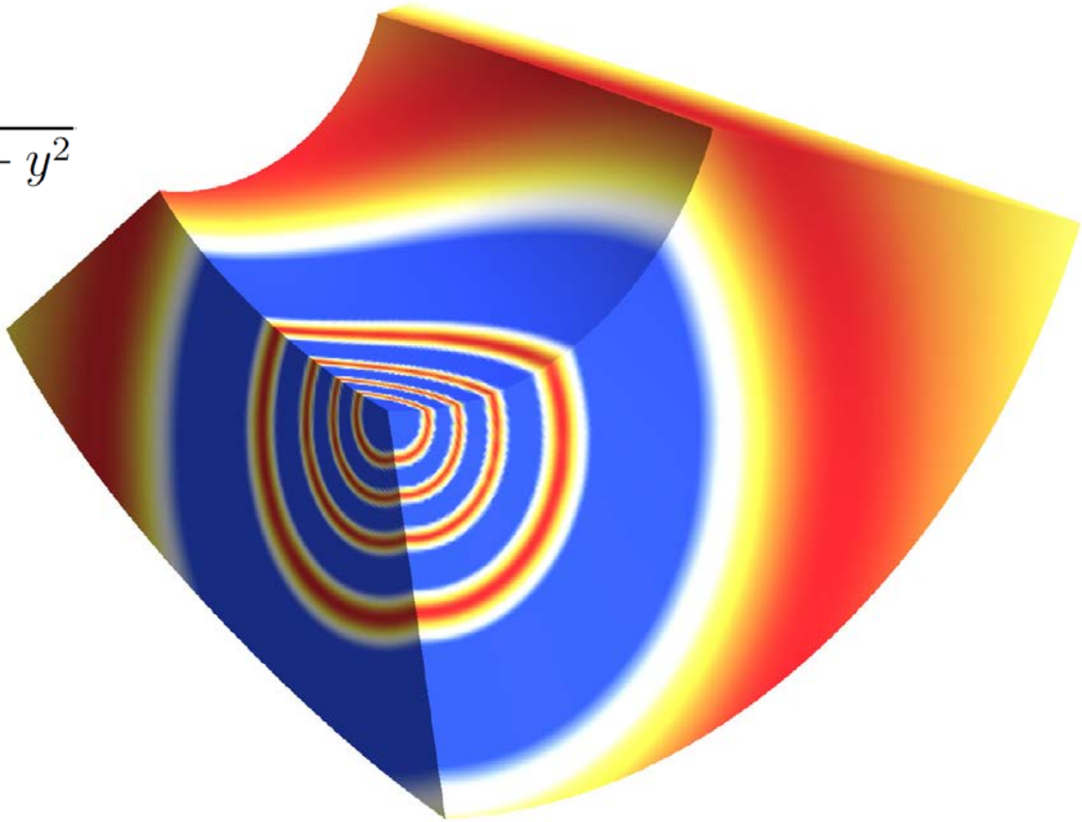
# Isogeometric analysis for 3D Poisson problem

Computational domain: sphere portion

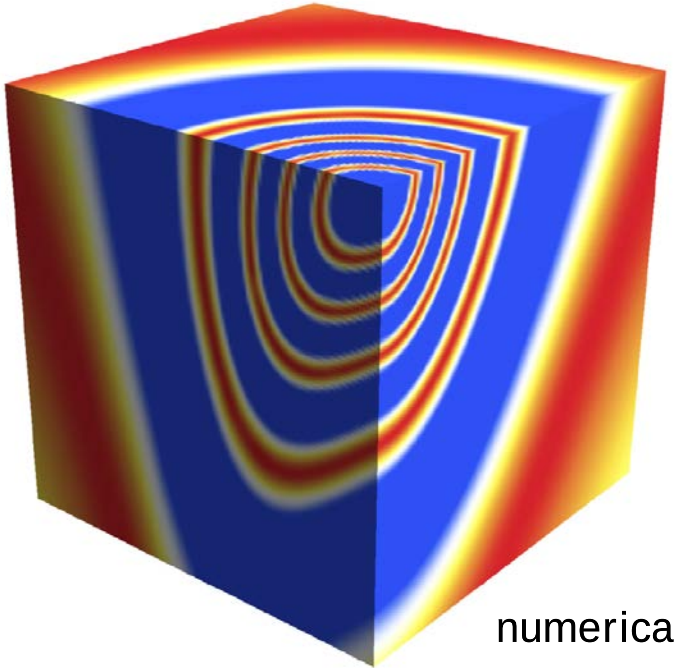
Analytical solution:

$$u(r) = \sin\left(\frac{1}{\alpha + r}\right), \quad \text{where } r = \sqrt{x^2 + y^2}$$

oscillation parameter $\alpha = 1/10\pi$



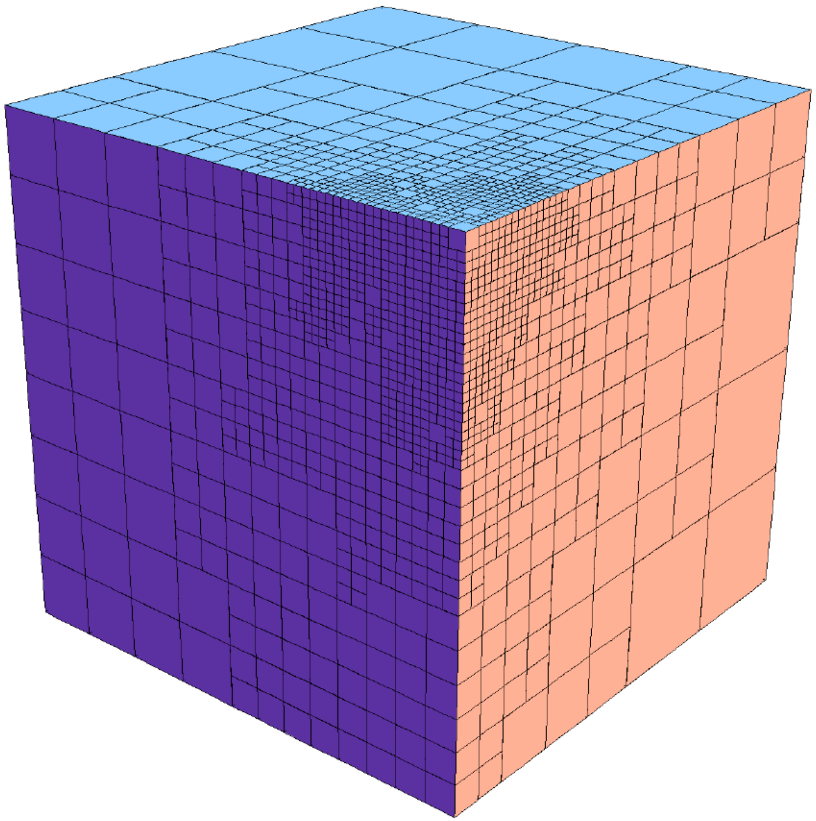
numerical solution in the physical domain

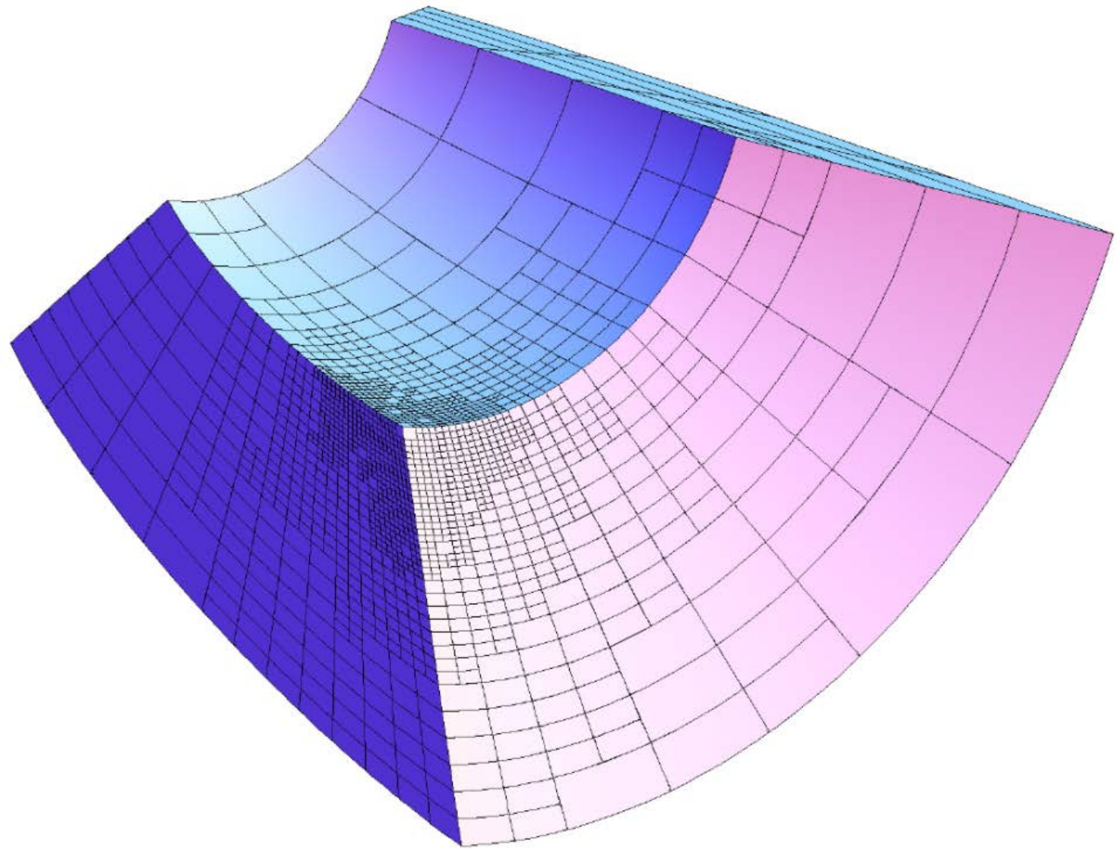


numerical solution in the parametric domain

# Adaptive refinement for 3D Poisson problem

## Adaptive refinement



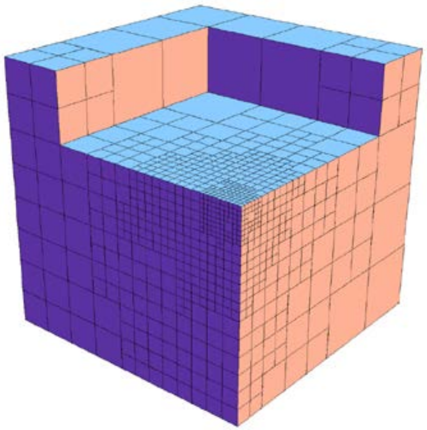


final refinement in the physical  domain

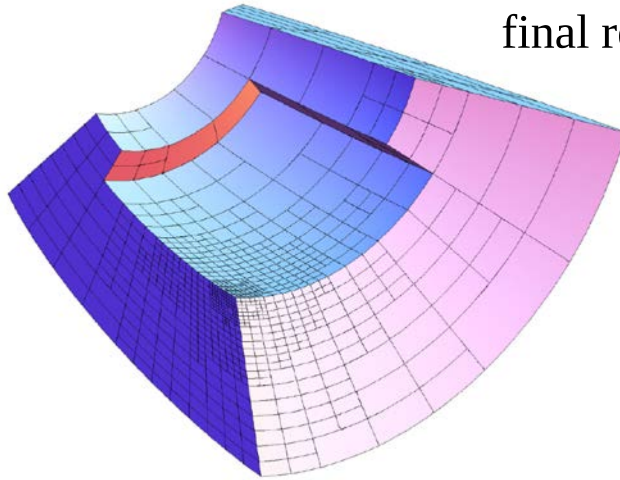
final refinement in the parametric domain

# Adaptive refinement for 3D Poisson problem

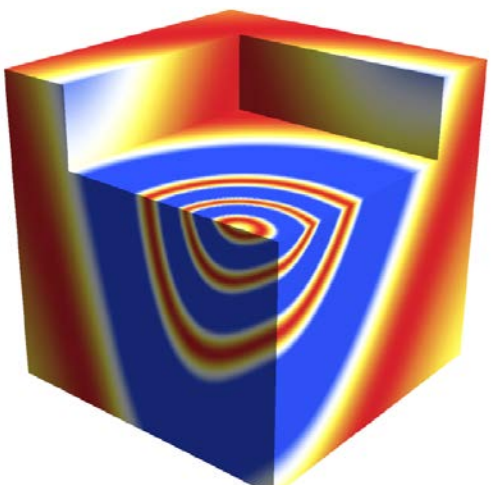
A section of the parametric and physical domain



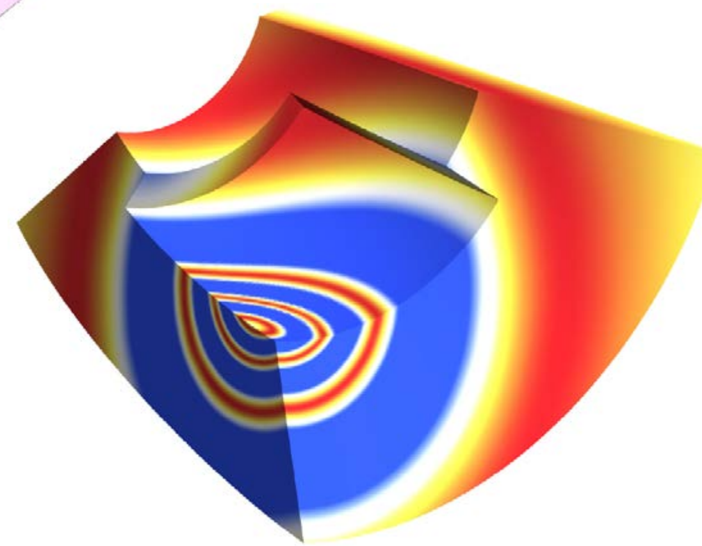
final refinement in the parametric domain



final refinement in the physical domain



numerical solution in the parametric domain



numerical solution in the physical domain

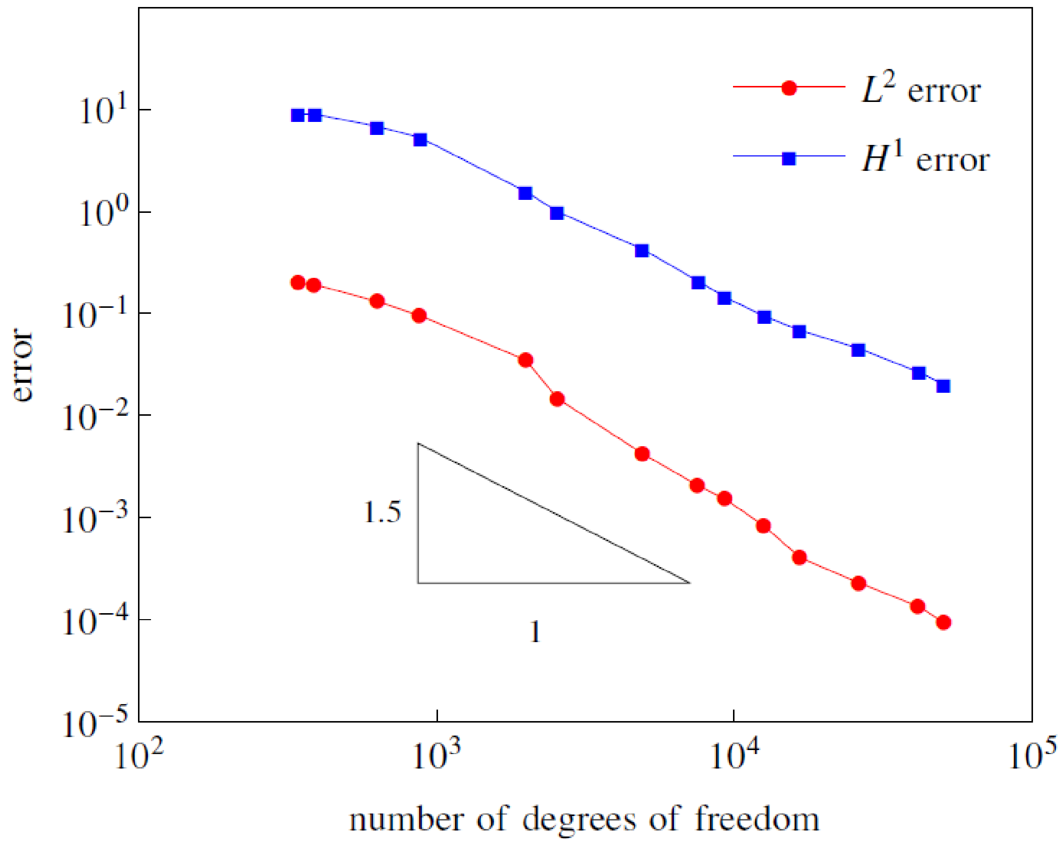
# Adaptive refinement for 3D Poisson problem

## Convergence behavior

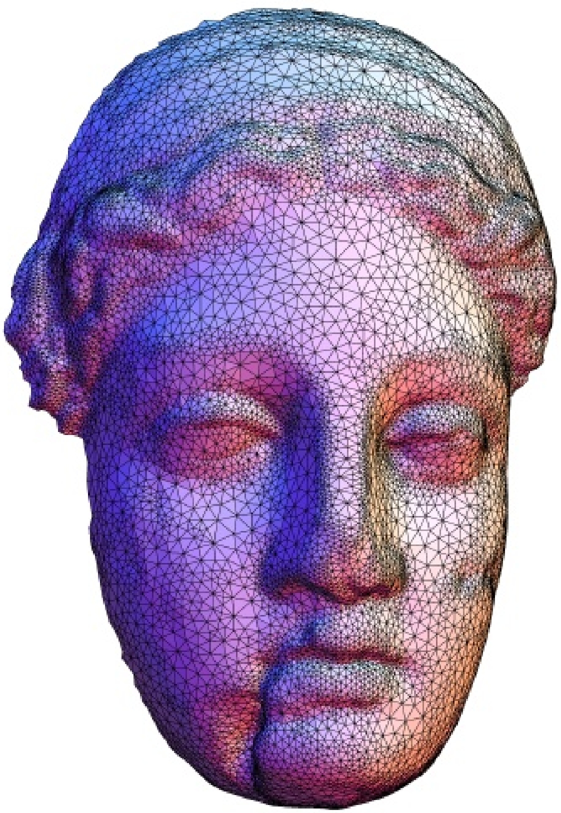
Convergence of $L^2$-norm and $H^1$-seminorm error

expected order for 3D problem:

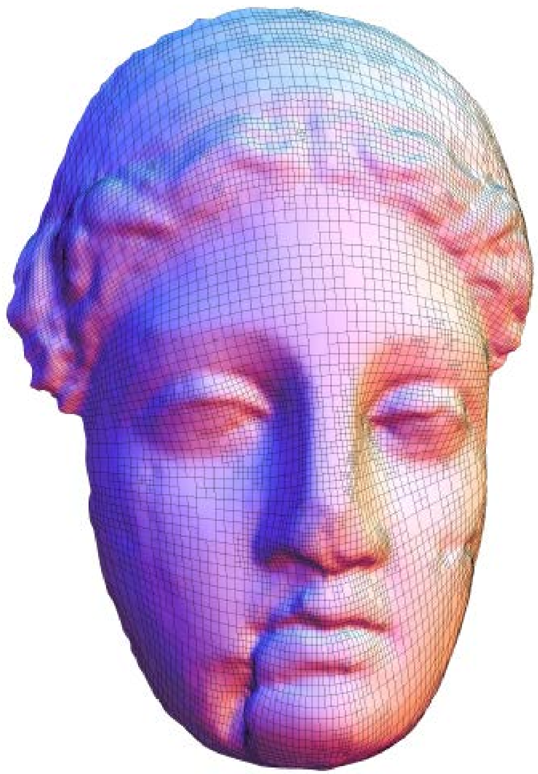
$L^2$-norm: 3/2
$H^1$-seminorm: 1



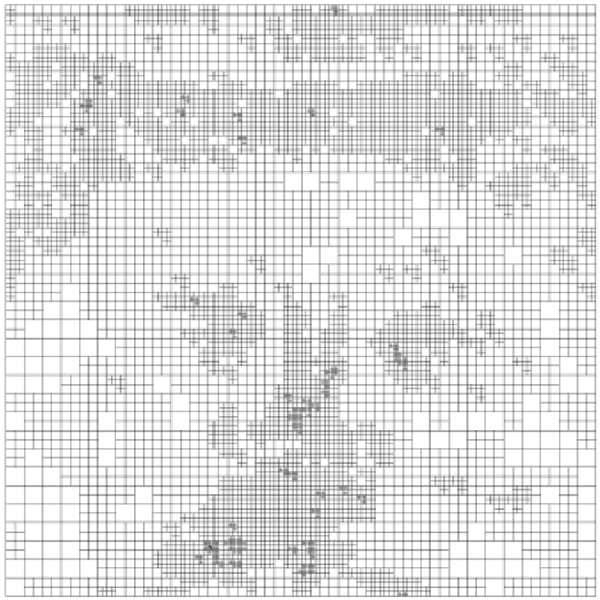
## Spline representation of the surface from its triangulation



input triangulation



spline representation



parametric T-mesh

# Spline spaces over T-meshes

## Limitations and future research

- **Excesive overlapping of function support  in some situations**

  Possible improvement of  the locallity of function supports

- **Rigorous proof of the properties of the spaces defined with our strategy:**

• linear independence
• nestedness of the spaces

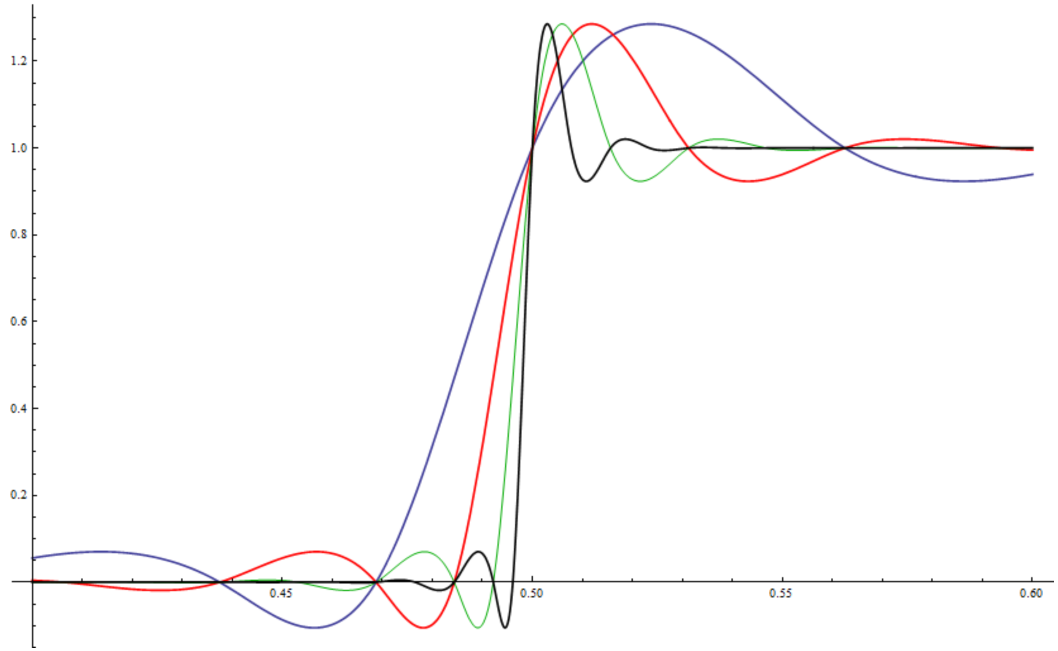
**And another interesting question.**

## Is IGA always better than FEM ?

## $C^2$ is alwais better than $C^0$ ?

It seems to be a lot better for approximation of very smooth function.

And not so smooth ??? Should be studied.




*Spline approximation of a steep function*

GRACIAS !