# The Meccano Method for Isogeometric Analysis of Planar Domains

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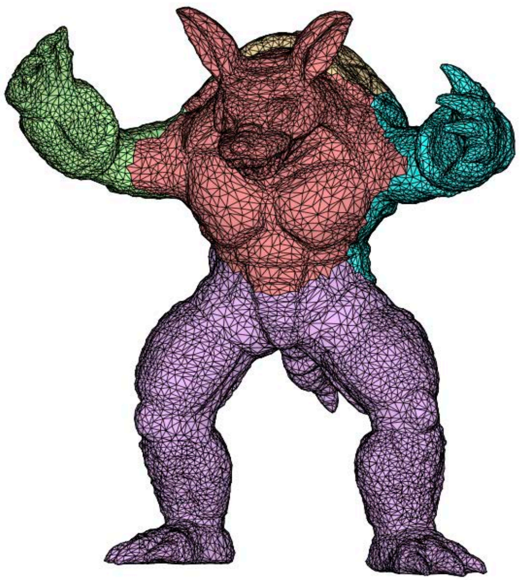
# Contents

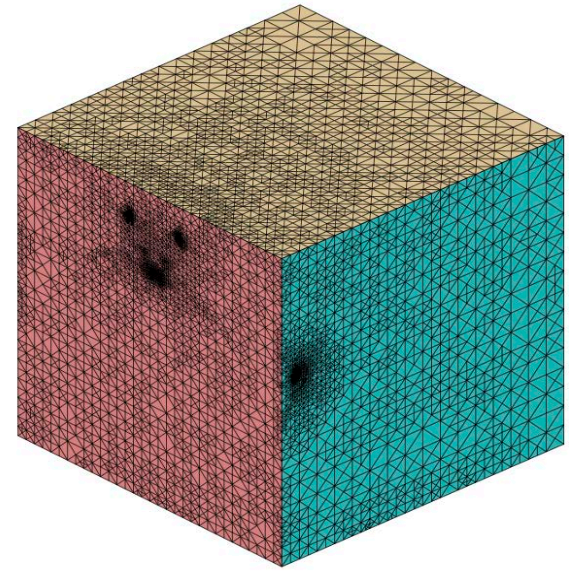


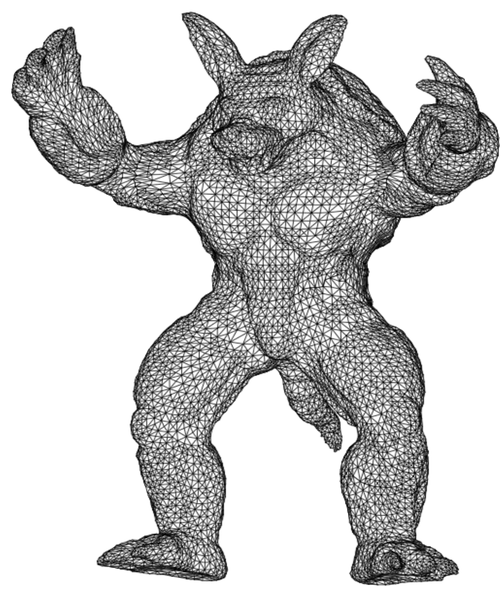
- The Meccano Method (based on tetrahedral mesh optimization)
  - The initial algorithm for tetrahedral mesh generation
  - Volumetric parameterization of tetrahedral meshes
  - Application to finite element and isogeometric solid modeling and analysis
- The Meccano Method (based on a new 2D T-mesh optimization)
  - The new algorithm for two-dimensional T-mesh generation
  - T-spline parameterization of 2D geometries
  - Geometries with several materials or holes
  - Application to isogeometric modeling and analysis
- Comments and Future Research

# Meccano Method for Complex Solids

## Volume parameterization based on SUS of tetrahedral meshes







- Parameterization
- Refinement
- Untangling/Smoothing

# Meccano Method for Complex Solids

## Volume parameterization based on SUS of tetrahedral meshes

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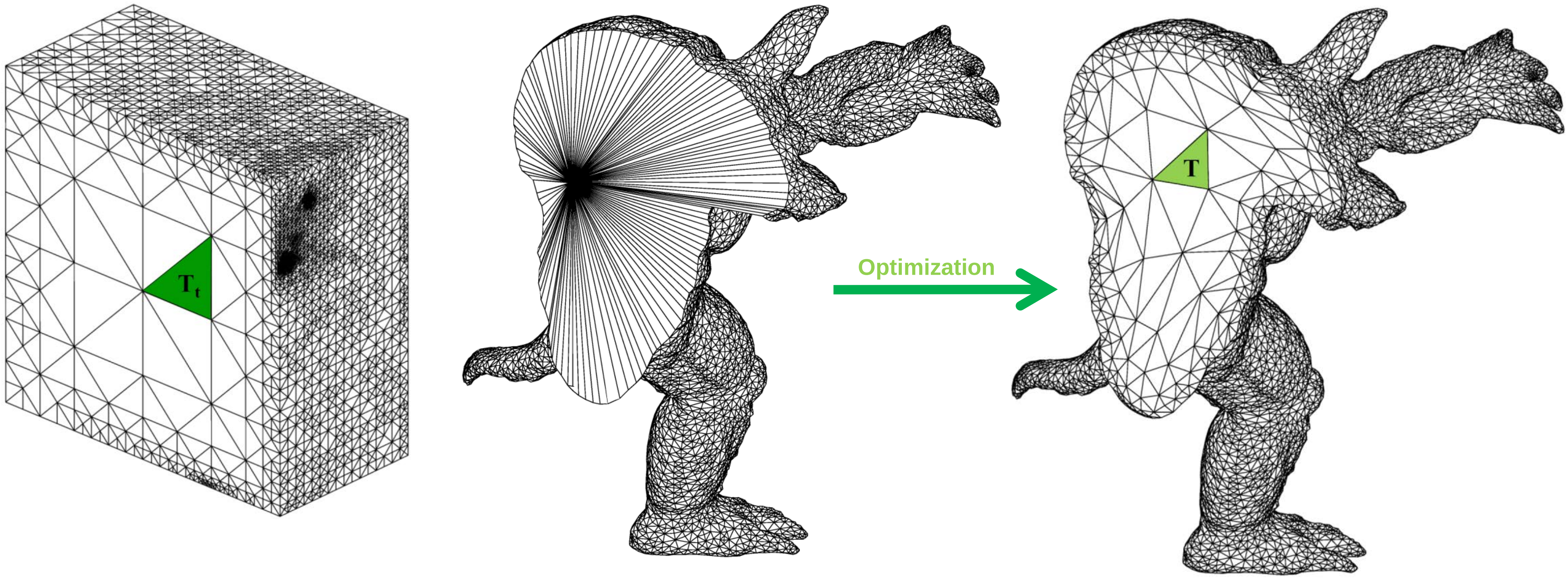



Parameter space
(meccano mesh)

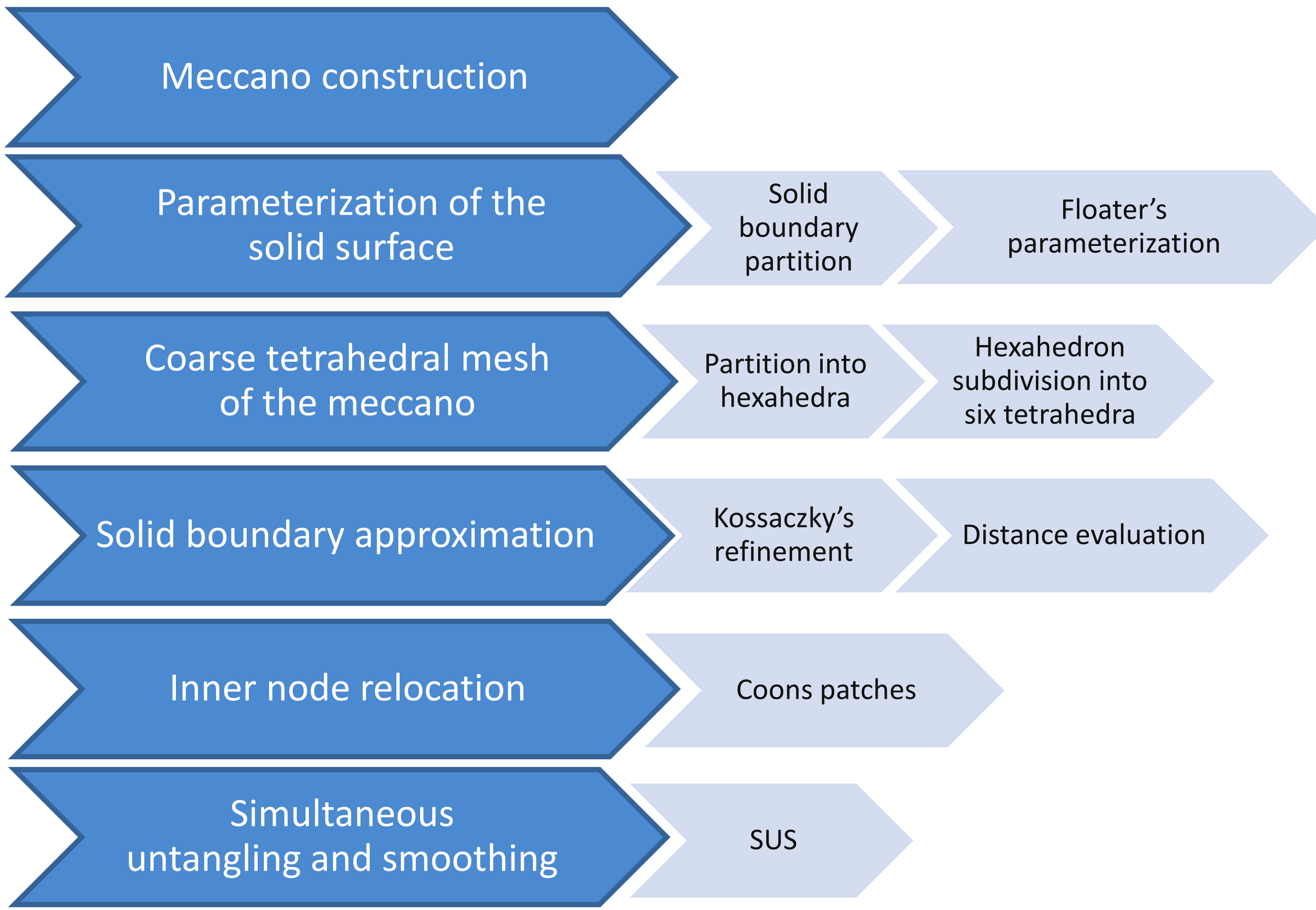
Physical space
(tangled mesh)

Physical space
(optimized mesh)

**Target Element $T_t$** ← Optimization — **Physical Element $T$** (to get less distortion in the parameterization)

# Meccano Method for Complex Solids

## Algorithm steps



# Adaptive Finite Element Solution

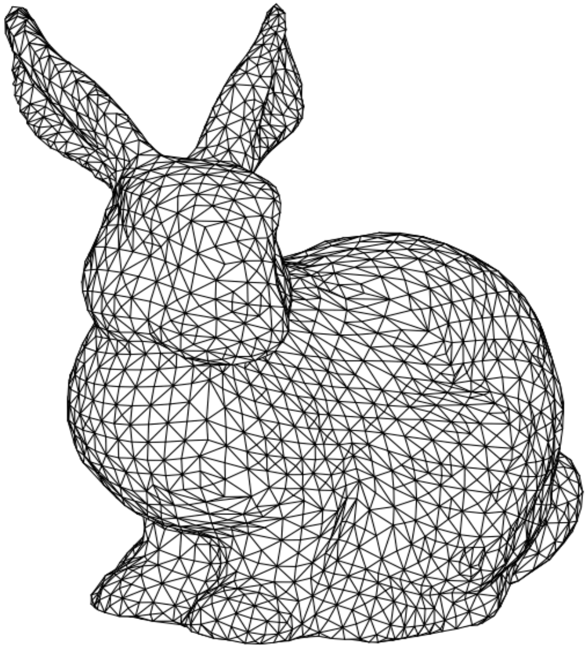
## Stanford Bunny Example

**Parabolic Problem:**

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u &= 0 && \text{on } \Omega \times \{0\}, \end{aligned}$$

Adaptive Algorithm: Solve → Estimate → Mark → Refine/Derefine

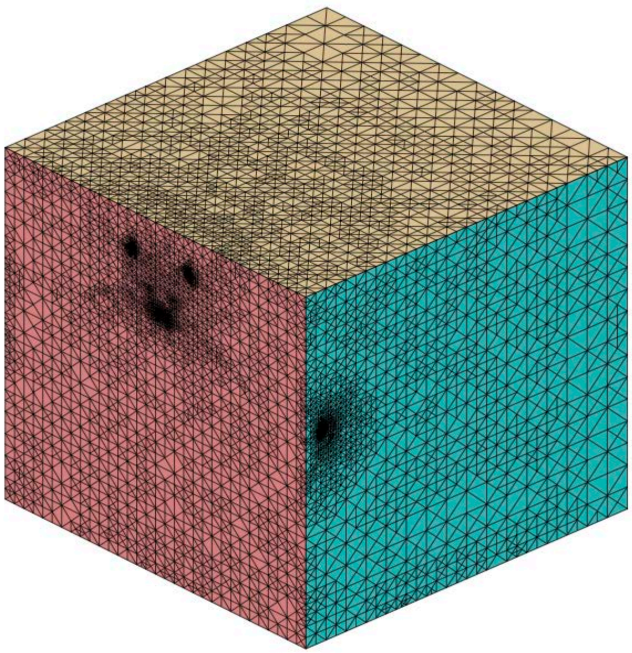
**3-D Domain:**



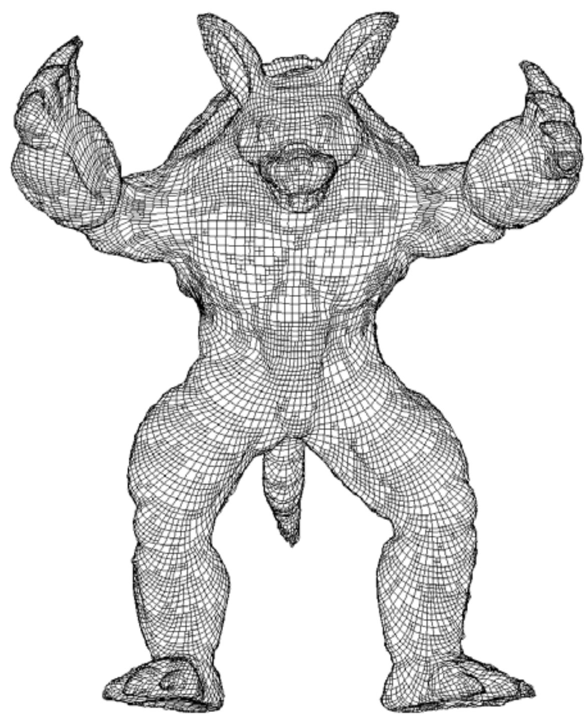
Stanford bunny

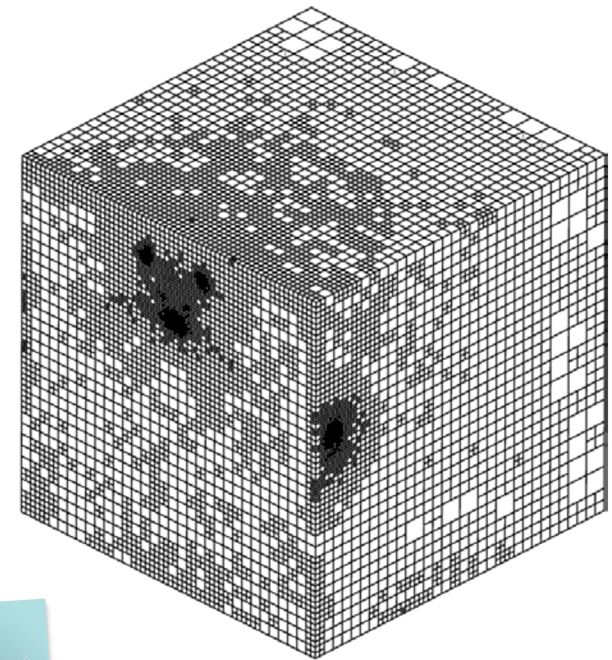
# Meccano Method for Complex Solids

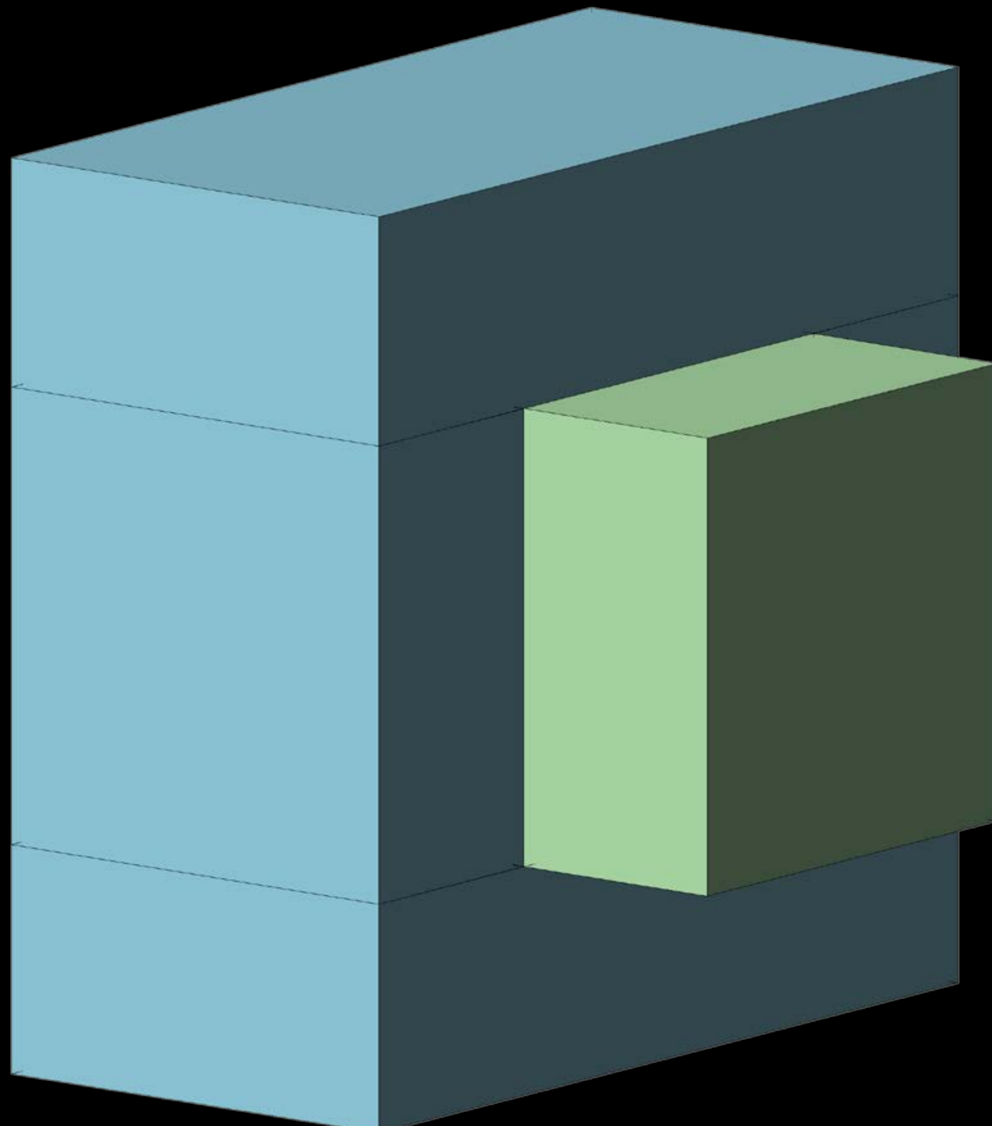
## Volume parameterization based on SUS of tetrahedral meshes

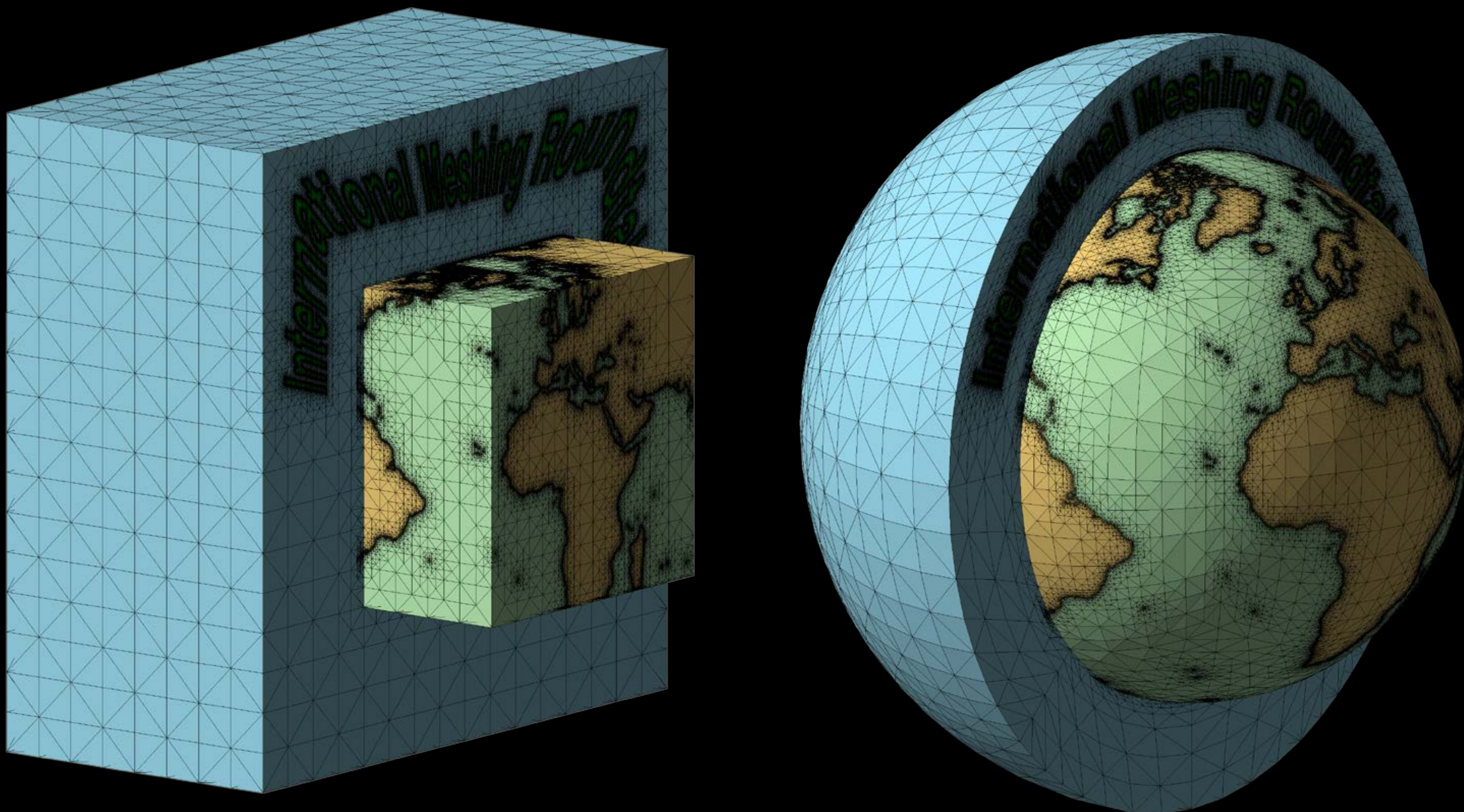


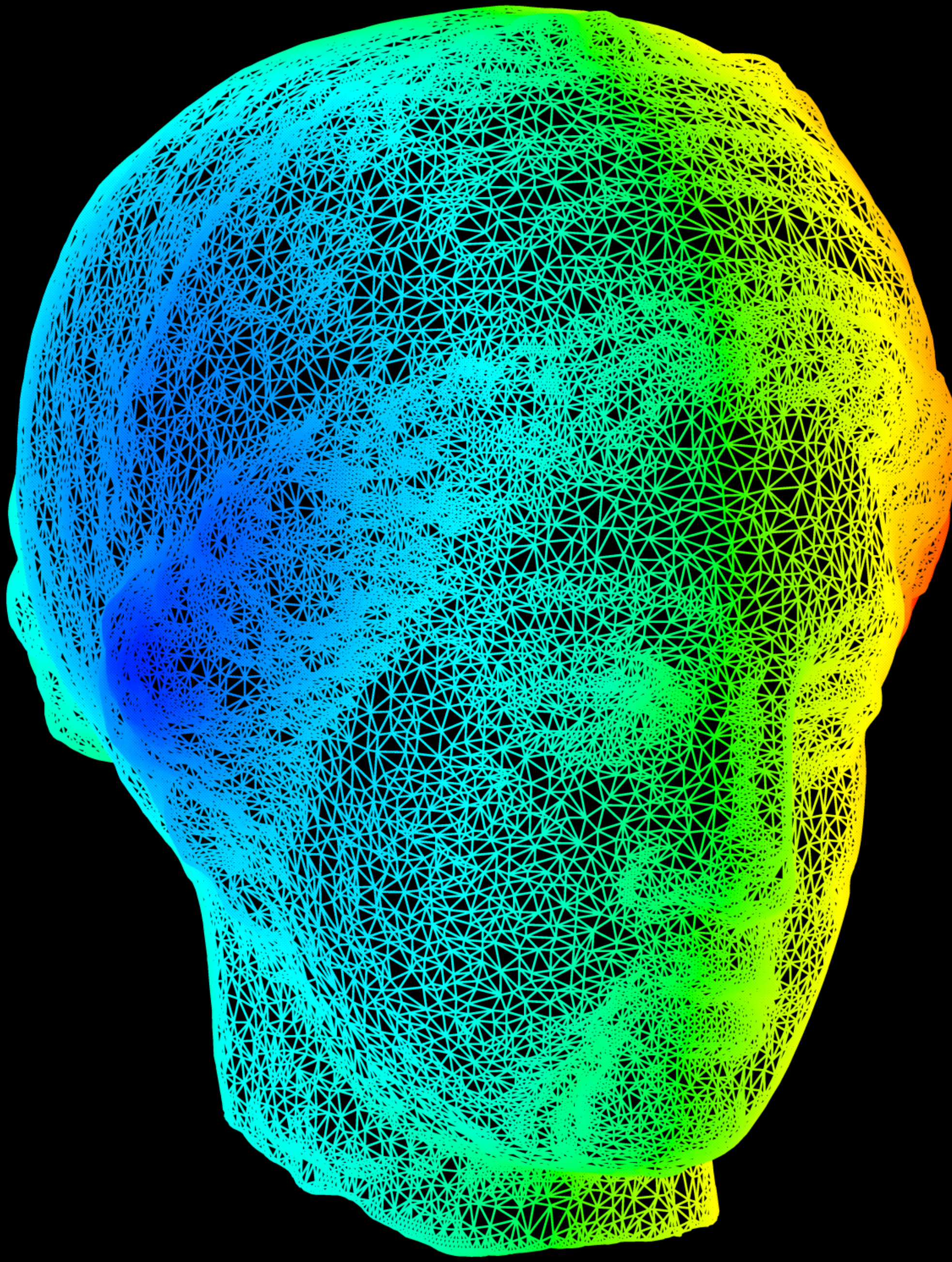
- Octree subdivision





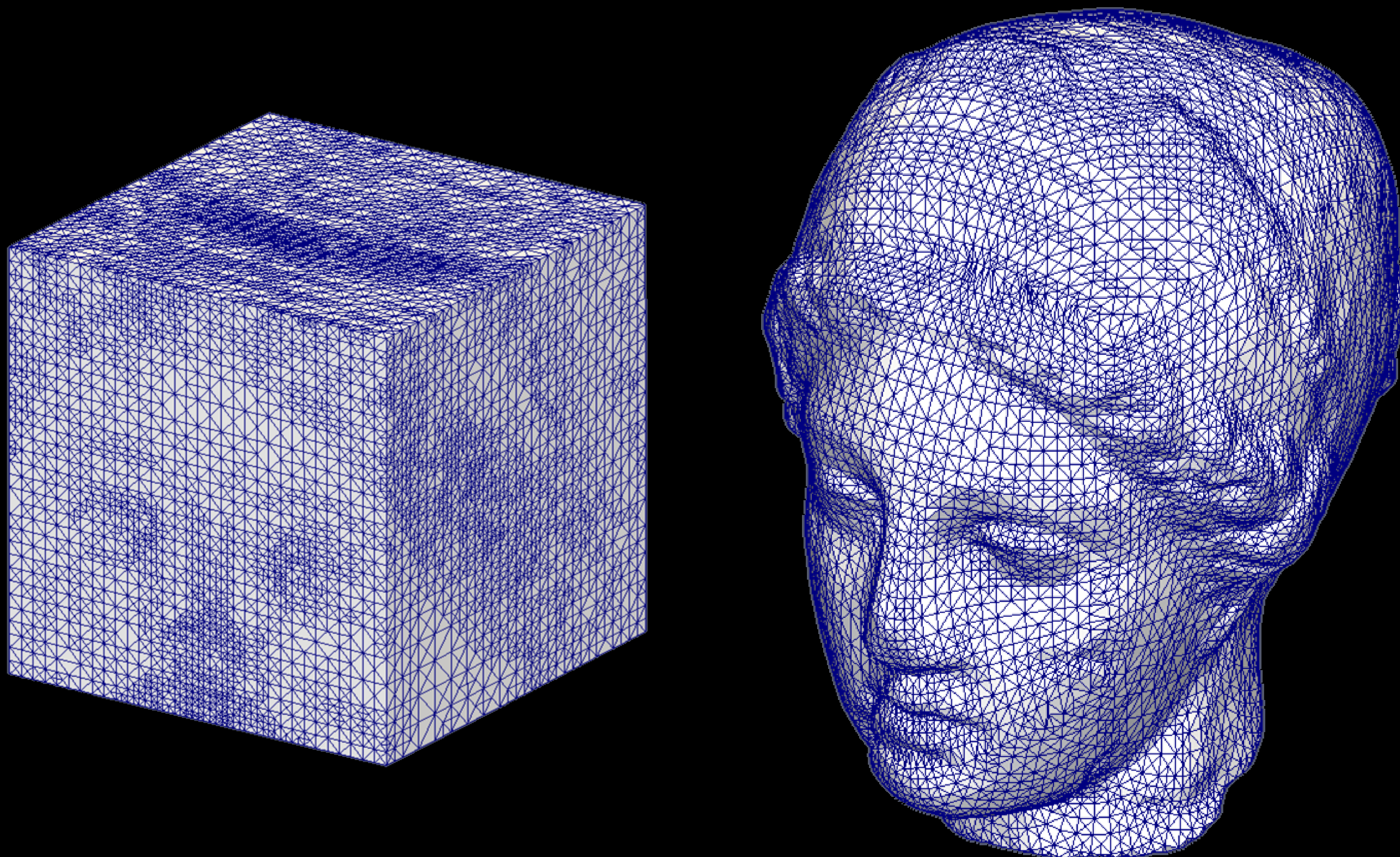



International Meshing Roundta



INPUT DATA: Surface Triangulation

http://www.cyberware.com/



# Adaptive Isogeometric Refinement (EWC 2012)

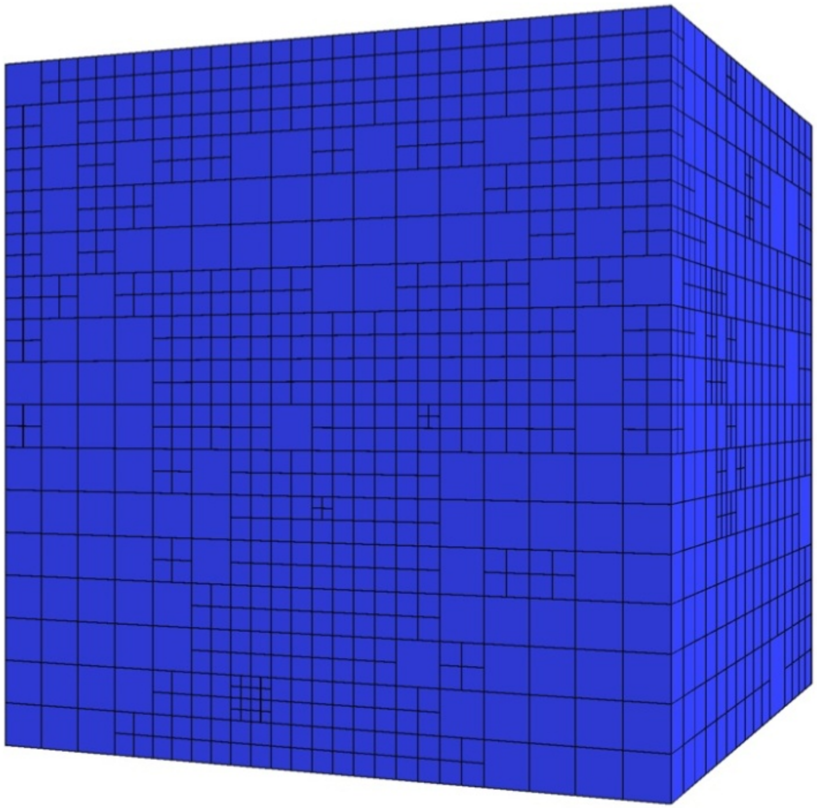
Application in Igea: Poisson problem with a central source

$$\Delta u = \frac{1}{25} e^{-\frac{\left(x^2+y^2+z^2\right)}{10}} \left(-15 + x^2 + y^2 + z^2\right) \quad \text{in } \Omega$$

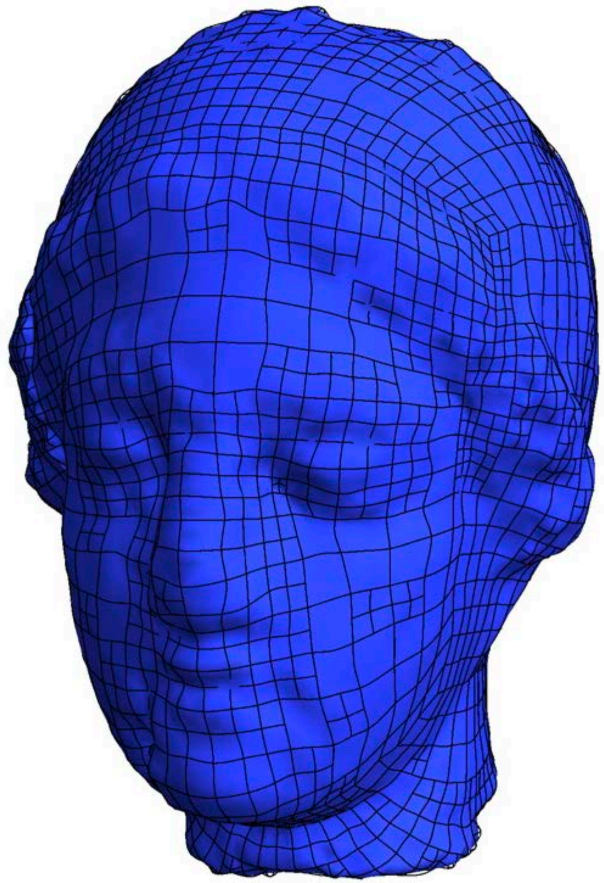
$$u\big|_{\partial\Omega} = 0$$

Exact solution:

$$u \approx e^{-\frac{\left(x^2+y^2+z^2\right)}{10}}$$

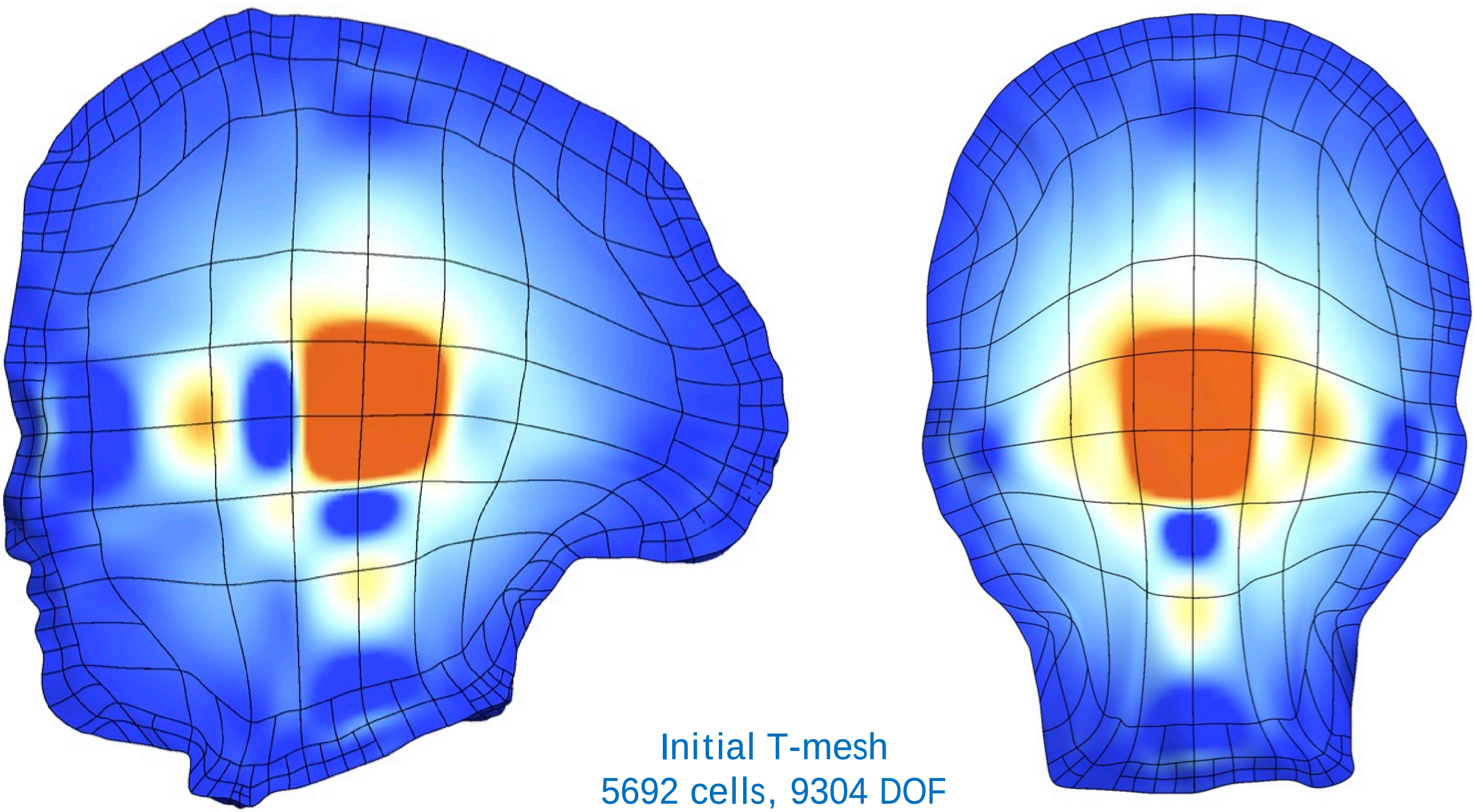



$\mathbf{S}(\xi)$



# Adaptive Isogeometric Refinement (EWC 2012)

Igea: T-spline of Numerical Solution

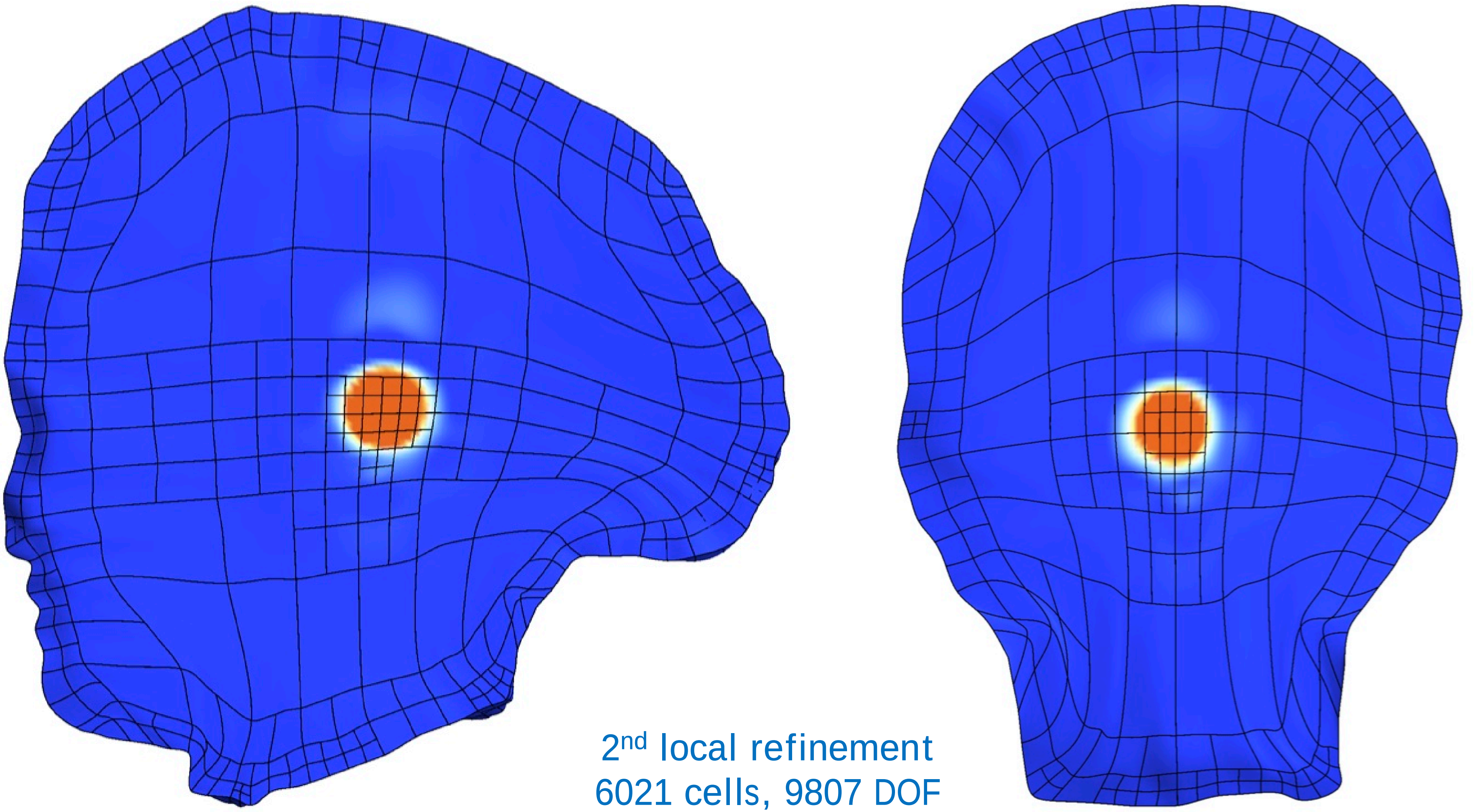


Initial T-mesh
5692 cells, 9304 DOF

# Adaptive Isogeometric Refinement (EWC 2012)

## Igea: T-spline of Numerical Solution

Error indicator : $\eta(\Omega_e)^2 = \int_{\Omega_e} h^2 (f + \Delta u_h)^2 \, d\Omega$



2nd local refinement
6021 cells, 9807 DOF

# Adaptive Isogeometric Refinement (EWC 2012)

## Igea: T-spline of Numerical Solution

Error indicator : $\eta\left(\Omega_e\right)^2 = \int_{\Omega_e} h^2 \left(f + \Delta u_h\right)^2 \, d\Omega$

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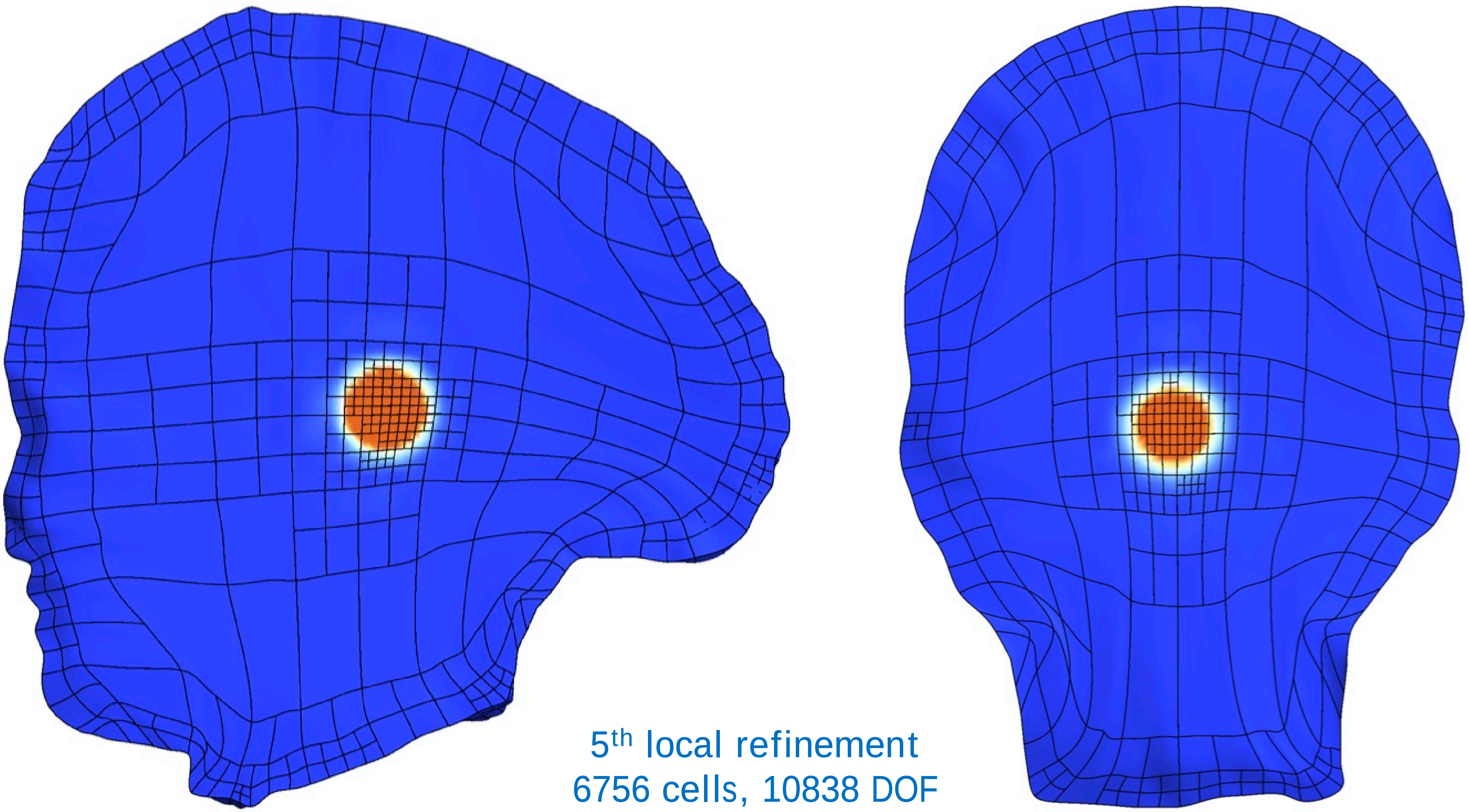

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1. The key of the meccano method is the simultaneous untangling and smoothing (SUS) procedure.

2. The quality of the T-spline mapping (i.e., positive Jacobian, good uniformity and orthogonality of the isoparametric curves) depends on the quality of the T-mesh in the physical space. We have to fix a quality metric for this mapping.

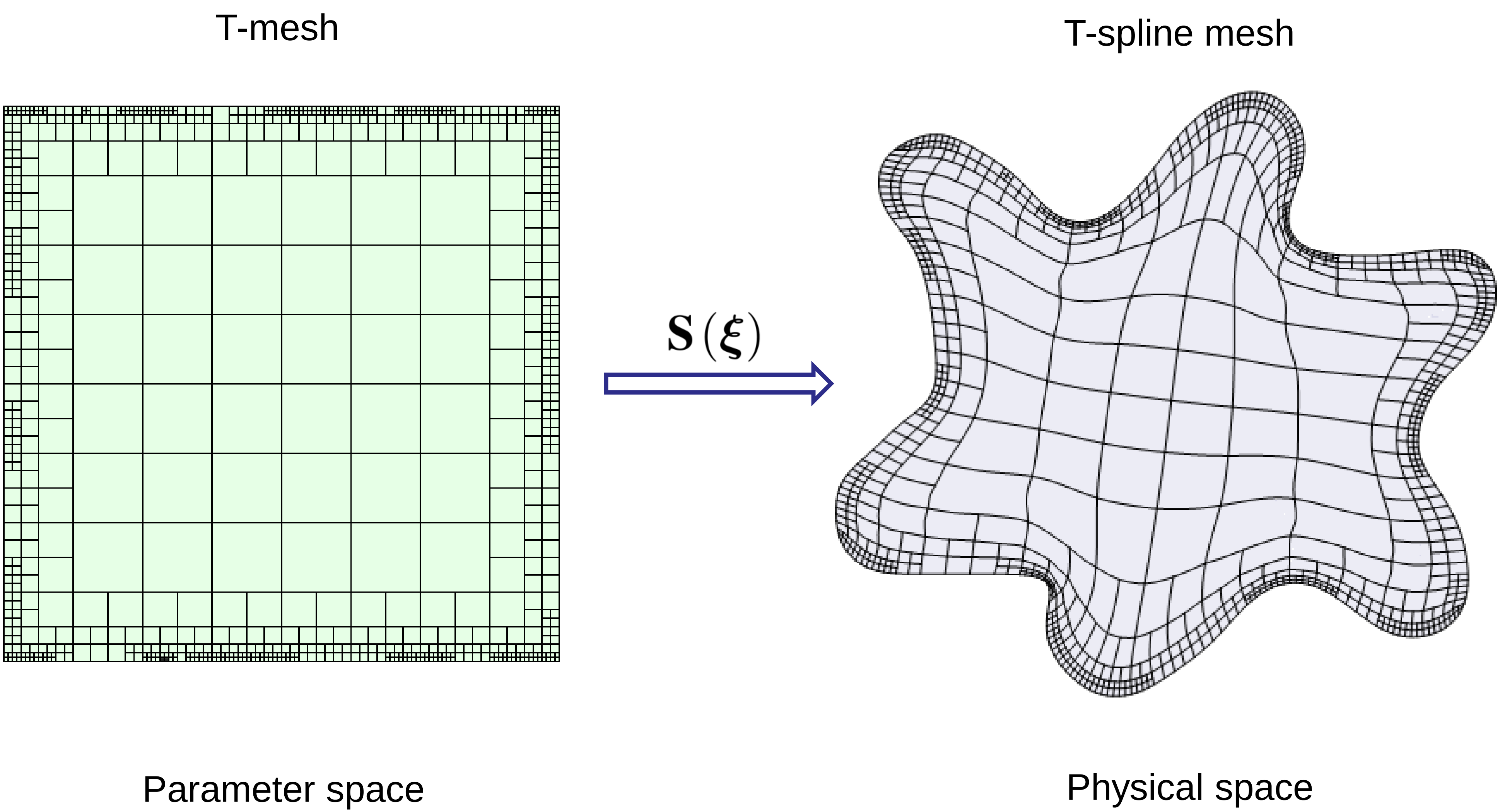
3. In order to simplify the procedure and to get less distortion in the volume parameterization, it should be interesting to **directly apply the meccano method on T-meshes instead of tetrahedral meshes**.

4. We have started analysing the problem in 2-D.

# The Meccano Method on T-meshes in 2-D

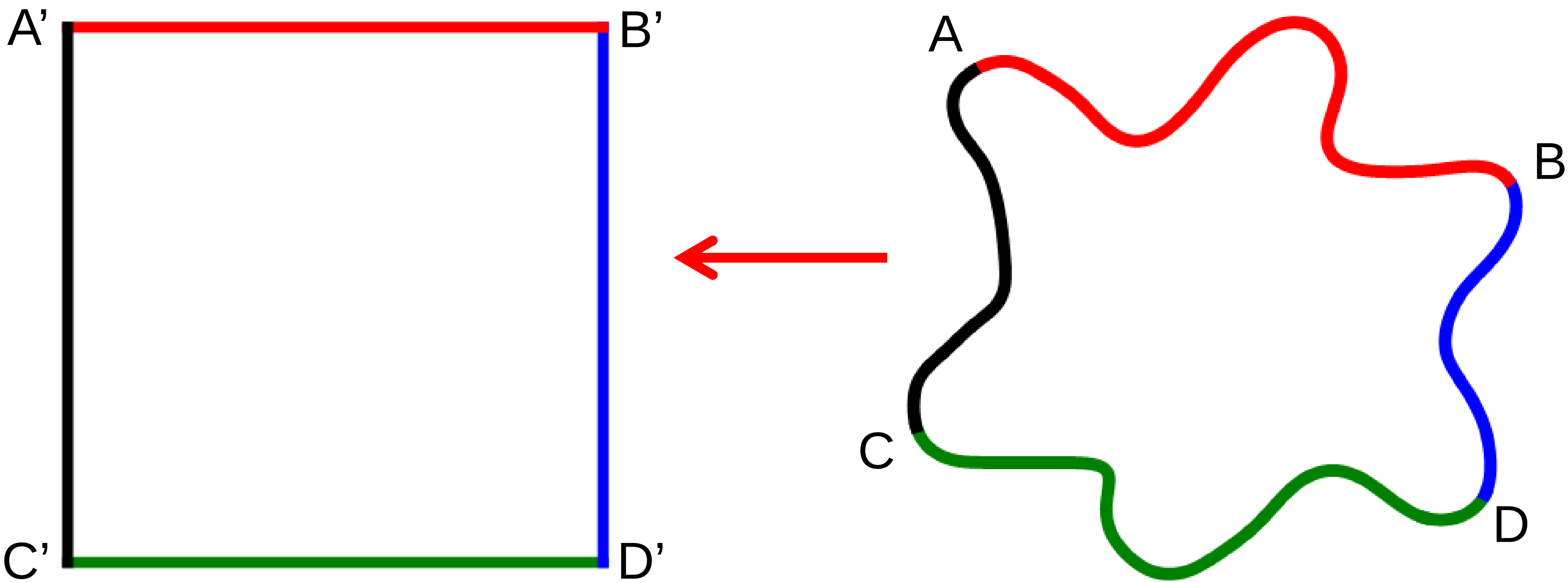
Input data: Boundary representation of the object

Objective: Construction of a high quality T-spline parameterization



# The Meccano Method on T-meshes in 2-D

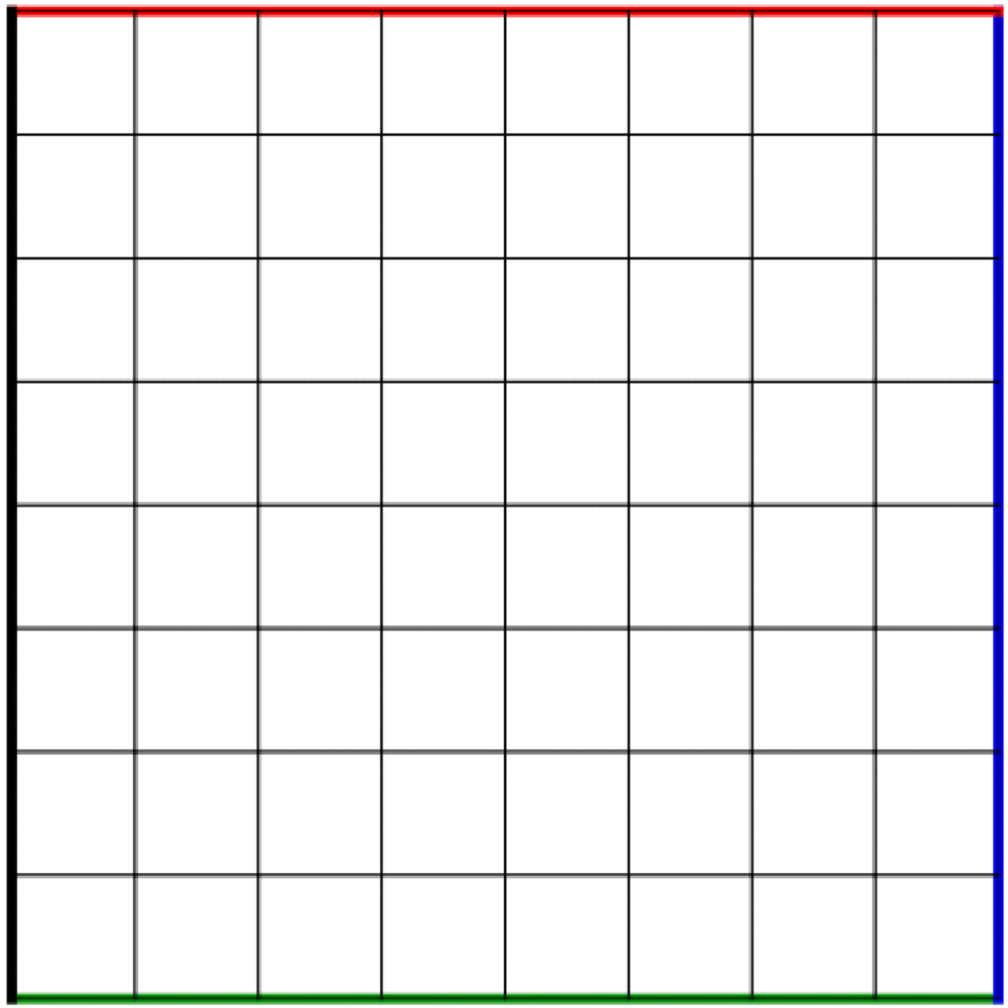
Step 1: Input boundary (image, polyline, curve, etc.) and boundary mapping




- Select four points (A, B, C, D) of the input boundary
- Boundary parameterization via chord-length

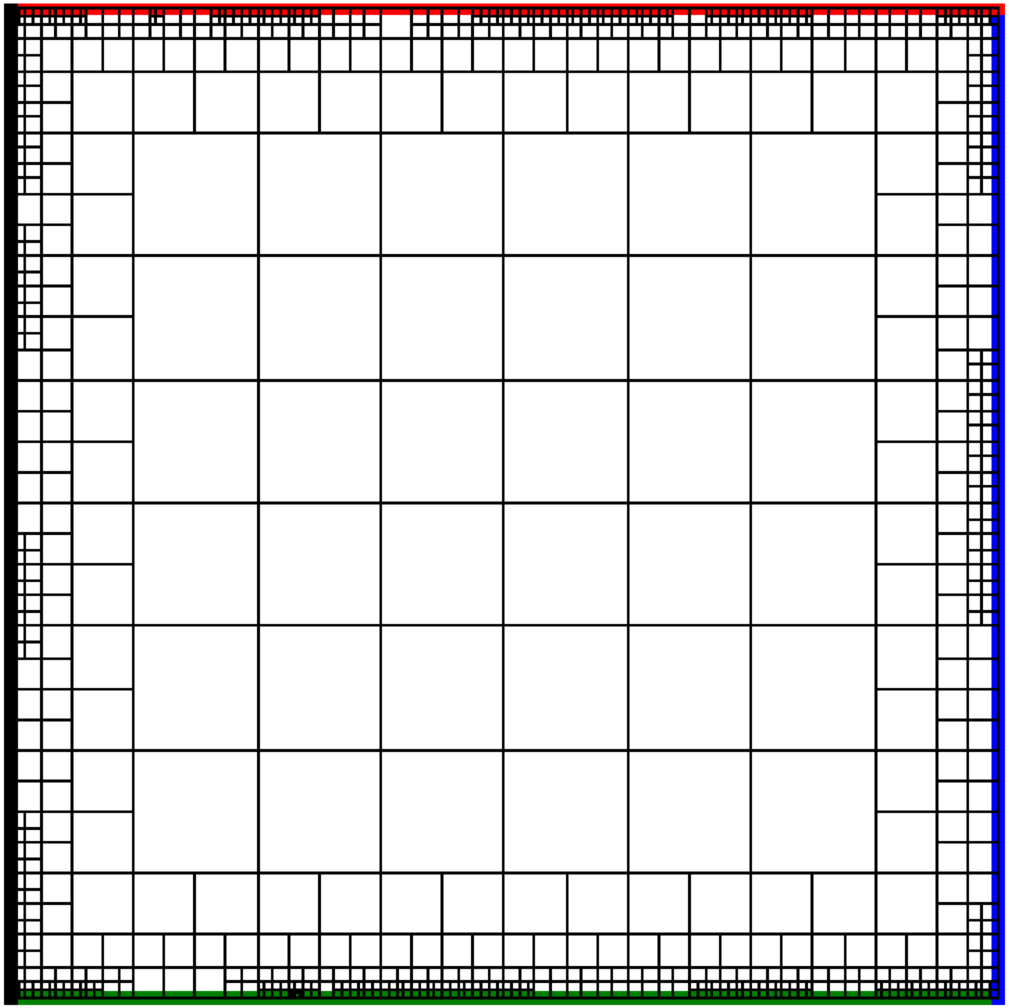
# The Meccano Method on T-meshes in 2-D

Step 2: Coarse quadrilateral mesh of the meccano (parameter space)



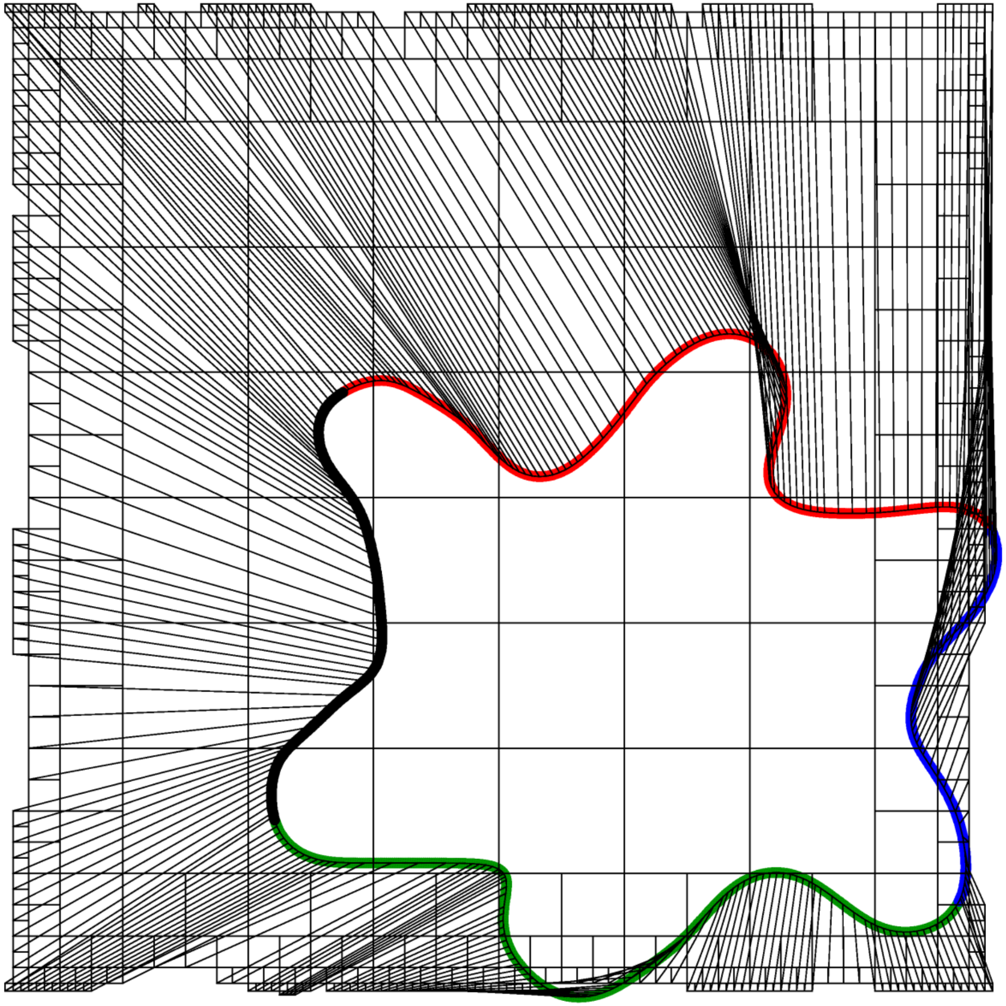
# The Meccano Method on T-meshes in 2-D

Step 3: Refine mesh with quadtree subdivisions to approach the boundary



# The Meccano Method on T-meshes in 2-D

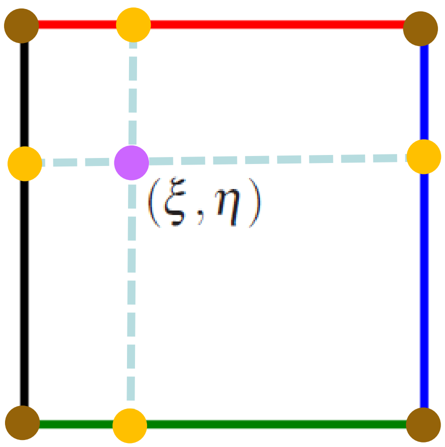
Step 4: Move the meccano boundary nodes to the object boundary

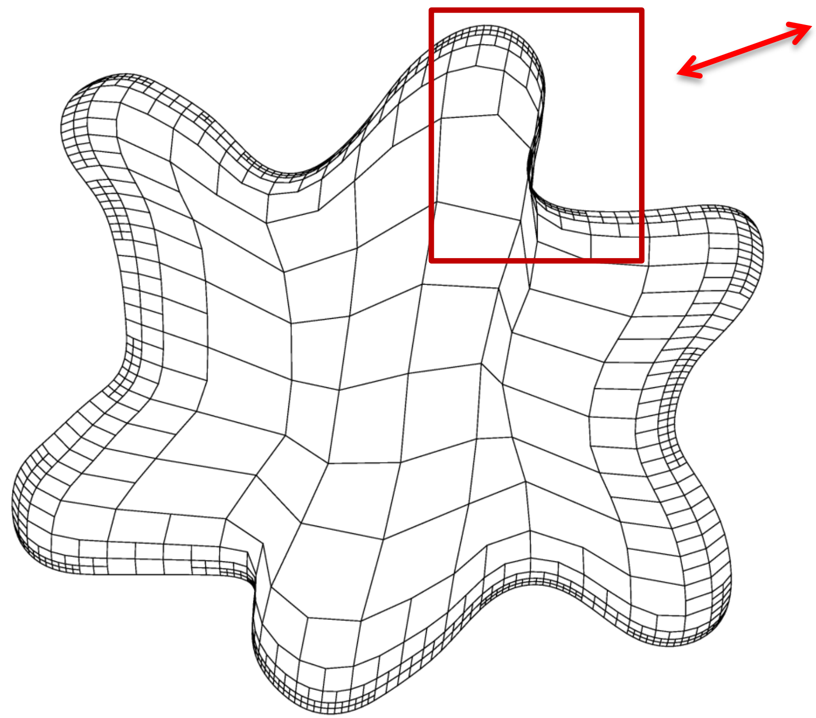


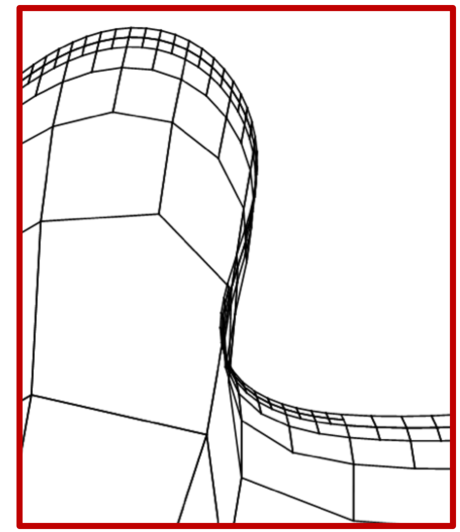
# The Meccano Method on T-meshes in 2-D

## Step 5: Inner node relocation with Coons patch to facilitate the optimization

$$
\begin{aligned}
\mathbf{x}(\xi,\eta) =& (1-\xi)\mathbf{x}(0,\eta)+\xi\mathbf{x}(1,\eta) \\
&+(1-\eta)\mathbf{x}(\xi,0)+\eta\mathbf{x}(\xi,1) \\
&-\begin{bmatrix}1-\xi & \xi\end{bmatrix}\begin{bmatrix}\mathbf{x}(0,0) & \mathbf{x}(0,1)\\ \mathbf{x}(1,0) & \mathbf{x}(1,1)\end{bmatrix}\begin{bmatrix}1-\eta\\ \eta\end{bmatrix}
\end{aligned}
$$



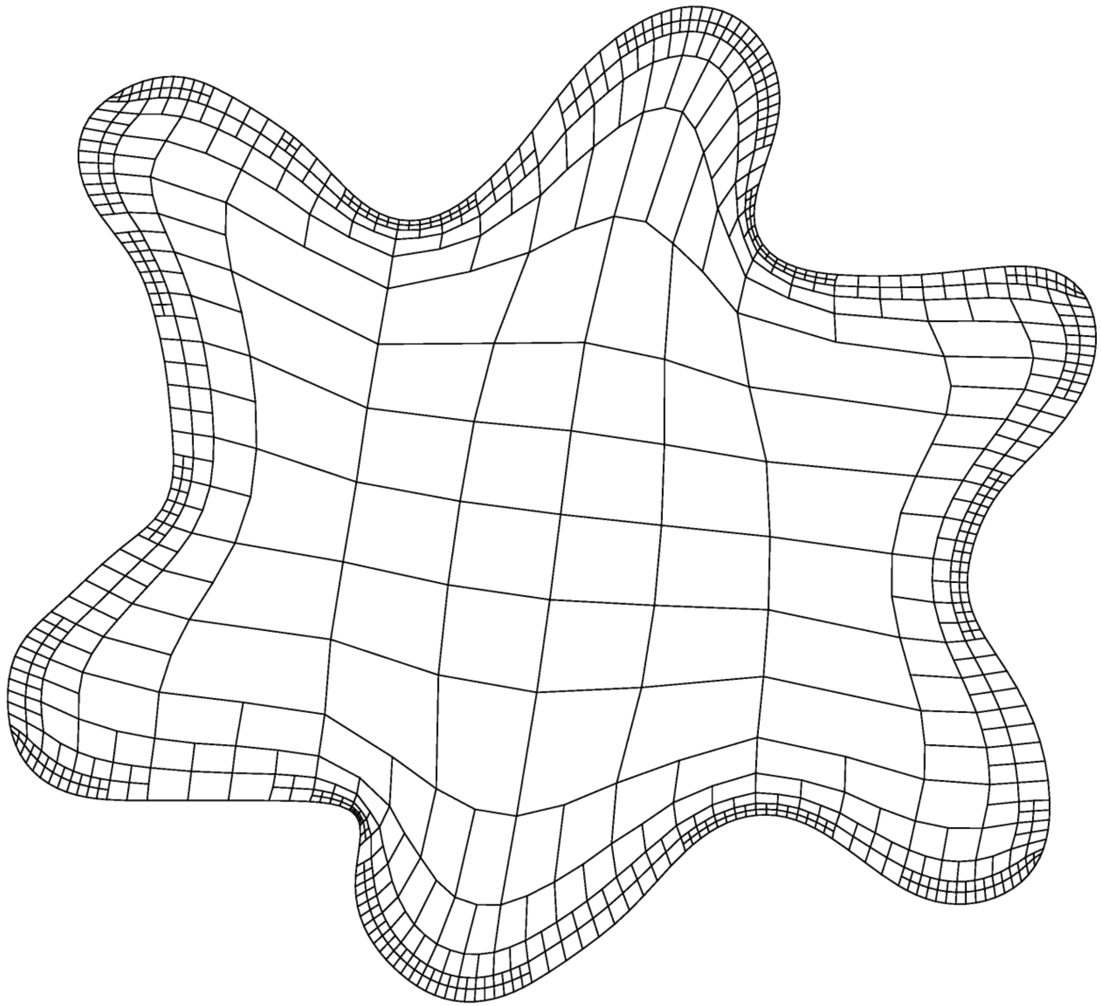





Mesh folder

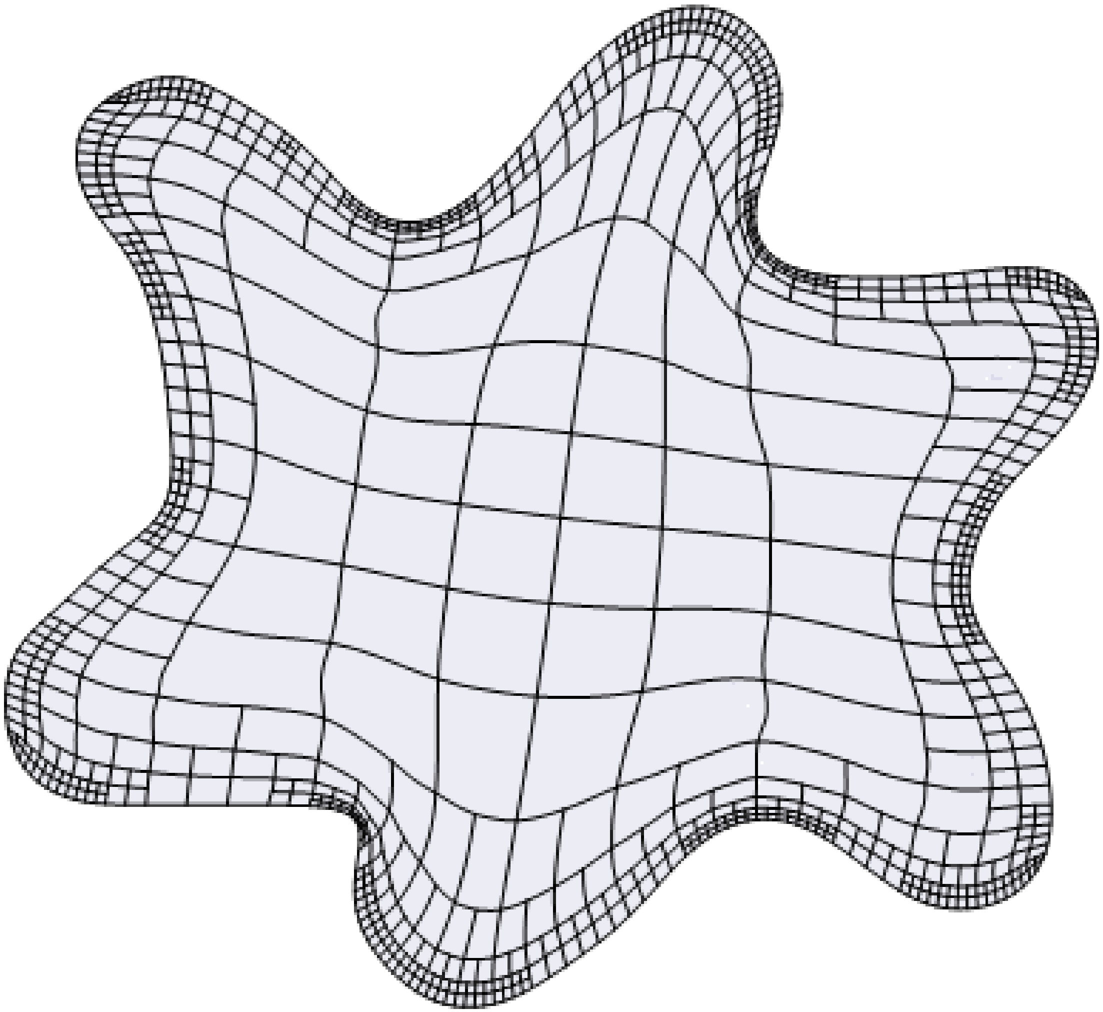
# The Meccano Method on T-meshes in 2-D

Step 6: Simultaneous Untangling and Smoothing (SUS) of the T-mesh



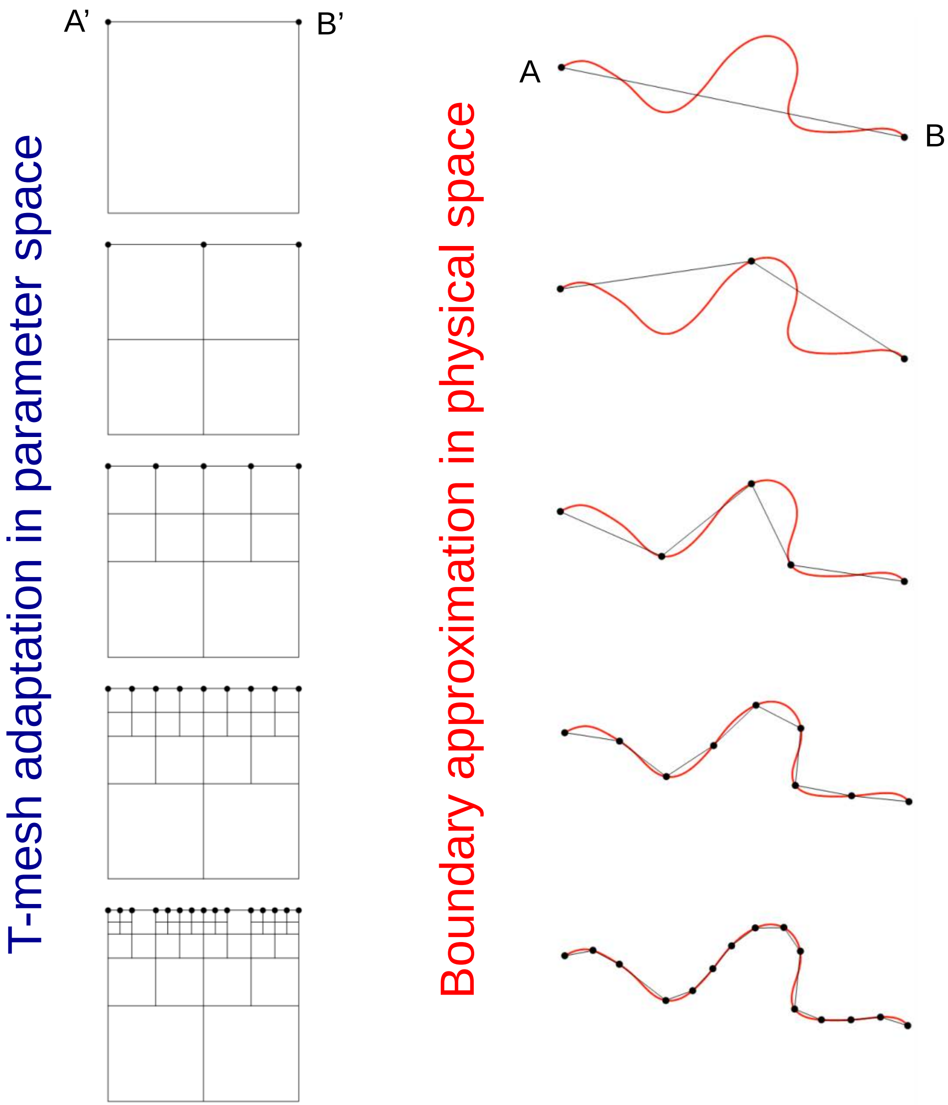
# The Meccano Method on T-meshes in 2-D

## Step 7: T-spline representation of the spot

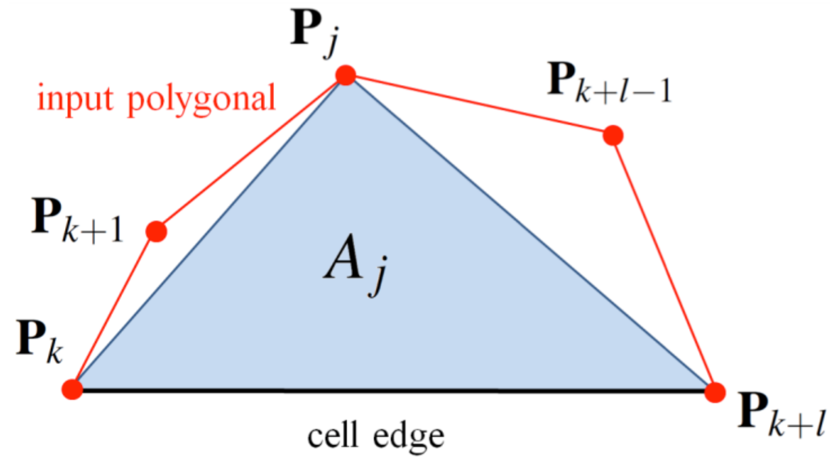


# Boundary Approach in 2-D

Input data: Boundary polyline approximation (red color line)




## Boundary edge refinement criterion:



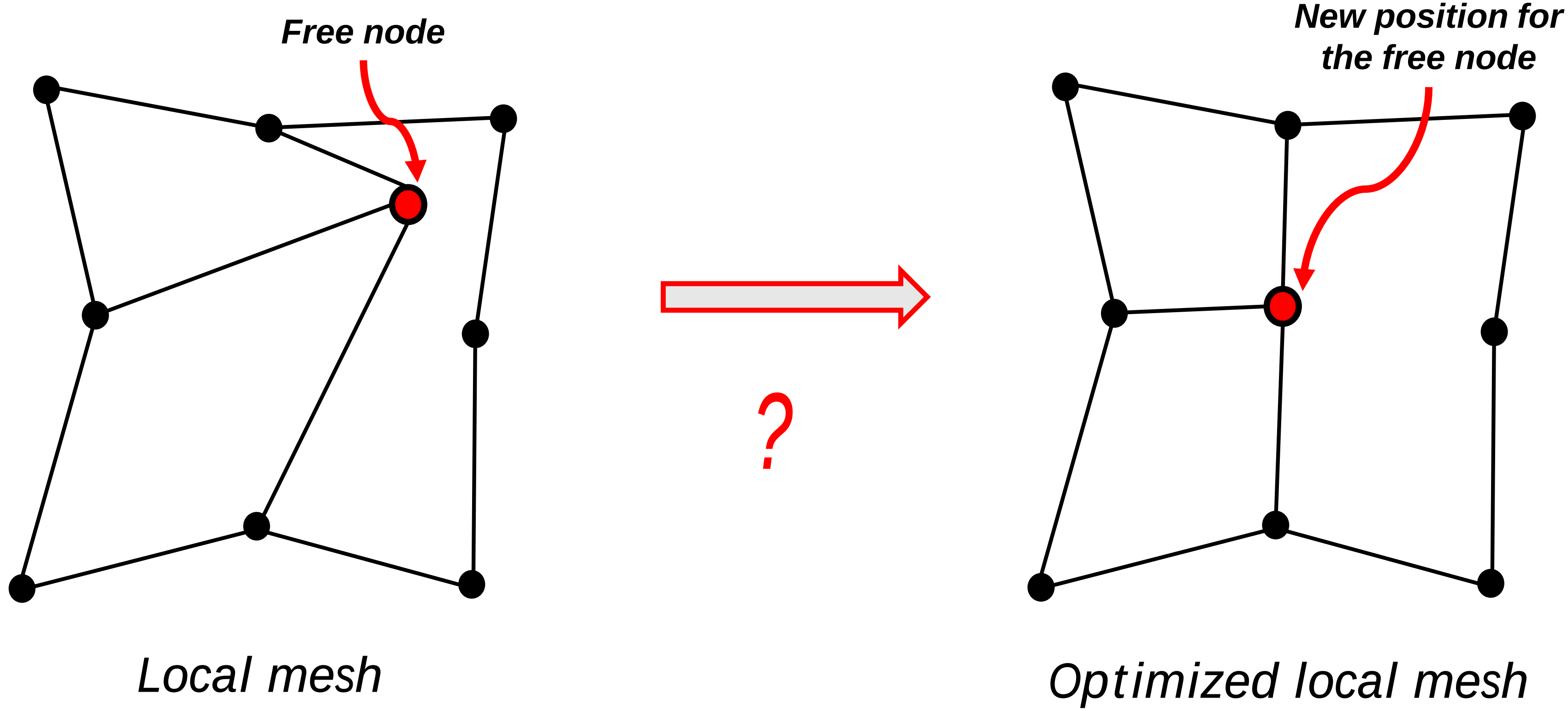

$$\exists j : \; A_j > \varepsilon \Rightarrow \text{refine}$$

$$A_j = \text{Area}(\mathbf{P}_k, \mathbf{P}_j, \mathbf{P}_{k+l})$$

$$k+1 \leqslant j \leqslant k+l-1$$

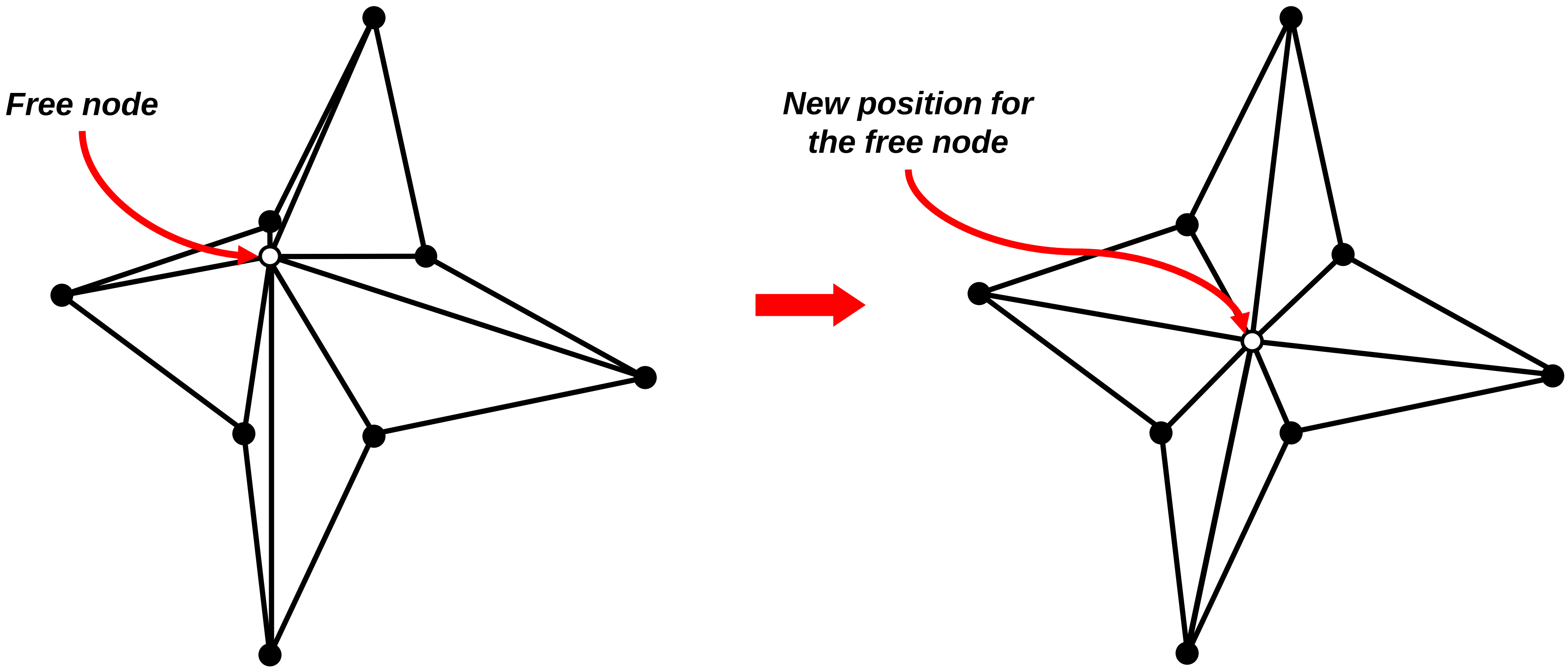
## Local optimization

***Objective:* Improve the quality of the local mesh by minimizing an objective function**



## Local optimization

***Objective:* Improve the quality of the local mesh by minimizing an objective function**

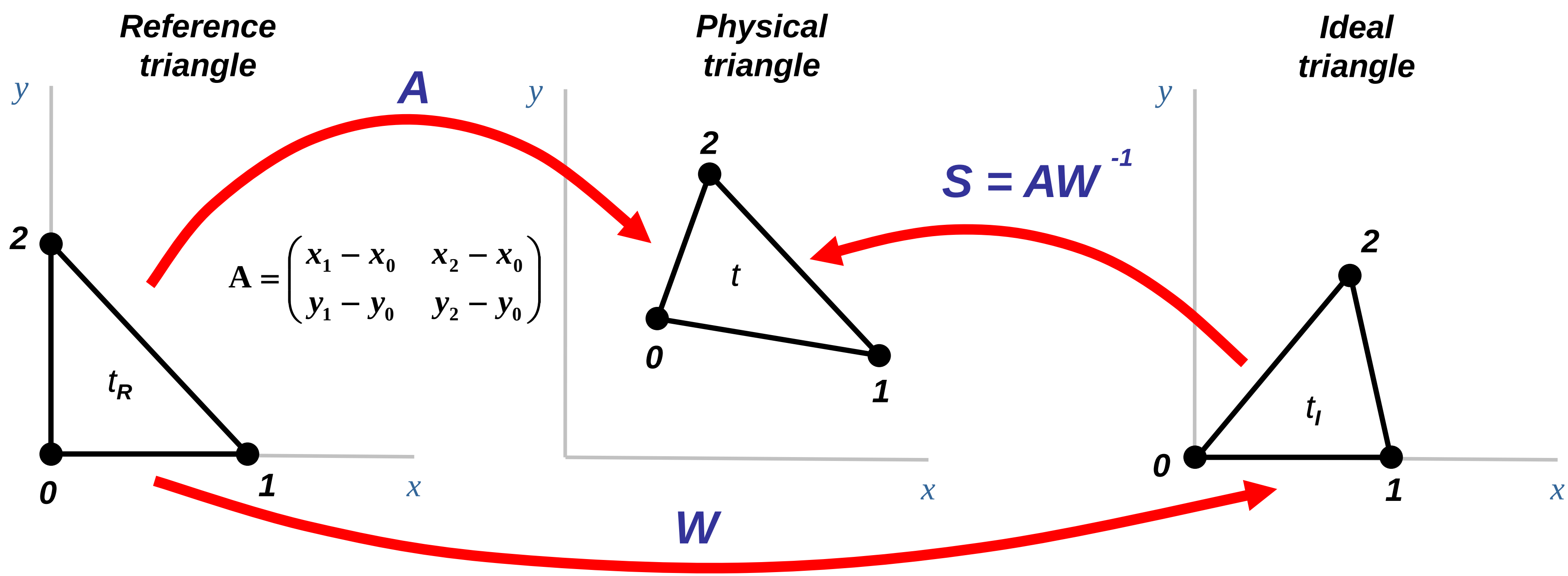



*Local mesh*

*Optimized local mesh*

# Simultaneous Untangling and Smoothing (CMAME 2003)

## Weighted Jacobian Matrix on a Plane




$t_I \xrightarrow{S} t$ $\quad S = AW^{-1}$: **Weighted Jacobian matrix**

***An algebraic quality metric of t (mean ratio)***

$$\mathbf{q} = \frac{2\boldsymbol{\sigma}}{\|\mathbf{S}\|^2} = \frac{1}{\eta}$$

**where:**

$$\|\mathbf{S}\| = \sqrt{\mathbf{tr}\left(\mathbf{S}^T\mathbf{S}\right)}$$

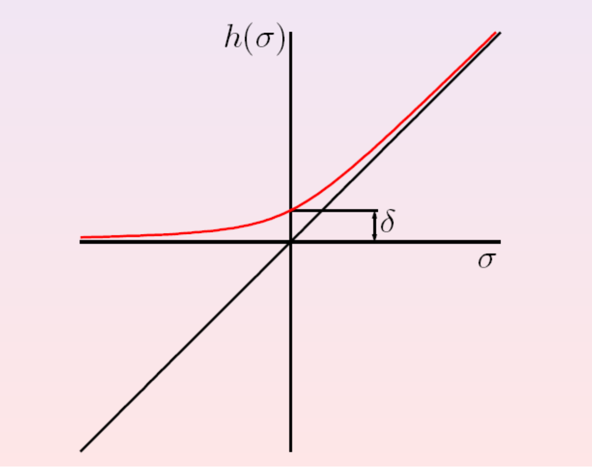
$$\boldsymbol{\sigma} = \det(\mathbf{S})$$

# Simultaneous Untangling and Smoothing (CMAME 2003)

Local objective function for plane triangulations

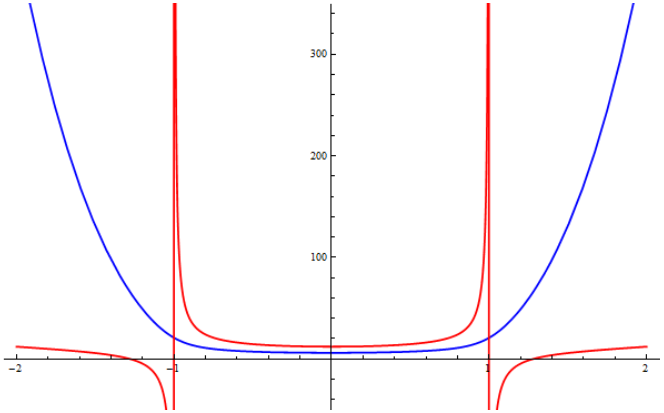
SUS Code: Freely-available in http://www.dca.iusiani.ulpgc.es/proyecto2012-2014

**Original function:** $$K(\mathbf{x}) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2\,\sigma_m}$$




$$h(\sigma) = \tfrac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\delta^2}\right)$$

**Modified function:** $$K^*(\mathbf{x}) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2h(\sigma_m)}$$

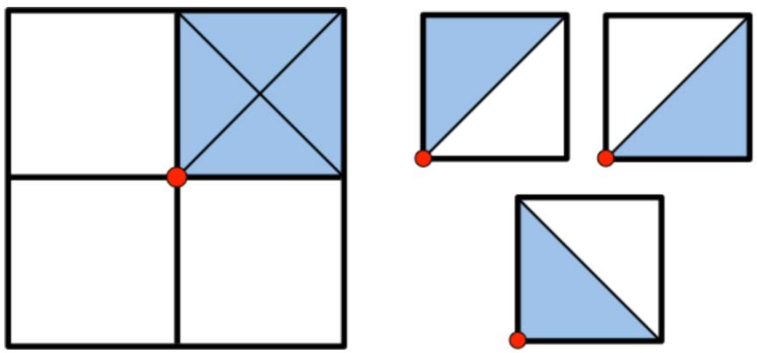


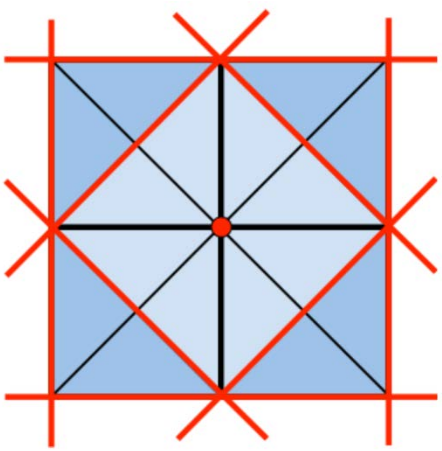

Modified function (blue) is regular in all $R^2$ and it approximates the same minimum that the original function (red). Moreover, it allows a simultaneous untangling and smoothing of triangular meshes

# Simultaneous Untangling and Smoothing of T-meshes

Triangle decomposition of the T-mesh cells

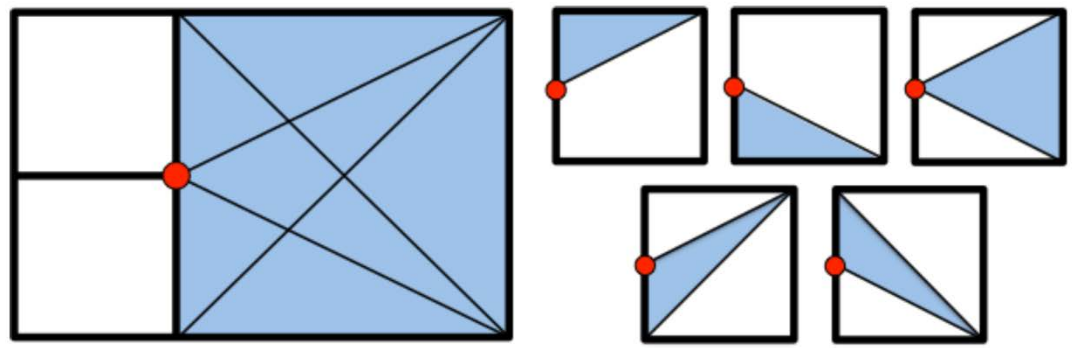
## Case 1: Free node is a regular node

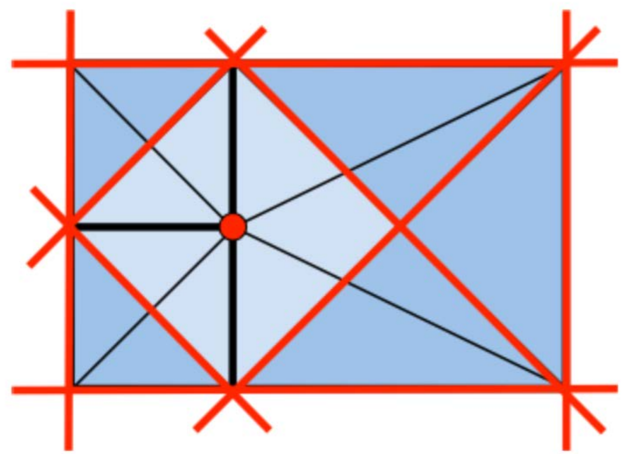




*Barriers and feasible region for a regular node*

## Case 2: Free node is a hanging node





*Barriers and feasible region for a hanging node*

# Simultaneous Untangling and Smoothing of T-meshes

Optimization is guided by the parametric T-mesh

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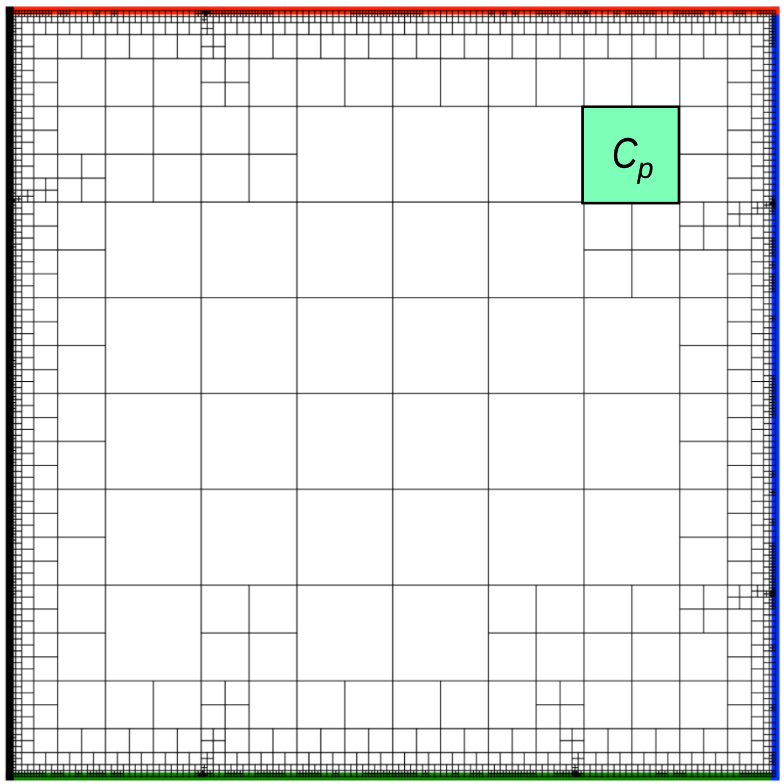
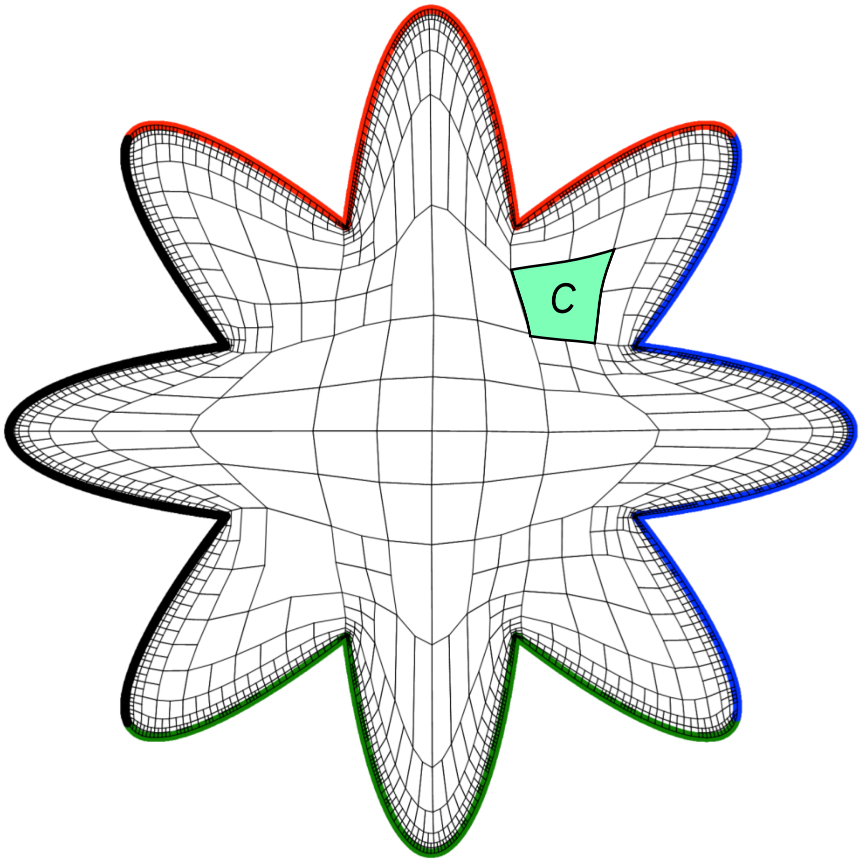
***Physical cell C must be as similar as possible to the counterpart in the parametric space $C_p$***




Parametric T-mesh

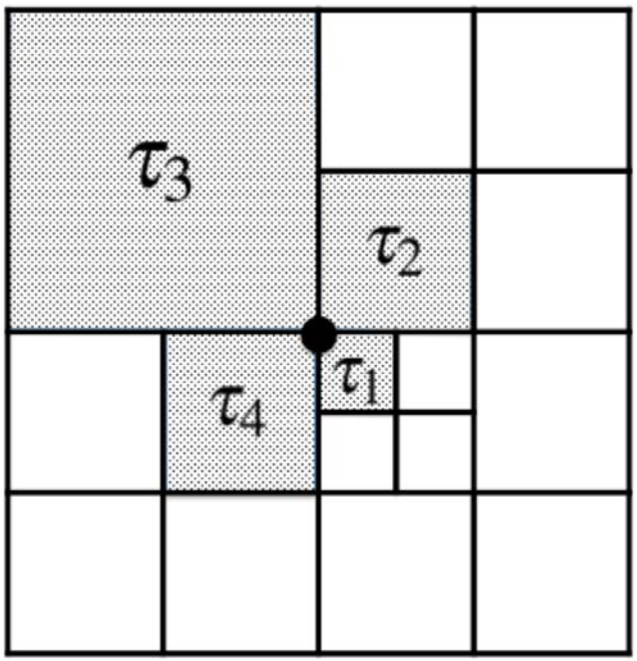



Optimized physical T-mesh

# Simultaneous Untangling and Smoothing of T-meshes

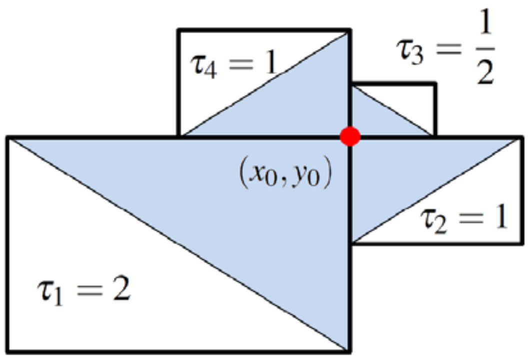
## Weighted objective functions (regular node)

$$K_\tau^*(\mathbf{x}) = \tau_1 \sum_{m=1}^{3} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_2 \sum_{m=4}^{6} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_3 \sum_{m=7}^{9} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_4 \sum_{m=10}^{12} \frac{\|S_m\|^2}{2h(\sigma_m)}$$

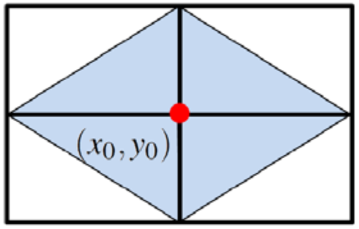



All possible weights for balanced quadtrees:

$\Rightarrow$ $\tau_1 = 1,\ \tau_2 = \tau_4 = 2$ y $\tau_3 = 4$




Non-conformal mesh

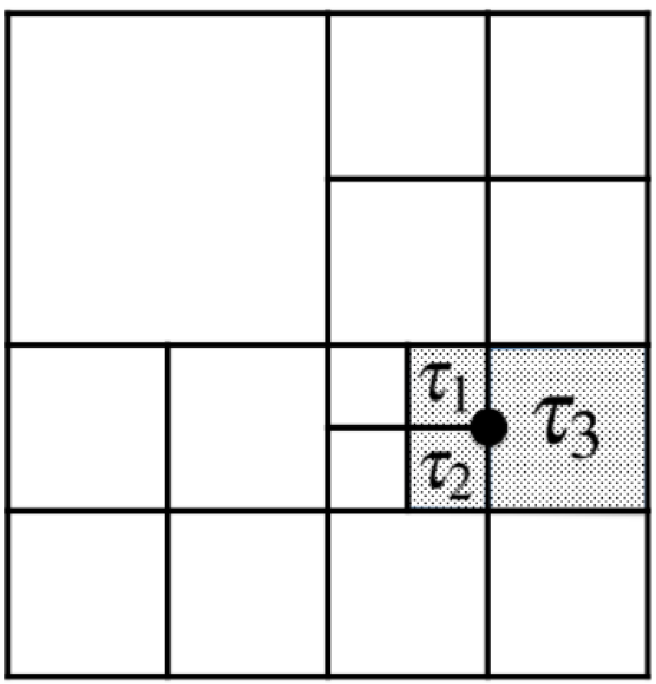



Conformal mesh

# Simultaneous Untangling and Smoothing of T-meshes

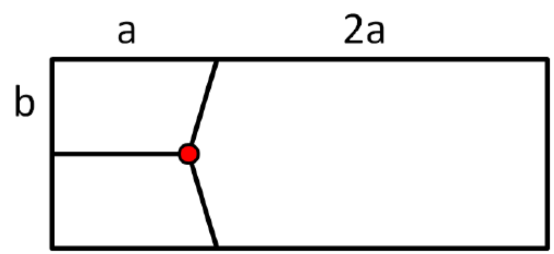
## Weighted objective functions (hanging node)

$$K_{\tau}^{*}(\mathbf{x}) = \tau_1 \sum_{m=1}^{3} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_2 \sum_{m=4}^{6} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_3 \sum_{m=7}^{11} \frac{\|S_m\|^2}{2h(\sigma_m)}$$

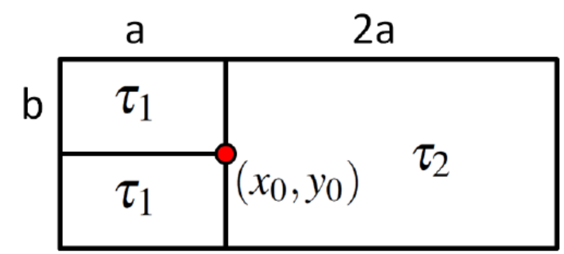



All possible weights for balanced quadtrees:

$$\tau_1 = \tau_2 = 1 \text{ y } \tau_3 = \frac{8}{5}$$




Optimized mesh
without weights



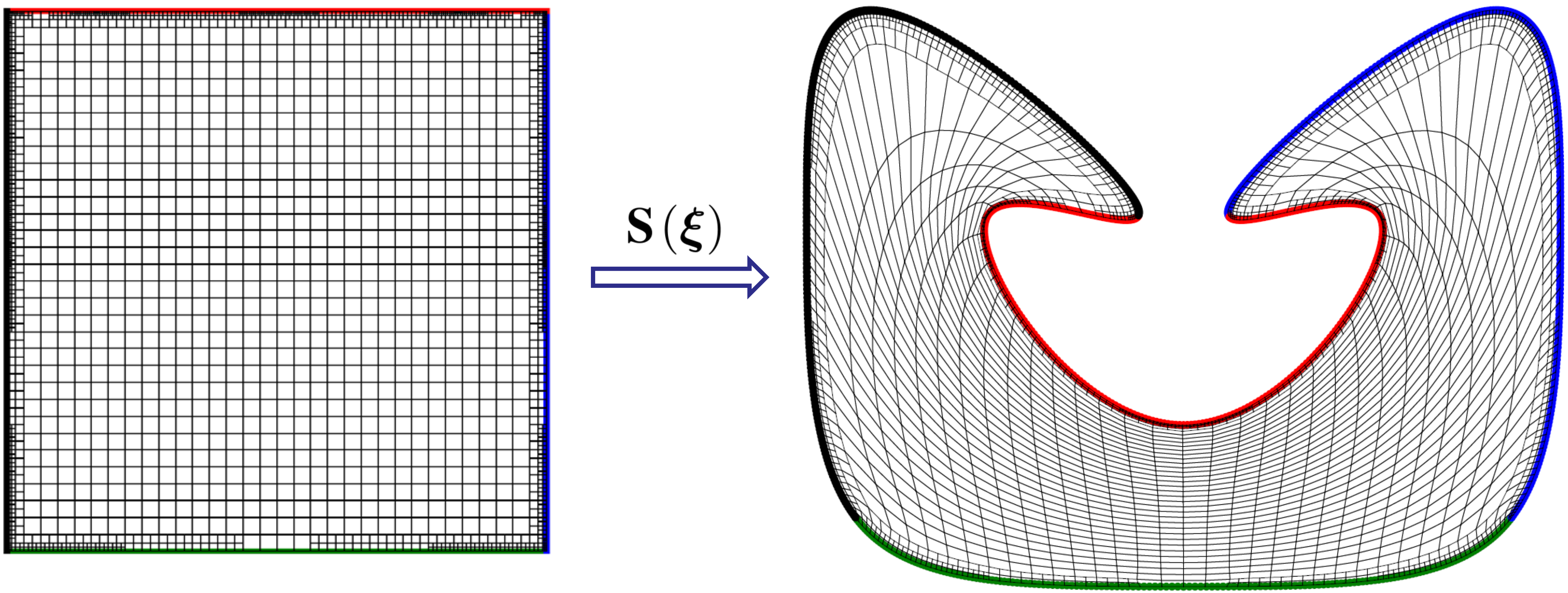

Optimized mesh
with weights

# Simultaneous Untangling and Smoothing of T-meshes

## T-mesh transformation along the SUS process: Example

T-mesh

T-spline

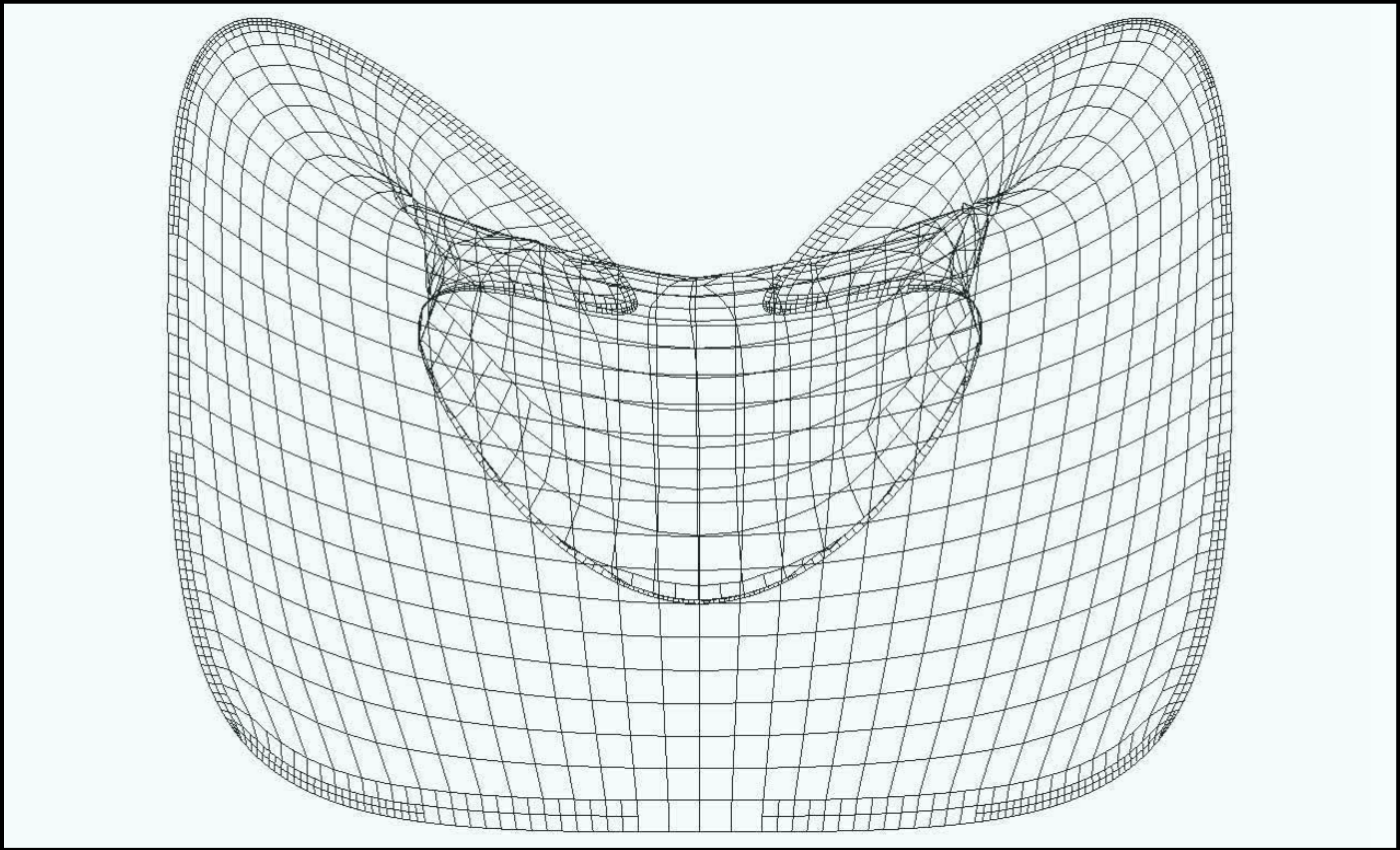



Parameter space

Physical space

# Simultaneous Untangling and Smoothing of T-meshes

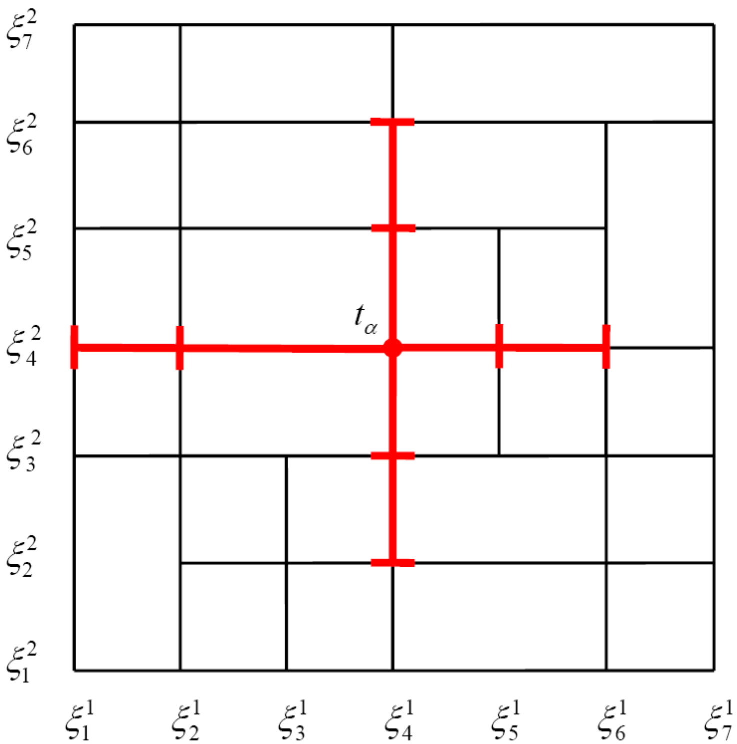
T-mesh transformation along the SUS process: Video



# T-spline Basis Functions in 2-D

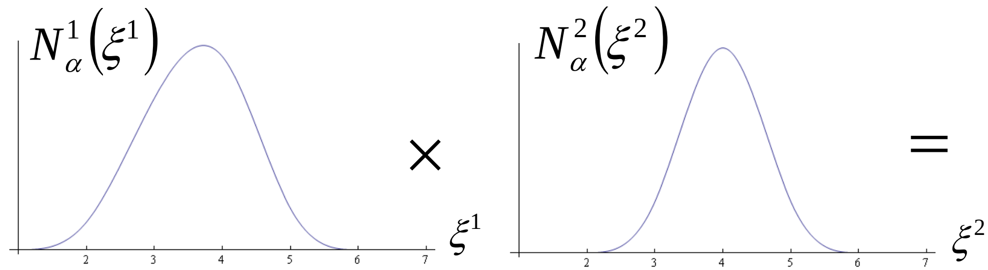
## Example of a bivarite  basis on a T-mesh

***T-mesh and anchor $t_a$***



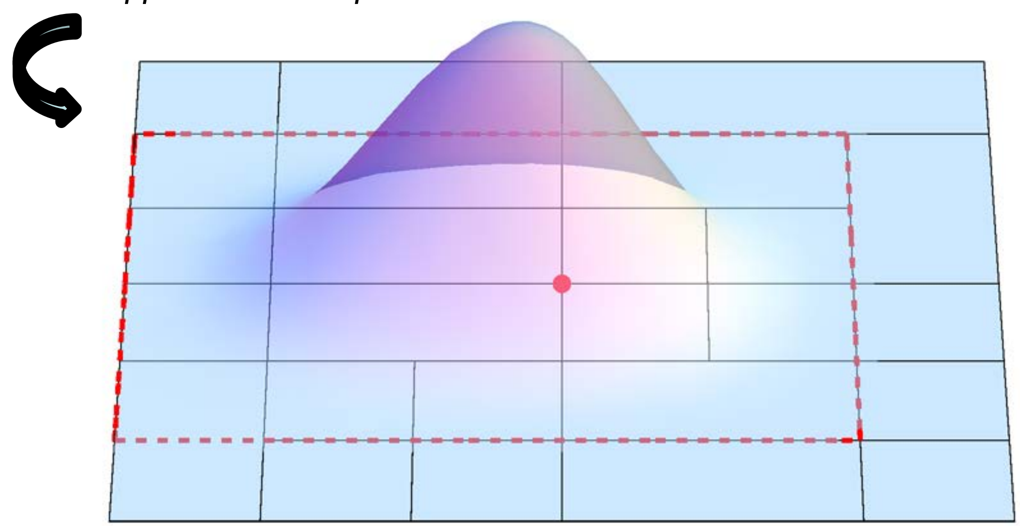

***Knots associated to anchor $t_a$ :***

$$\Xi_\alpha^1 = \left\{\xi_1^1, \xi_2^1, \xi_4^1, \xi_5^1, \xi_6^1\right\} \qquad \Xi_\alpha^2 = \left\{\xi_2^2, \xi_3^2, \xi_4^2, \xi_5^2, \xi_6^2\right\}$$




$$B_\alpha\left(\xi^1, \xi^2\right) = N_\alpha^1\left(\xi^1\right) N_\alpha^2\left(\xi^2\right)$$

*support of the T-spline*



***Bivariate Cubic T-spline Basis Function***

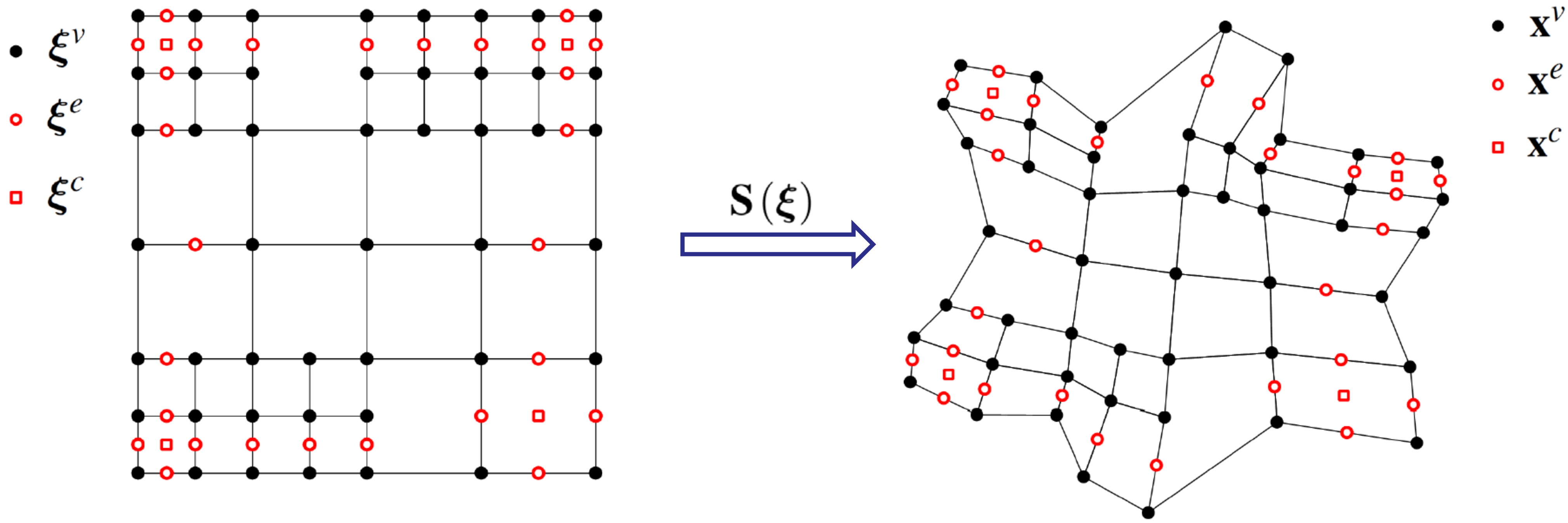
# T-spline Parameterization

Determination of control points by imposing interpolation conditions

$$\mathbf{S}\left(\boldsymbol{\xi}\right) = \sum_{\alpha \in A} \mathbf{P}_{\alpha}\, R_{\alpha}\left(\boldsymbol{\xi}\right)$$

where

$$R_{\alpha}(\boldsymbol{\xi}) = \frac{w_{\alpha} B_{\alpha}(\boldsymbol{\xi})}{\sum_{\beta \in A} w_{\beta} B_{\beta}(\boldsymbol{\xi})}$$



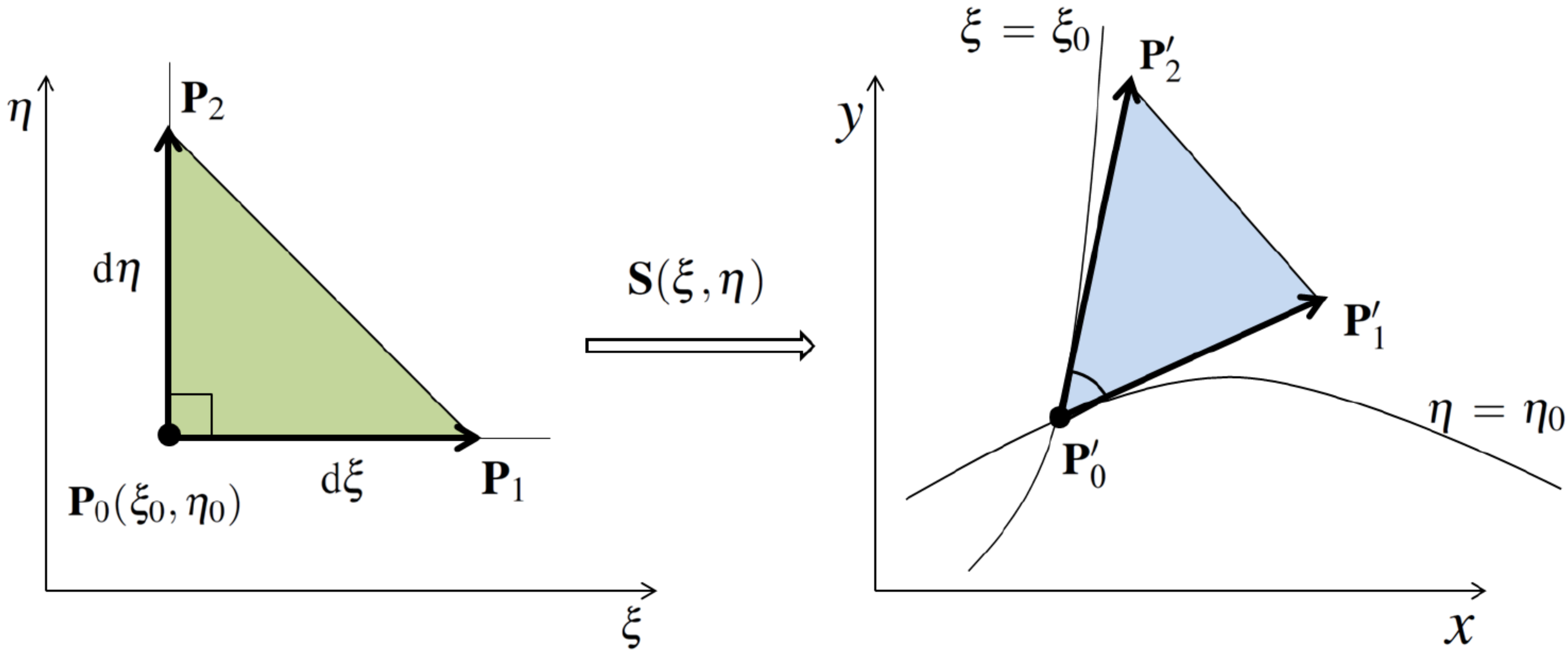
# Mean Ratio Jacobian $J_r(\xi)$ of Parametric Transformation

A quality metric of the T-spline mapping at any point $P_0$

SIANI

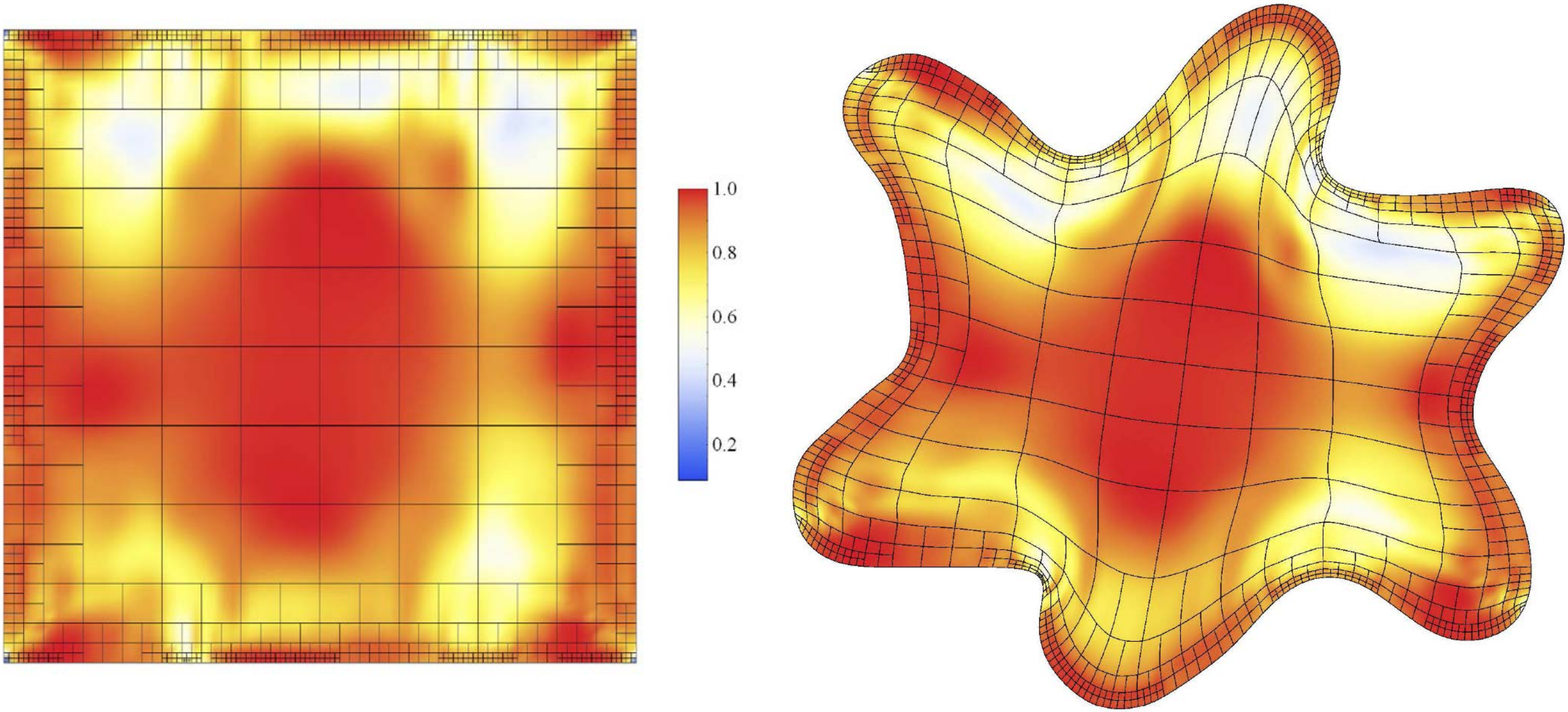
$$-1 \leq J_r(\boldsymbol{\xi}) = \frac{2\det(\boldsymbol{J})}{\|\boldsymbol{J}\|^2} \leq 1$$

where $\boldsymbol{J}$ is the jacobian matrix of the T-spline mapping $\mathbf{S}$



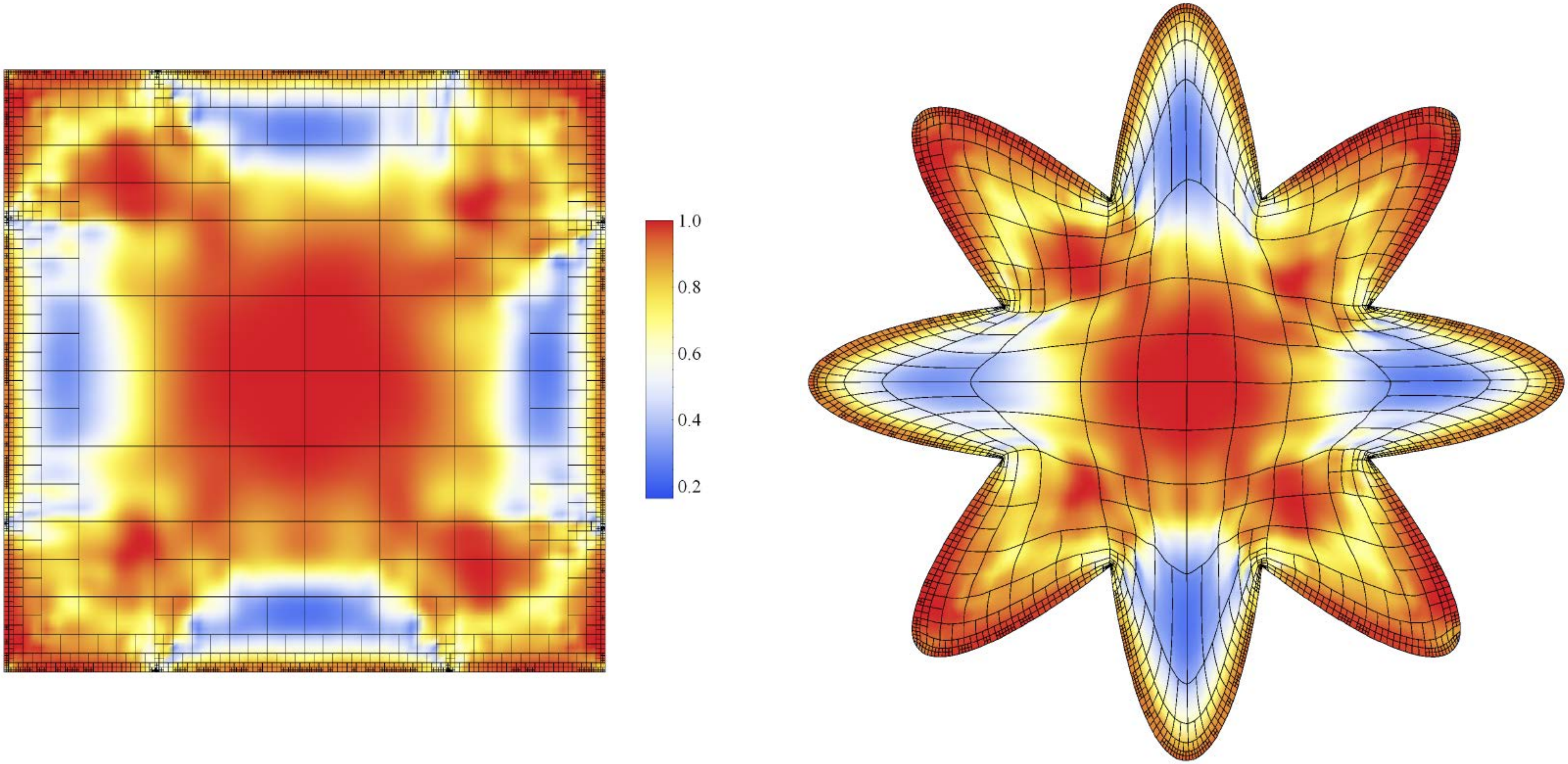
# Applications: Isogeometric Modeling

The Spot (Mean Ratio Jacobian)



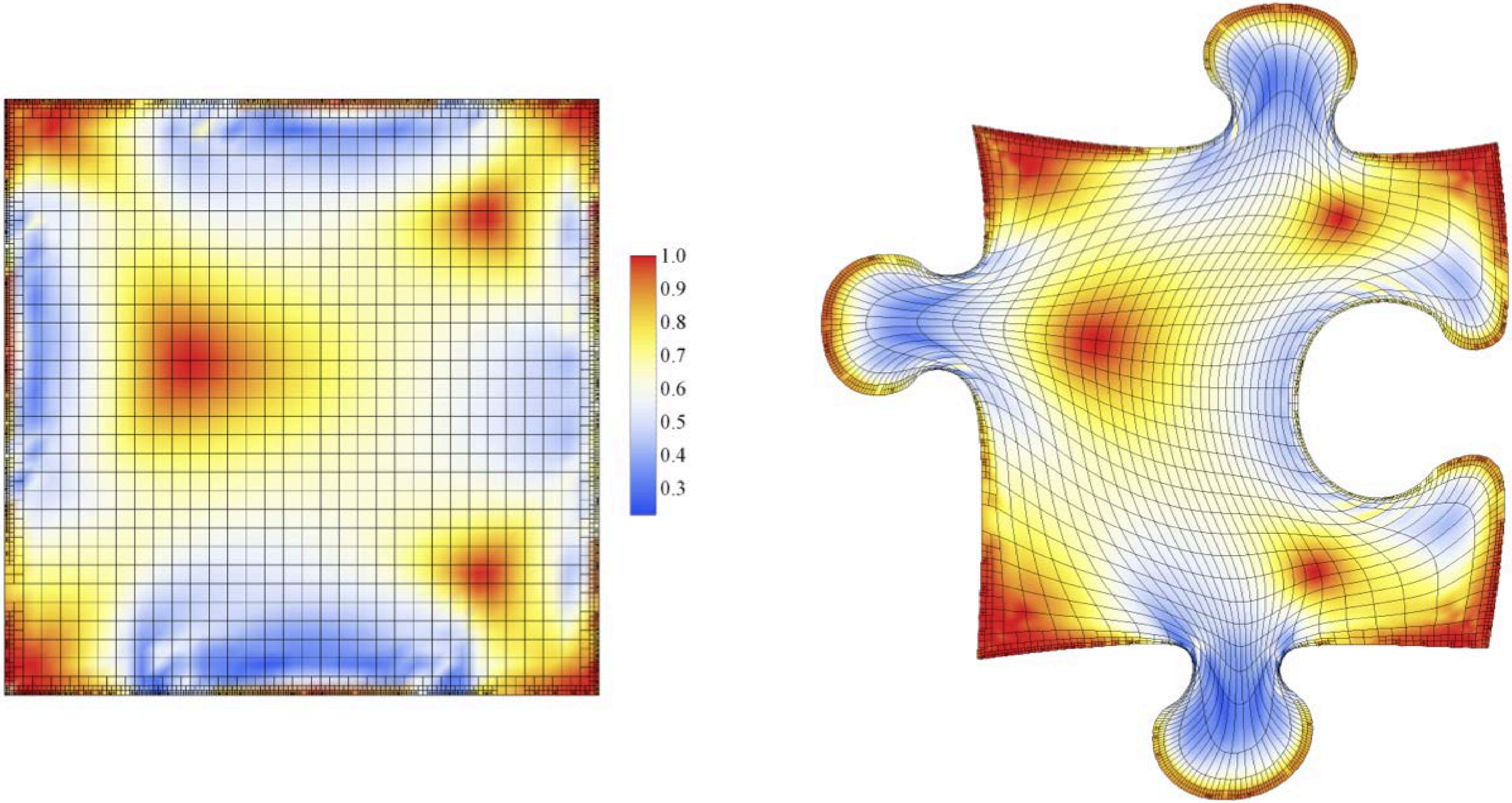
# Applications: Isogeometric Modeling

The Flower (Mean Ratio Jacobian)



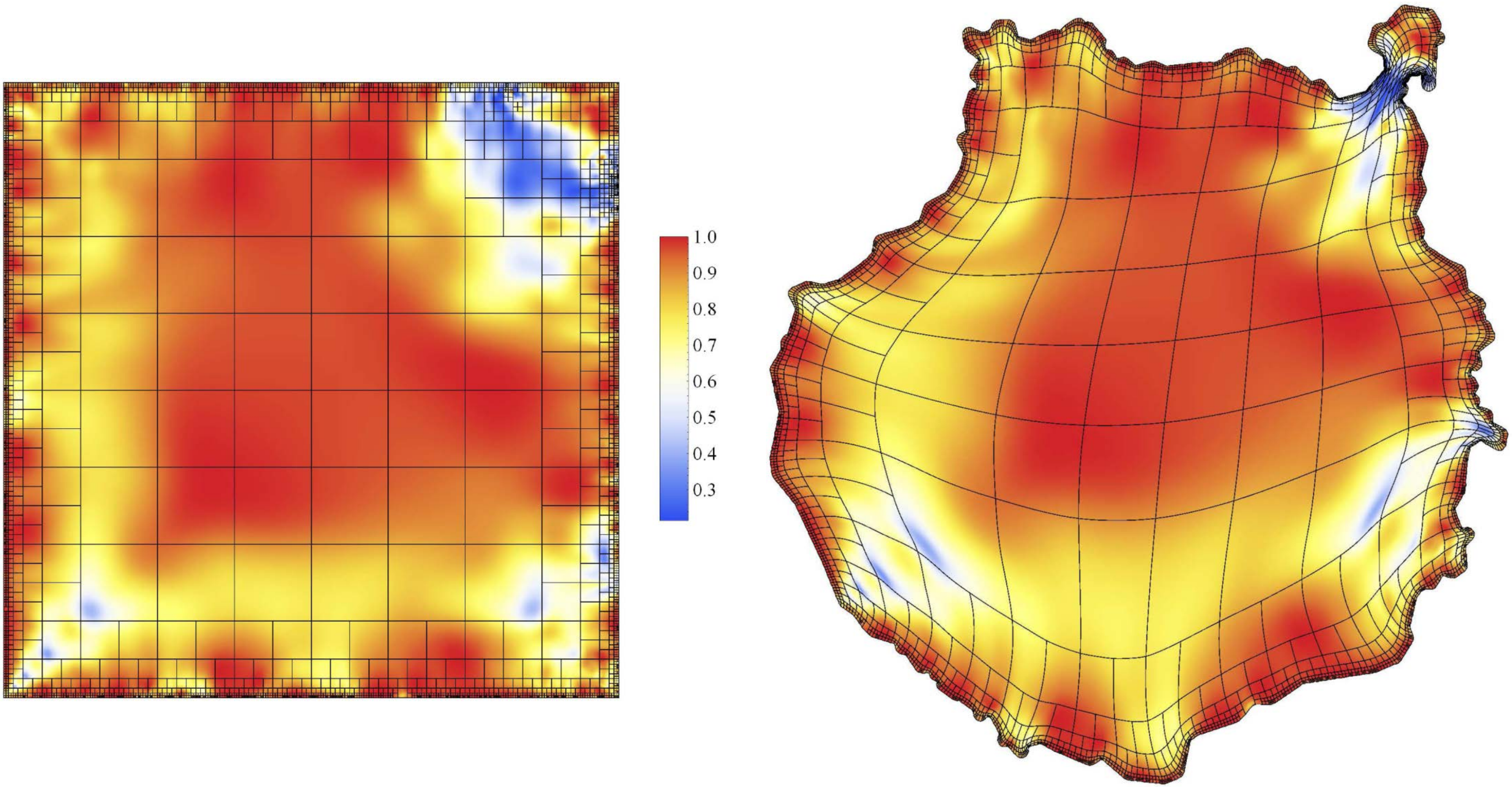
# Applications: Isogeometric Modeling

Puzzle Piece (Mean Ratio Jacobian)



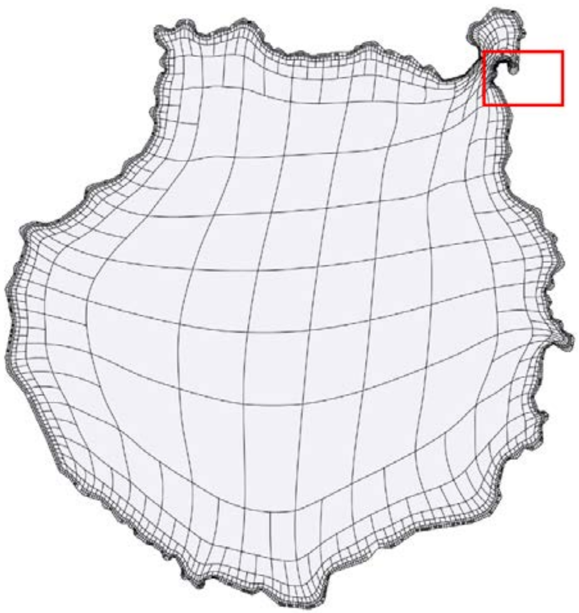
# Applications: Isogeometric Modeling

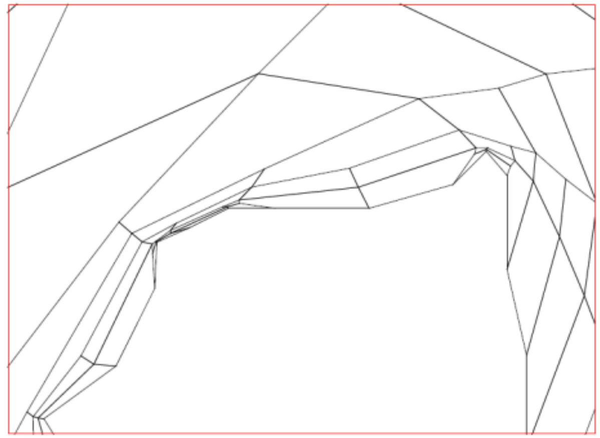
Gran Canaria Island (Mean Ratio Jacobian)



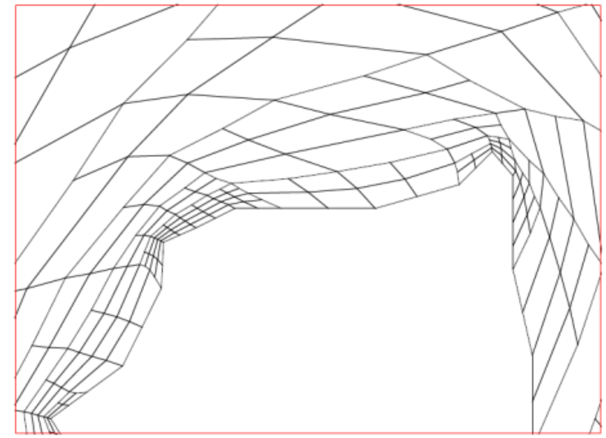
# Applications: Isogeometric Modeling

Gran Canaria Island (adaptive refinement to improve the mean ratio Jacobian)

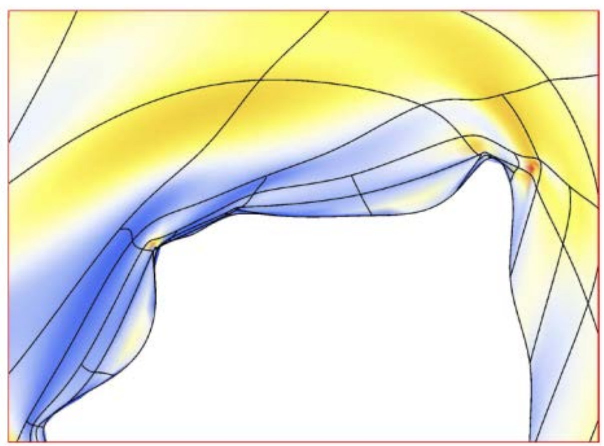




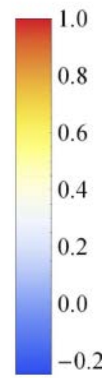
*Initial T-mesh*

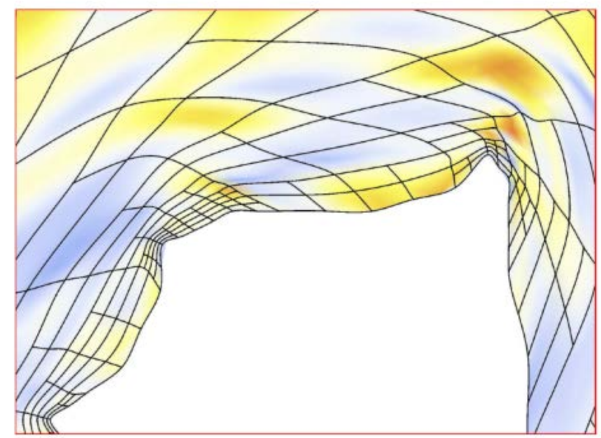


*Refined T-mesh*



*Initial T-spline & Mean ratio Jacobian*



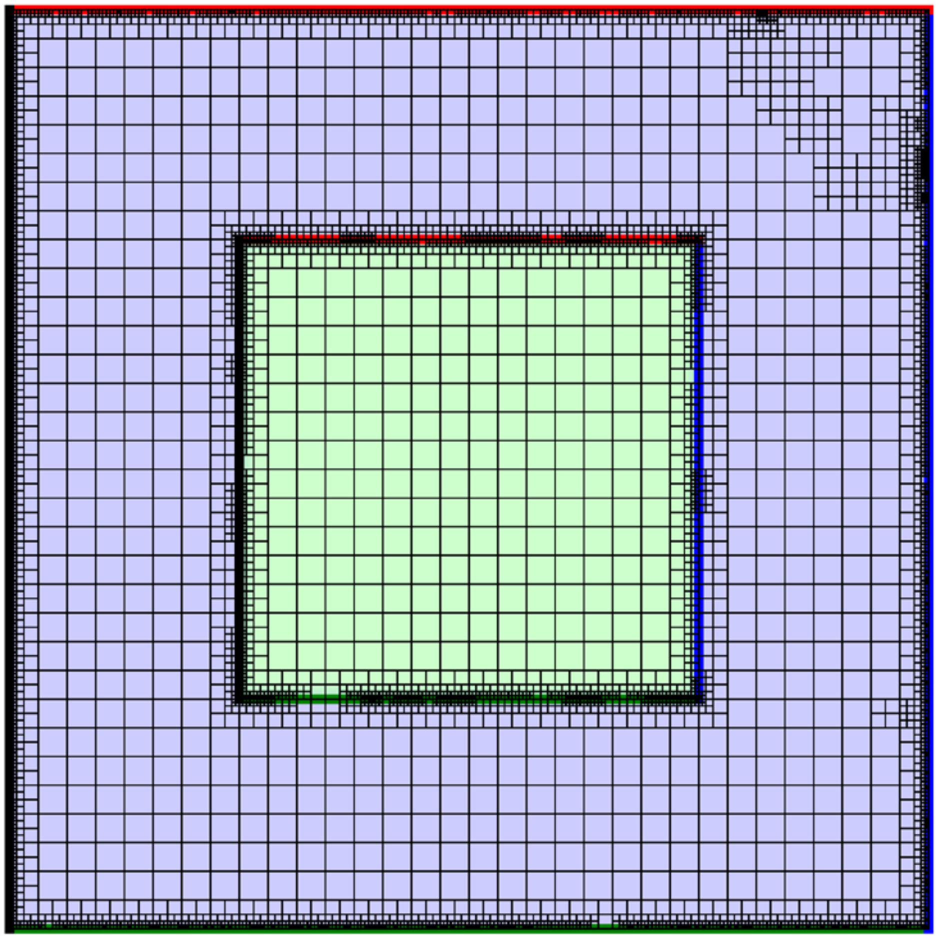



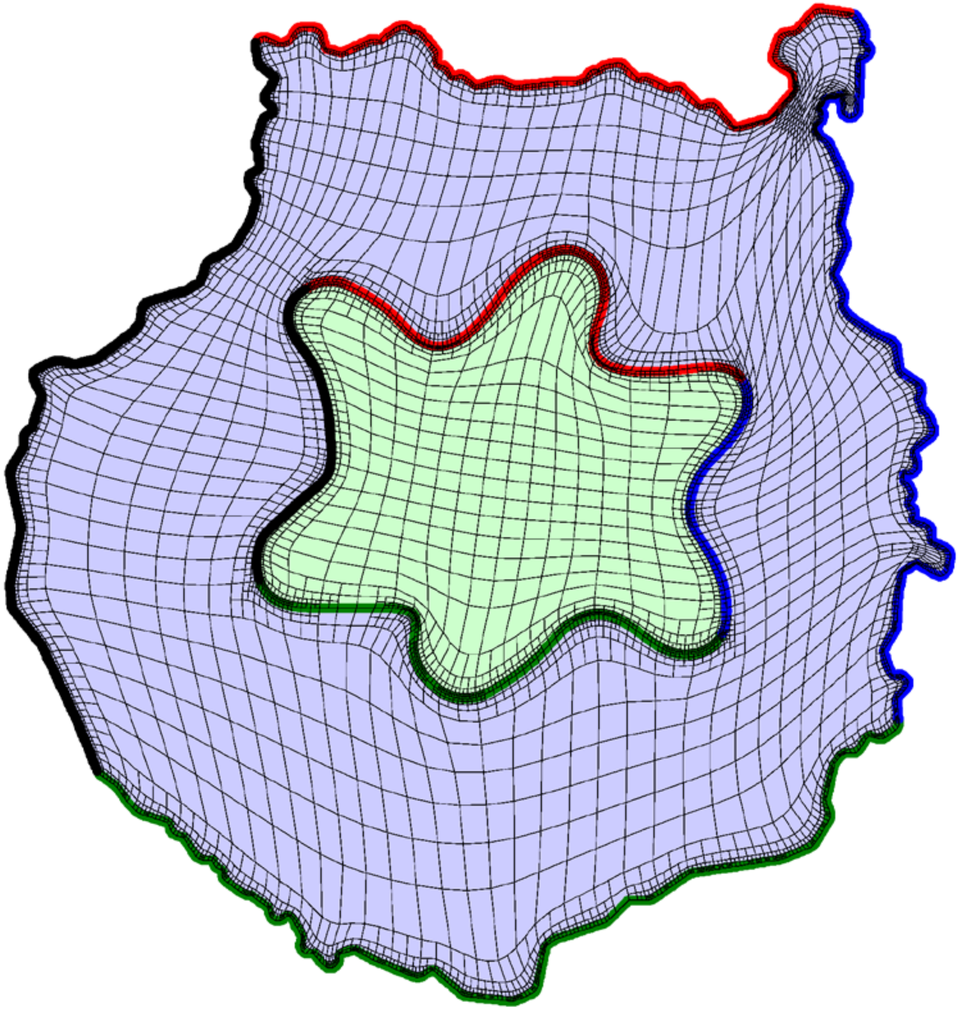
*Refined T-spline & Mean ratio Jacobian*

*No negative Jacobian after refinement!*

# Applications: Isogeometric Modeling

Geometries with several materials

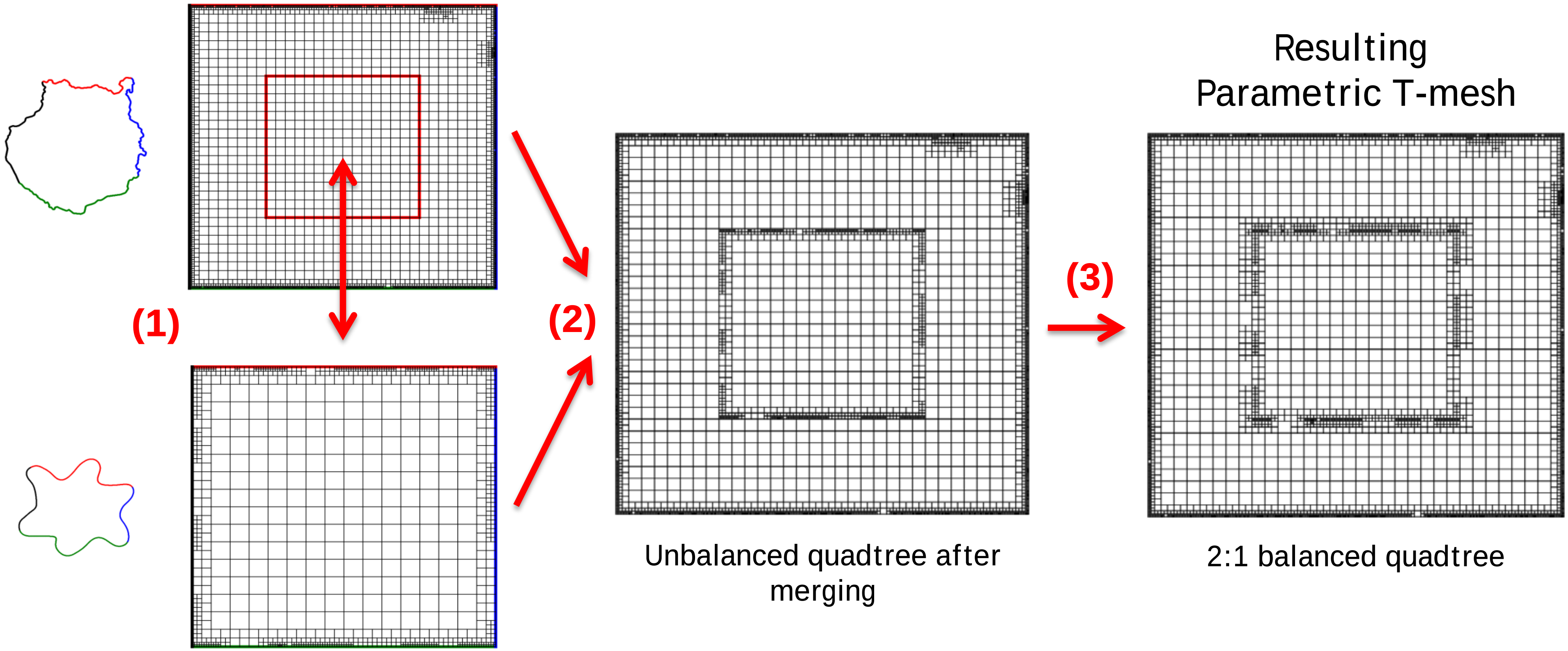




# Applications: Isogeometric Modeling

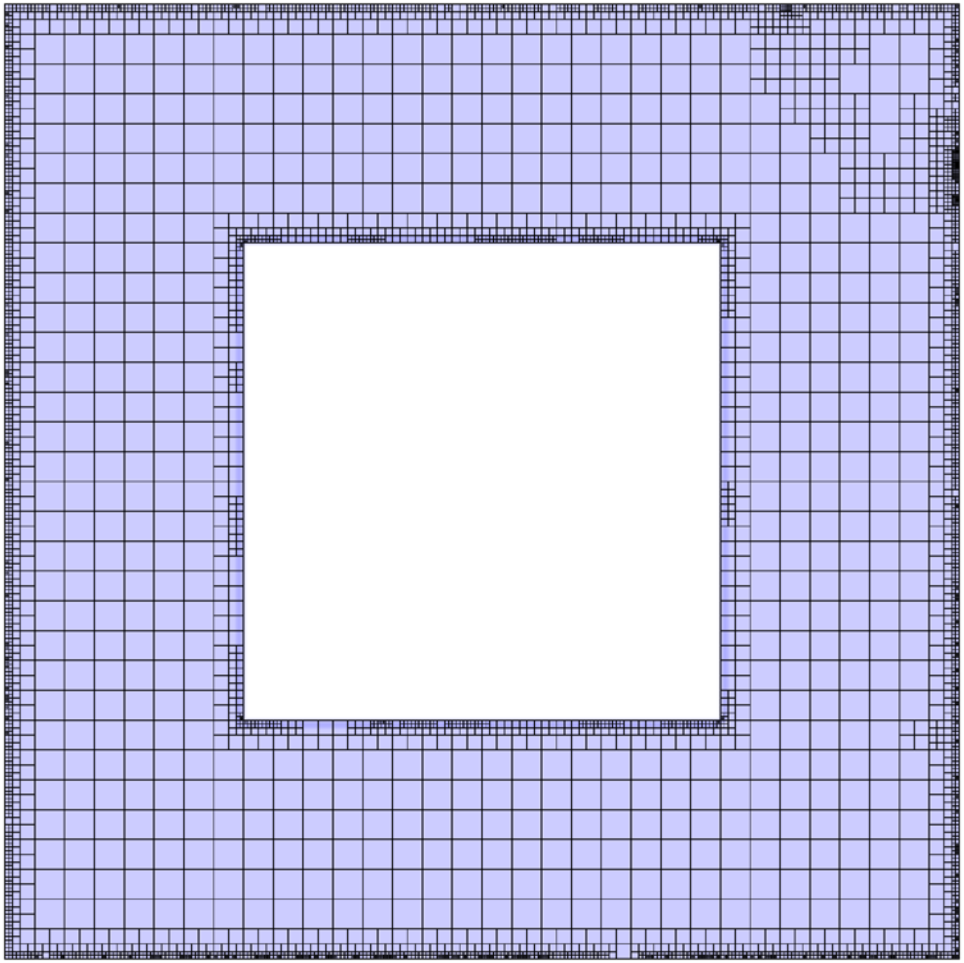
## Strategy for embedded planar geometries: Three steps

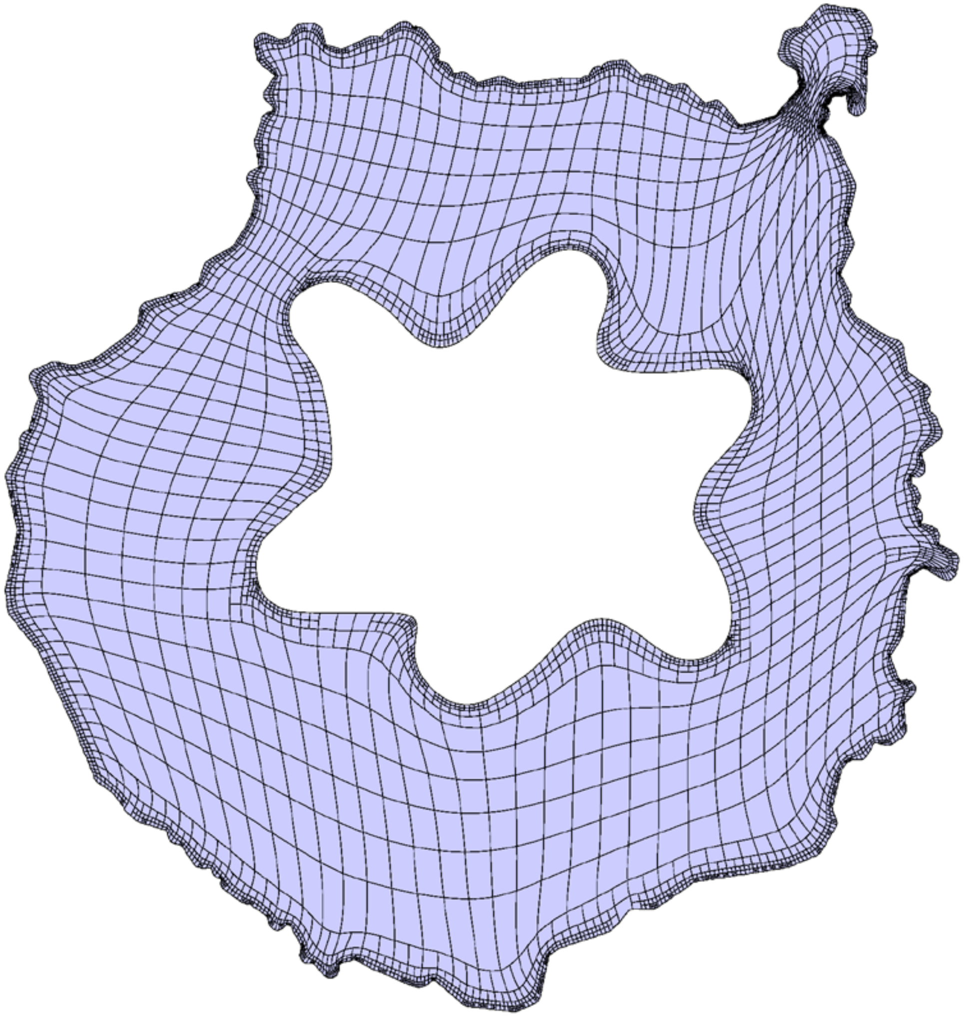
1. Individual quadtree approximation for each geometry (they can be processed in parallel)
2. Insert a quadtree in a region of another quadtree
3. Balance the resulting quadtree



# Applications: Isogeometric Modeling

## Geometries with holes





# Applications: Isogeometric Modeling

Embedded planar geometries (T-spline representation in physical space)

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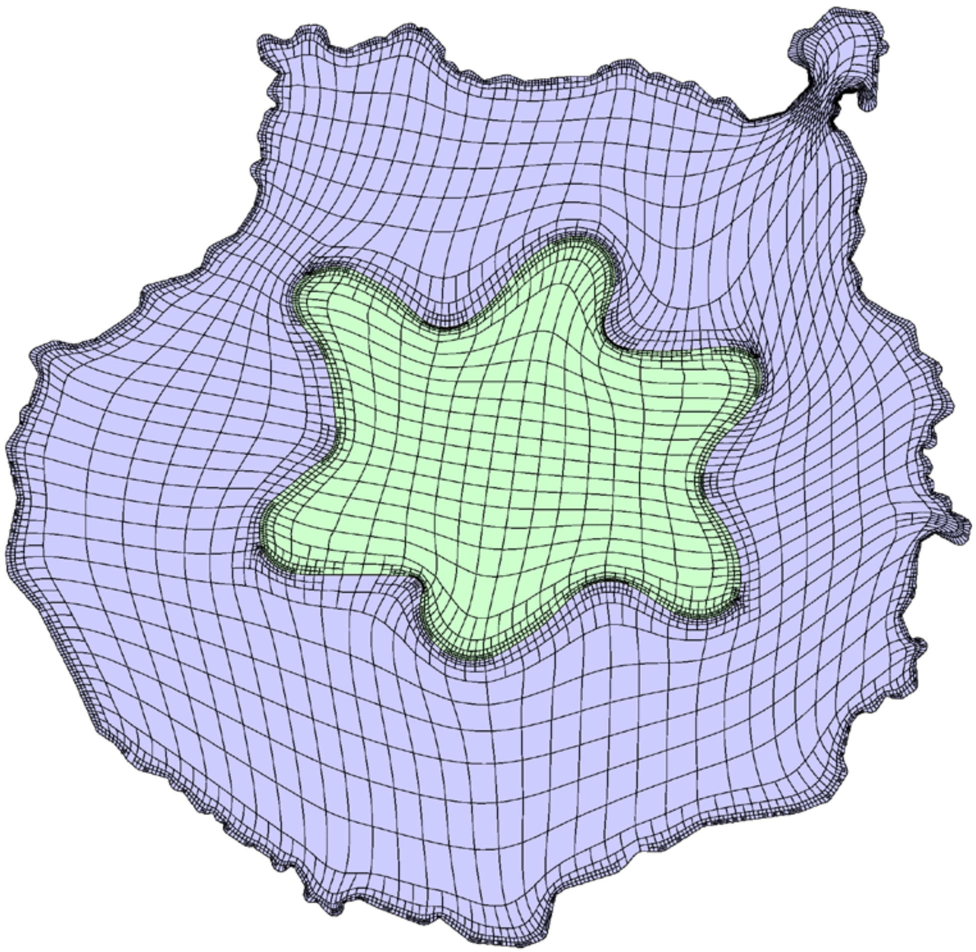
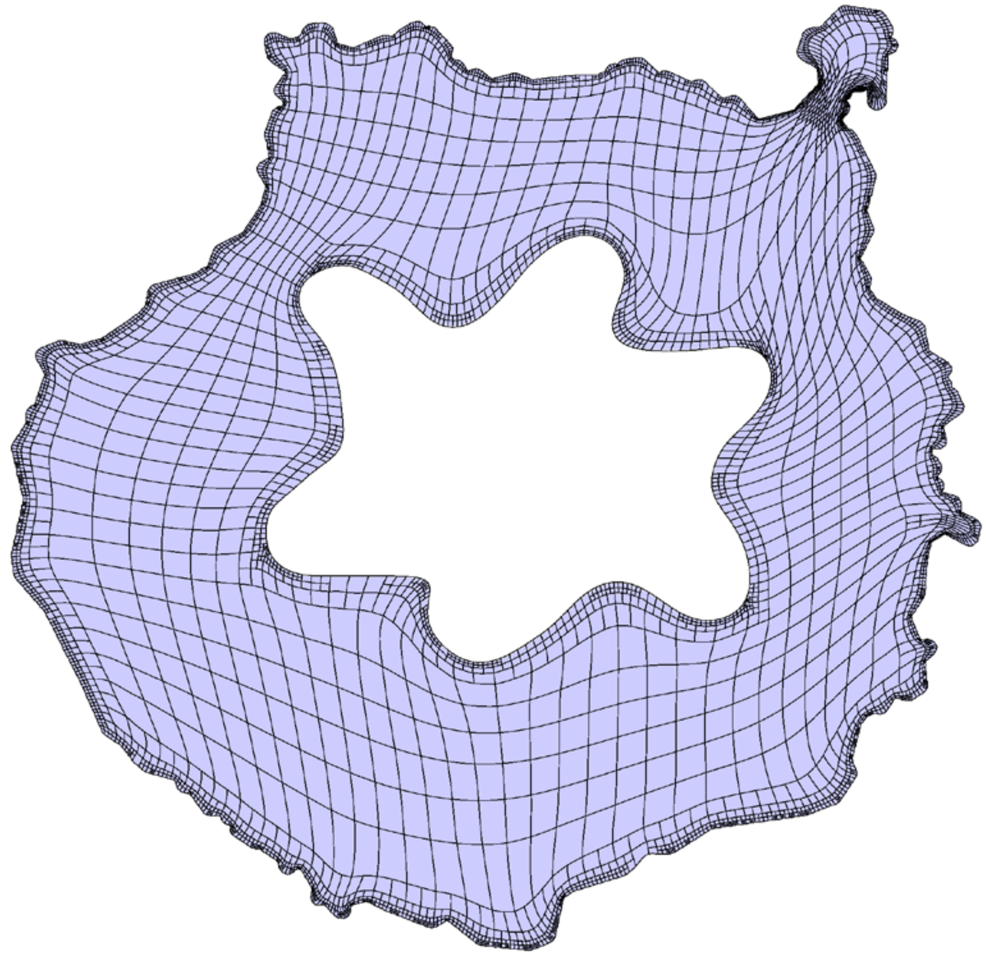


T-spline representation with different materials

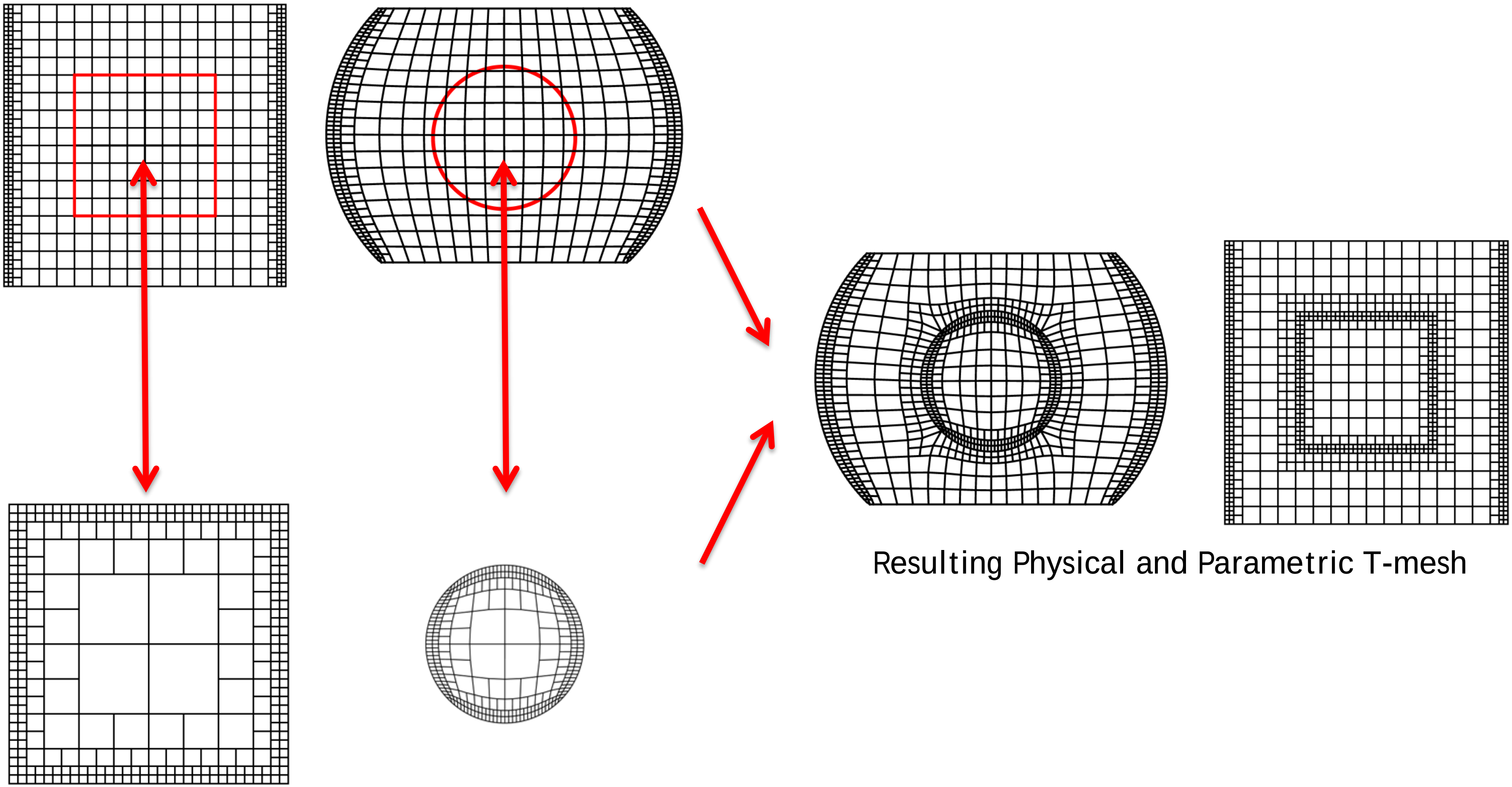


T-spline representation with a hole

# Applications: Isogeometric Modeling

## Strategy for embedded planar geometries: Other example

SIANI

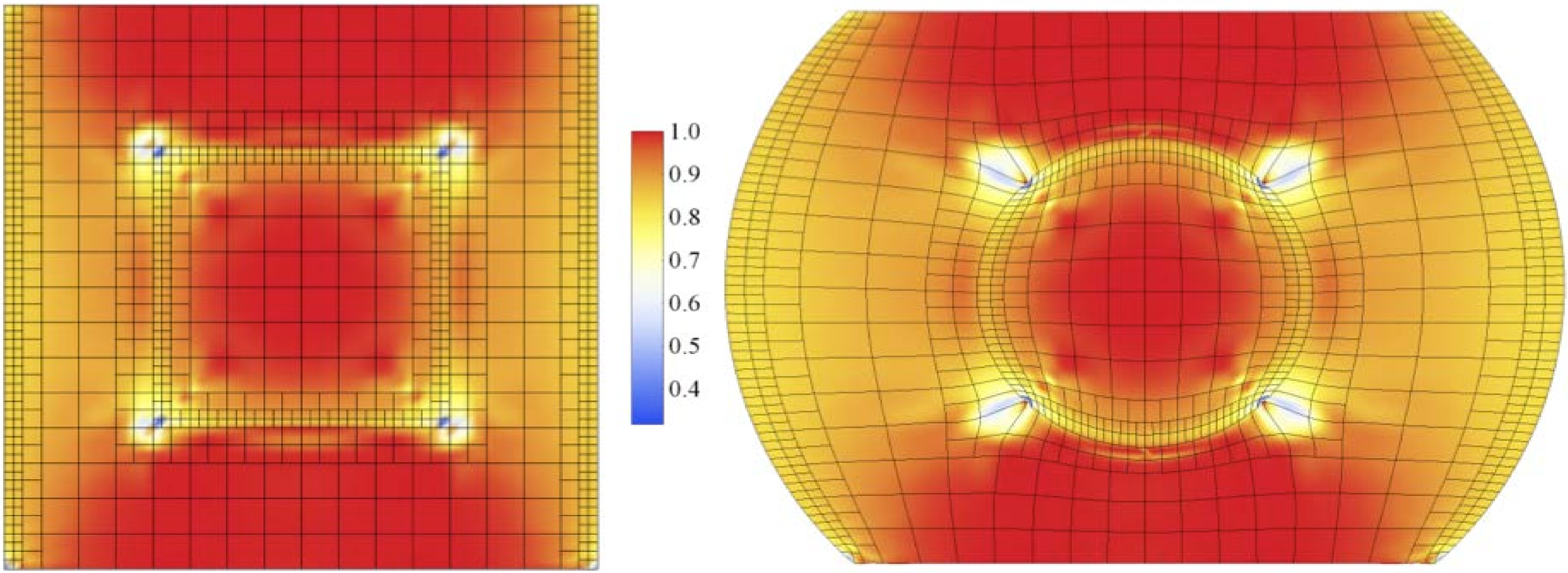


Resulting Physical and Parametric T-mesh

# Applications: Isogeometric Modeling

Strategy for embedded planar geometries: Other example

Mean ratio Jacobian

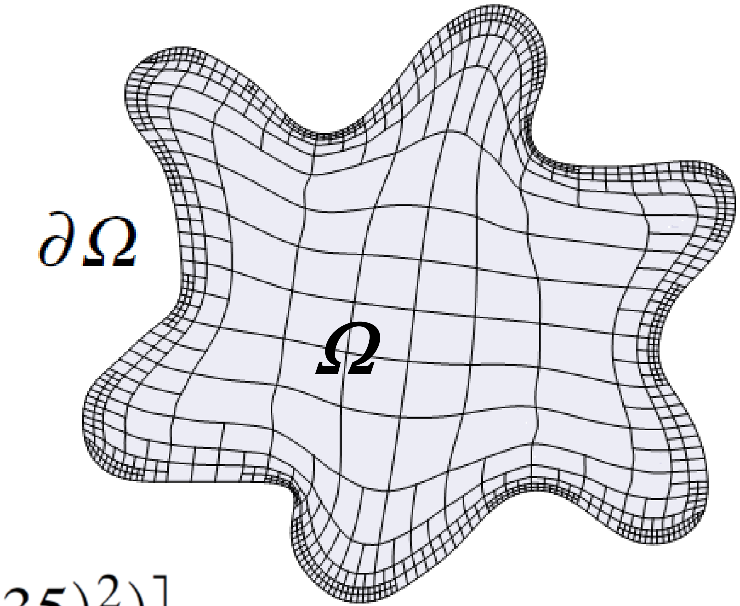


# Local Nested Adaptive Refinement

Numerical solution of a Poisson problem

Concentrate source in relation to the initial mesh size

$$\begin{aligned} -\triangle u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$




Exact solution:

$$u(x,y) = \exp\left[-10^3\left((x-0.6)^2 + (y-0.35)^2\right)\right]$$

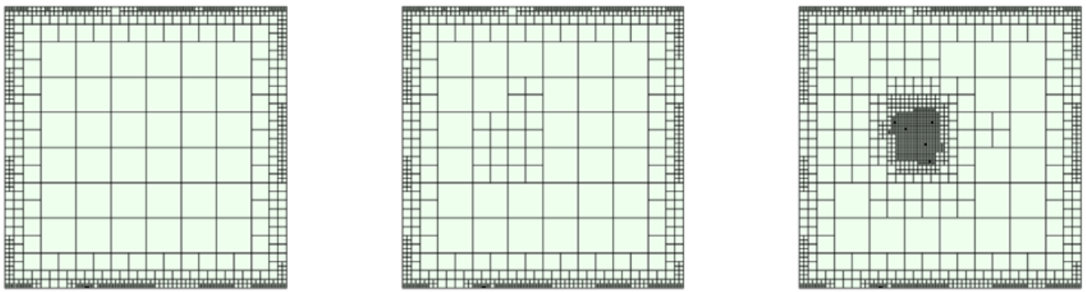
Residual-type error indicator:

$$\eta(\Omega_e)^2 = \|h(f+\Delta u_h)\|^2_{0,\Omega_e} = \int_{\Omega_e} h^2 (f+\Delta u_h)^2 \, \mathrm{d}\Omega$$

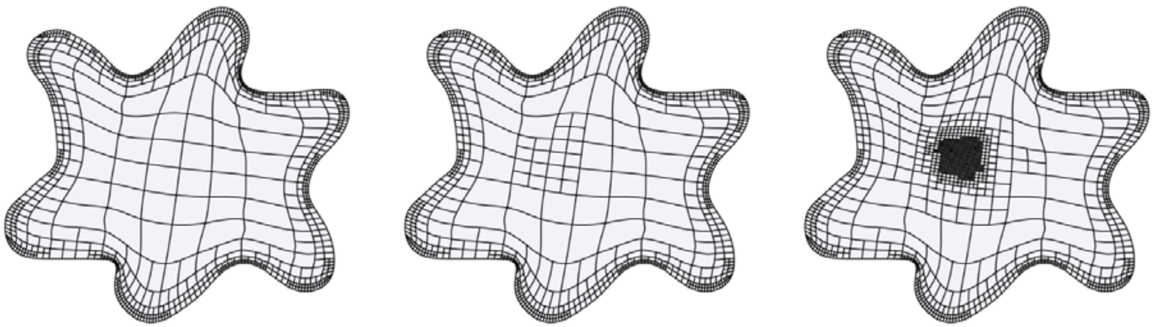
# Local Nested Adaptive Refinement

## Numerical solution of a Poisson problem

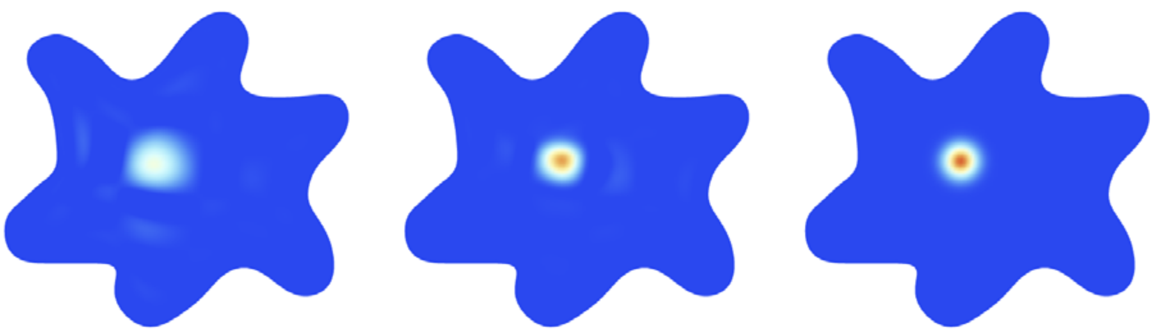
Error indicator : $\eta\left(\Omega_e\right)^2 = \int_{\Omega_e} h^2\left(f + \Delta u_h\right)^2 \, d\Omega$



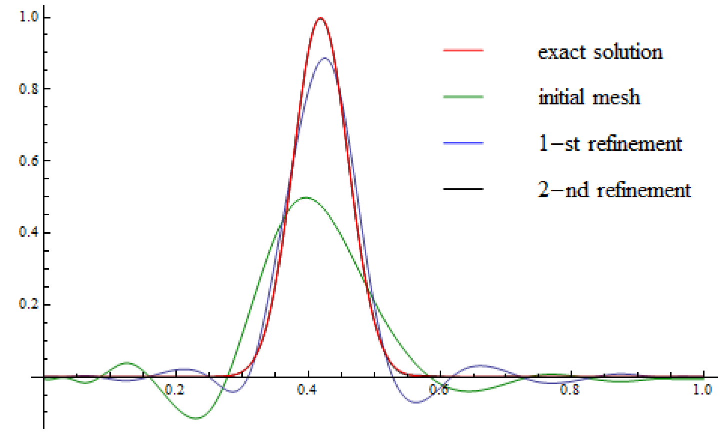
(a) 844 cells,1456 DOF (b) 859 cells, 1476 DOF (c) 1552 cells, 2233 DOF



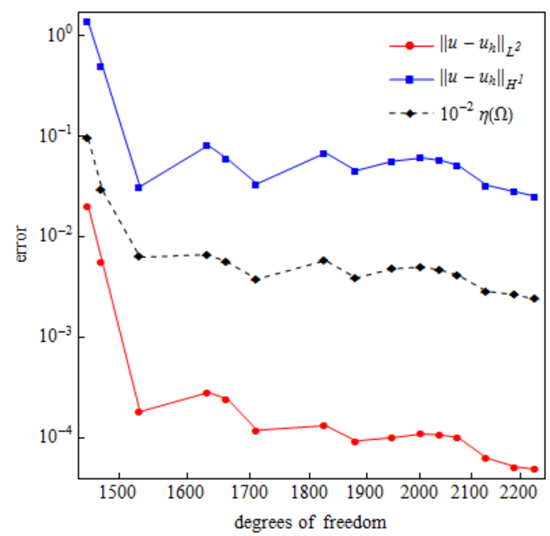
(d) Initial mesh (e) 1-st refinement (f) 14-th refinement



(g) Initial solution (h) 1-st refinement (i) 14-th refinement



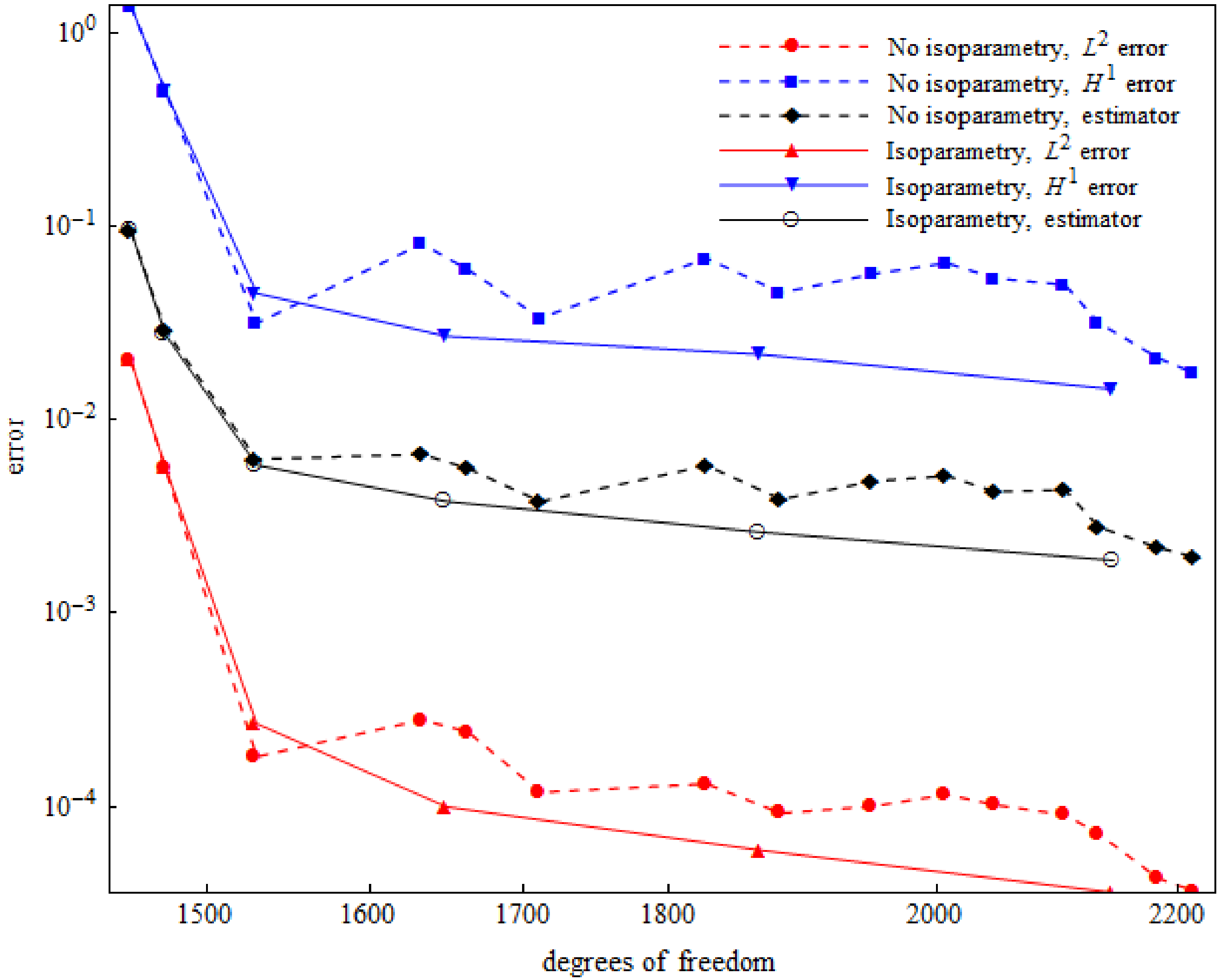

Numerical solution across a section




Convergence behavior

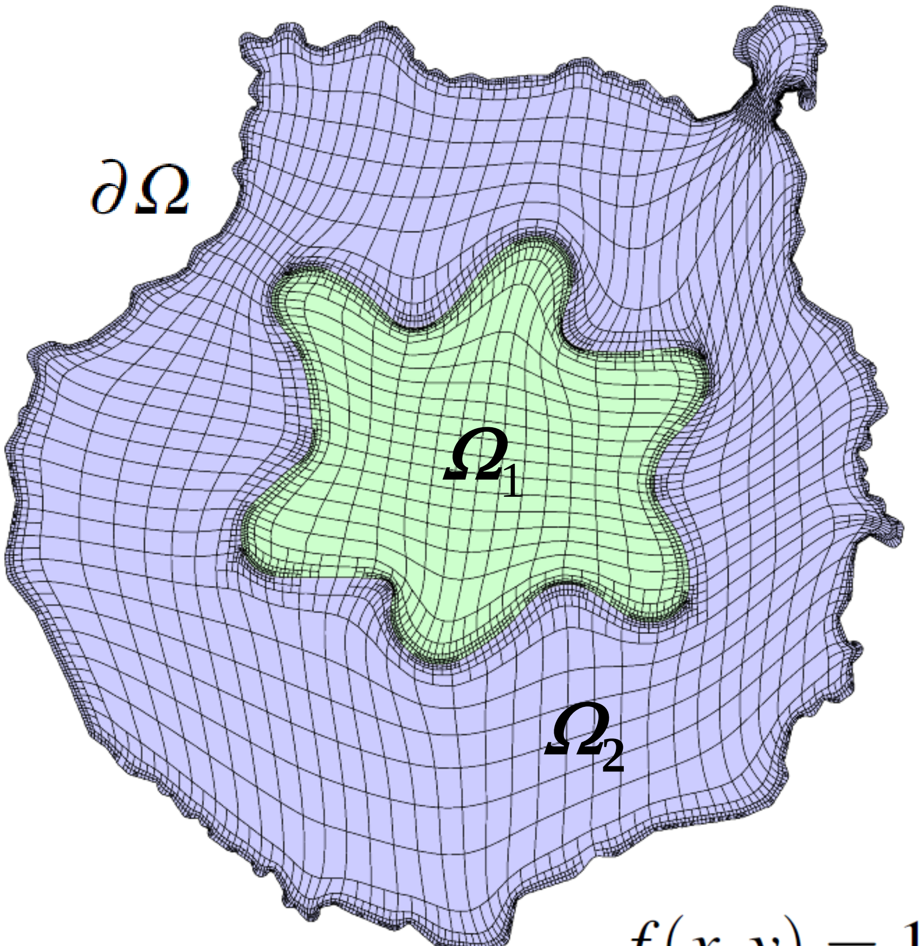
# Local Nested Adaptive Refinement

Influence of isoparametry in the convergence behavior



# Poisson Problem for a Domain with Two Materials

## Statement of the problem




$$
\begin{aligned}
-\nabla(k(\mathbf{x})\nabla u) &= f \qquad \text{in } \Omega, \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

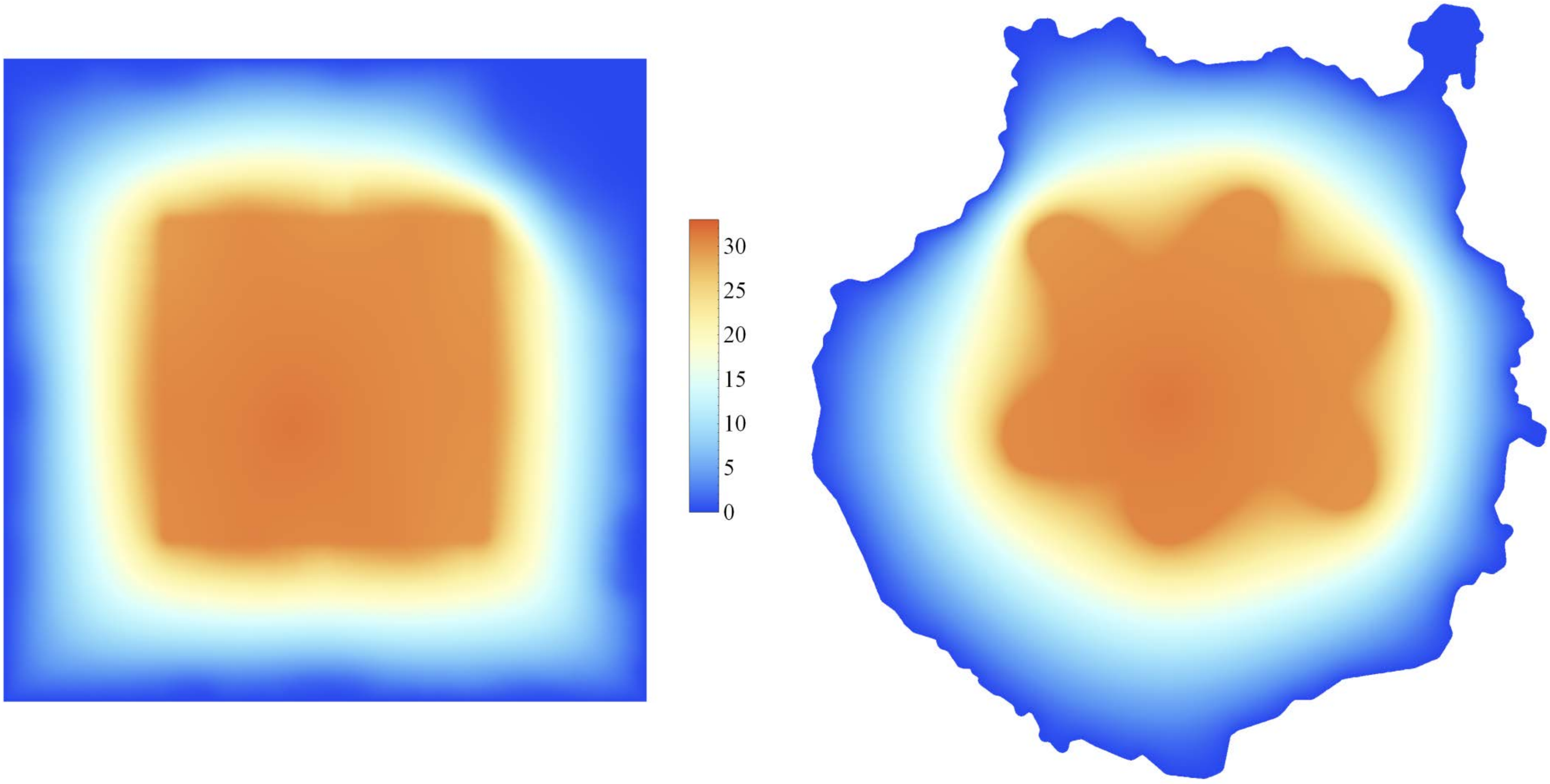
$$
k(\mathbf{x}) = \begin{cases} k_1 & \text{if } \mathbf{x} \in \Omega_1 \\ k_2 & \text{if } \mathbf{x} \in \Omega_2 \end{cases}
$$

$$
k_1 = 1 \quad >> \quad k_2 = 0.01
$$

$$
f(x,y) = 10^3 \exp\left[-10^3\left((x-0.5)^2 + (y-0.5)^2\right)\right]
$$

# Poisson Problem for a Domain with Two Materials

Numerical solution in parametric and physical domain



# Poisson Problem for a Domain with Two Materials

Dielectric cylinder in an uniform horizontal electric field $E_0$

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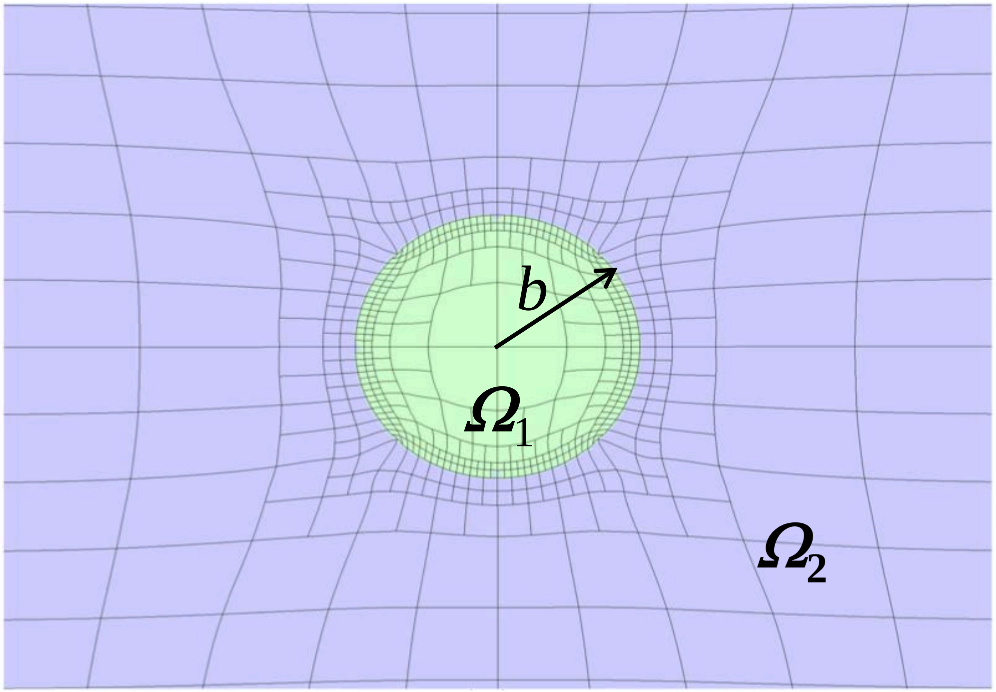



T-spline detail

$$-\nabla\left(k(\mathbf{x})\nabla u\right) = 0 \qquad \text{in } \Omega,$$
$$u = g \qquad \text{on } \partial\Omega.$$

$$k(\mathbf{x}) = \begin{cases} \varepsilon_0\varepsilon_r & \text{if } \mathbf{x} \in \Omega_1 \quad \rho < b \\ \varepsilon_0 & \text{if } \mathbf{x} \in \Omega_2 \quad \rho \geq b \end{cases}$$

The analytic solution in cylindrical coordinates $(\rho, \varphi)$ is given by:

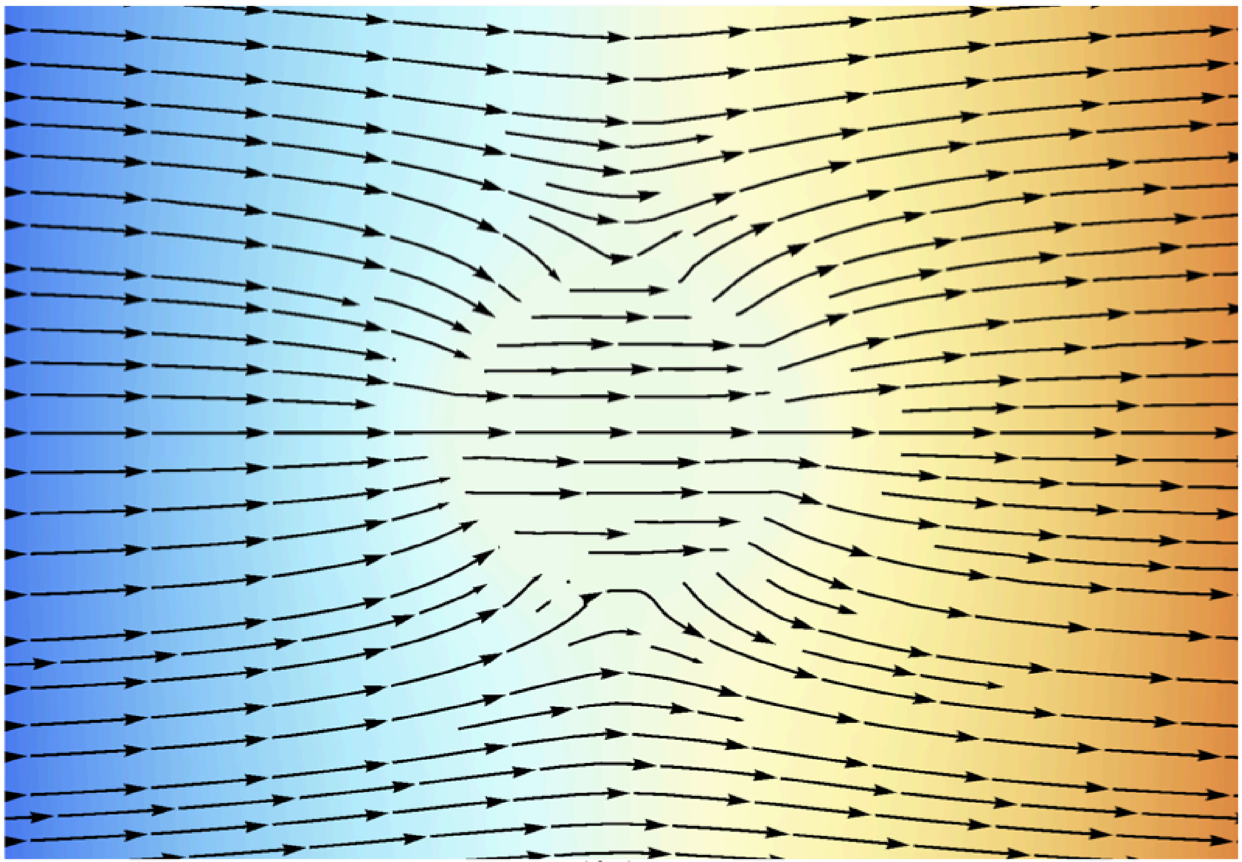
$$u_{\rho<b} = \frac{-2E_0\rho\cos\varphi}{(\varepsilon_r + 1)}$$

$$u_{\rho\geq b} = E_0\cos\varphi\left(-\rho + \frac{b^2(\varepsilon_r - 1)}{\rho(\varepsilon_r + 1)}\right)$$

# Poisson Problem for a Domain with Two Materials

Dielectric cylinder in an uniform electric field

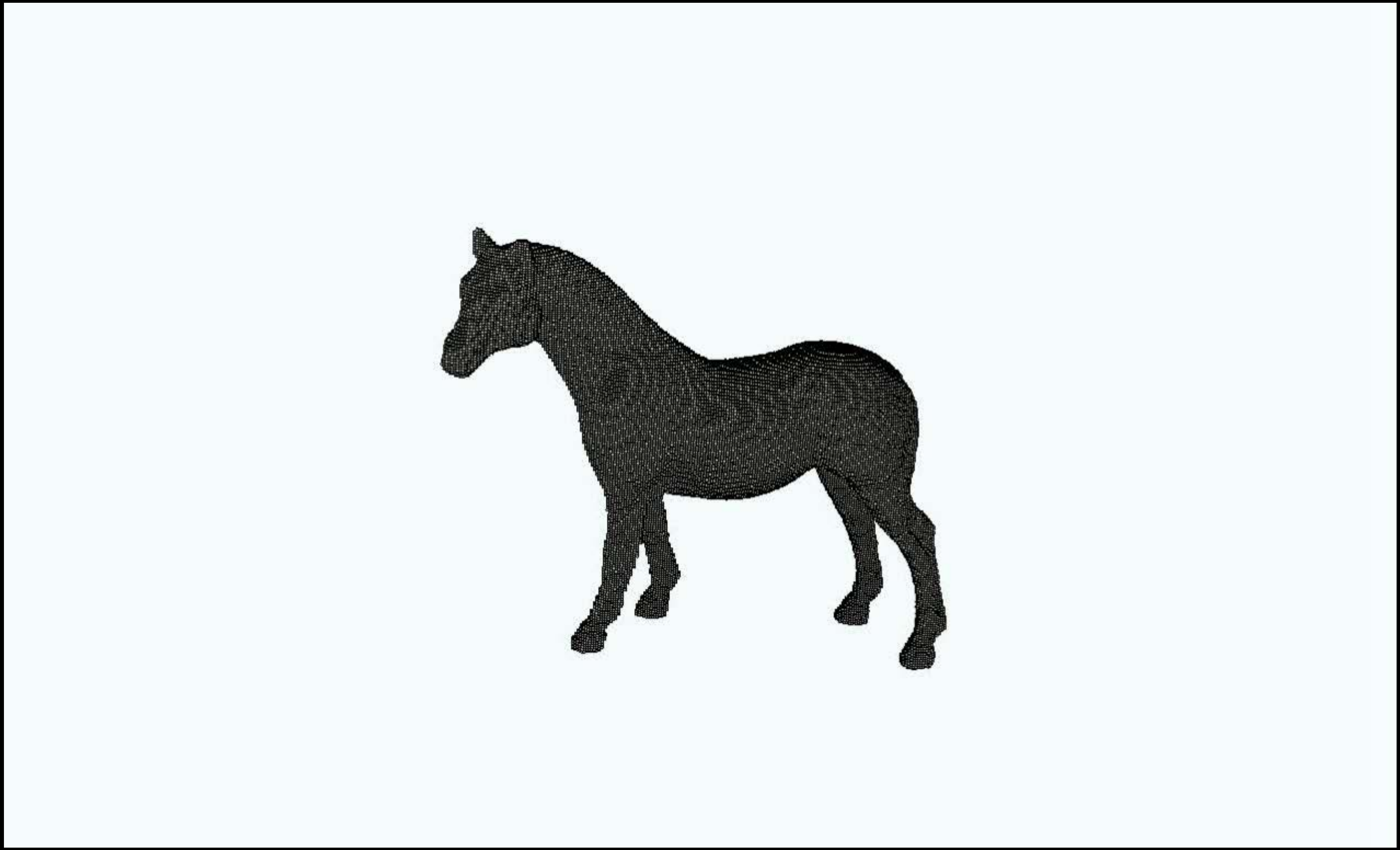
Numerical solution detail (potential and electric field) in physical domain



Comparing with the analytic solution, we have measured a **maximum error** in the potential of **0.88%**.

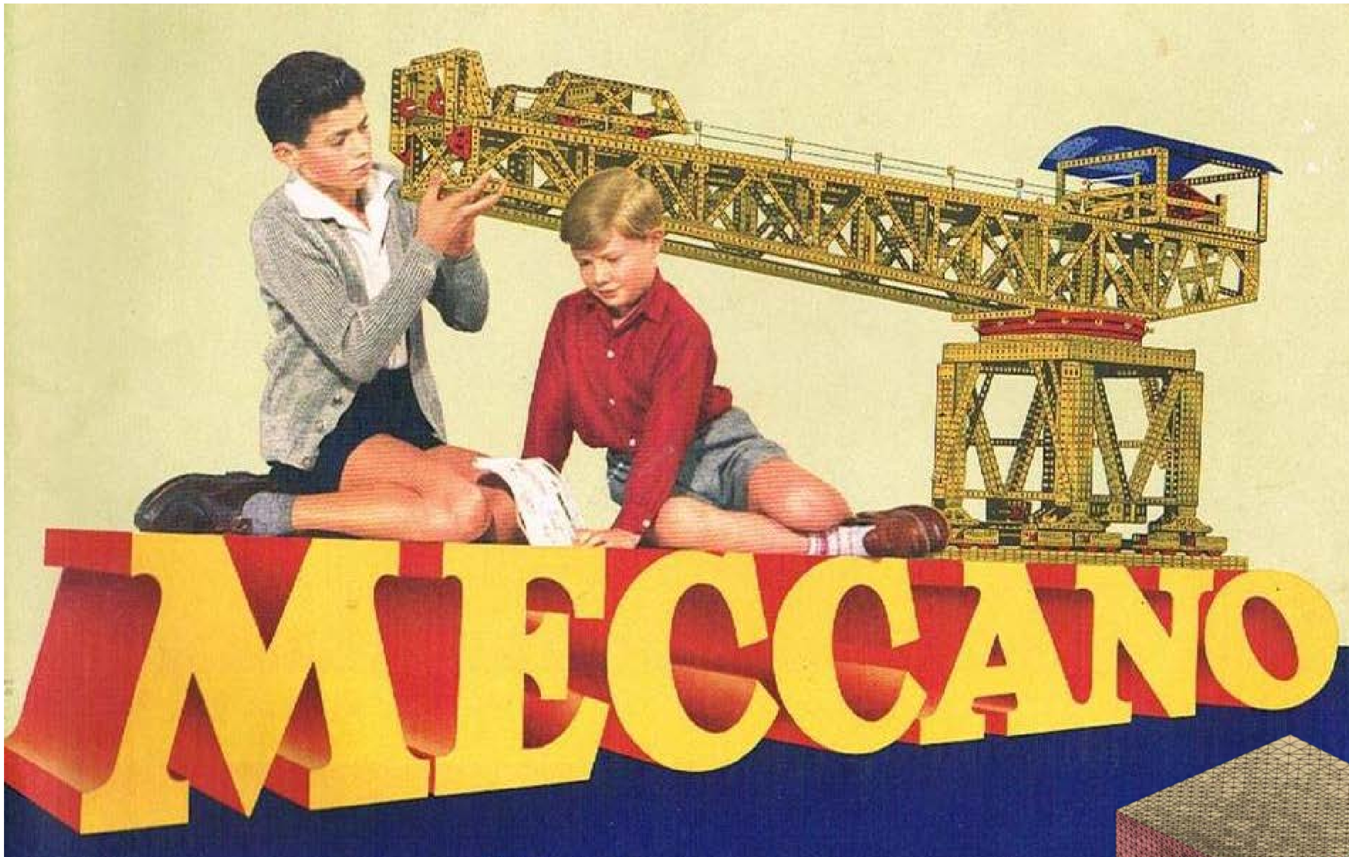
# Final Comments and Future Works

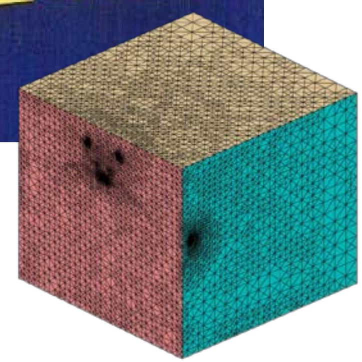
Automatic Construction of the Meccano



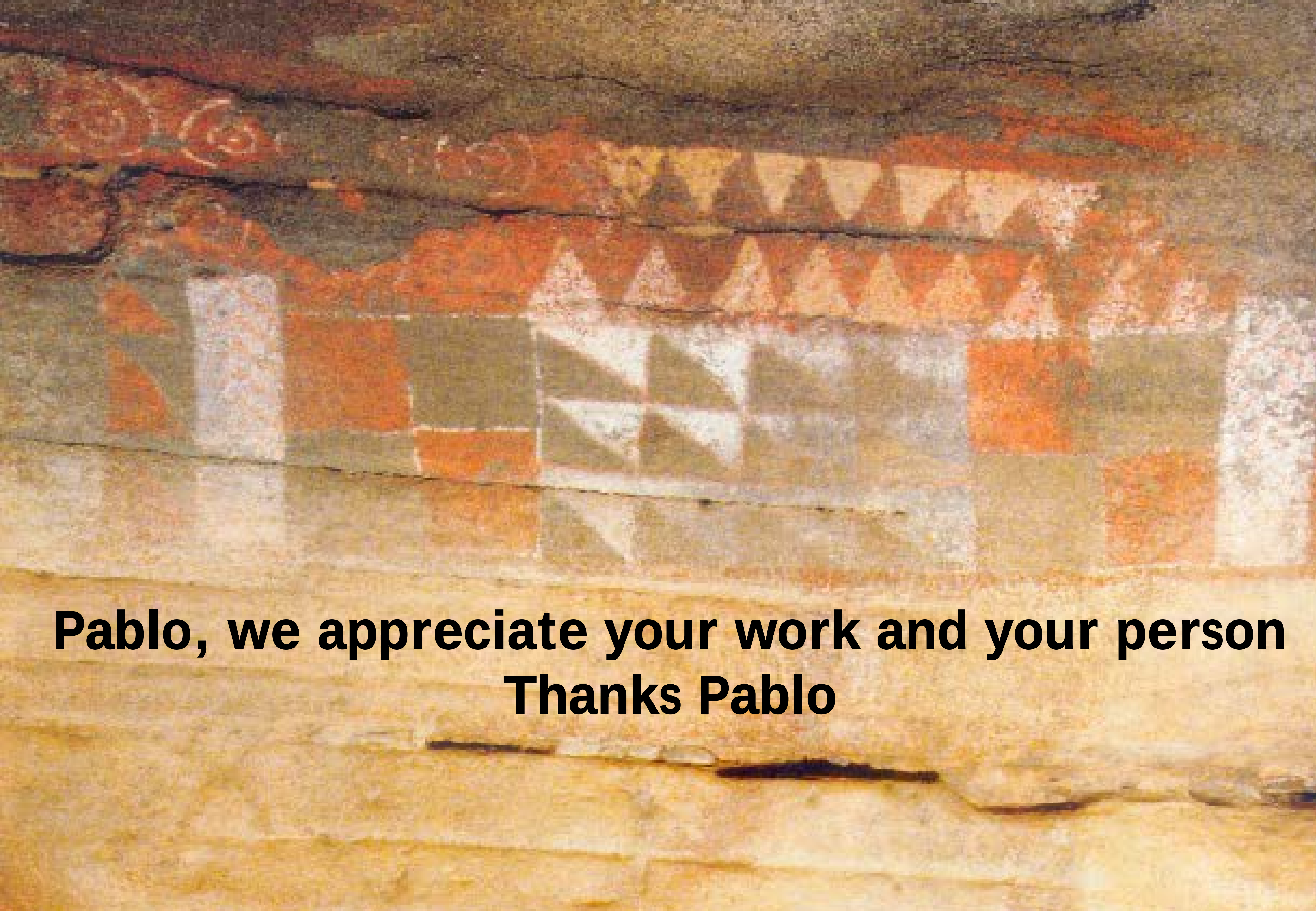
# Final Comments and Future Works

Automatic Construction of the Meccano



**Plovdiv, Bulgaria, 1993**


Pablo, we appreciate your work and your person
Thanks Pablo

# The Meccano Method for Isogeometric Analysis of Planar Domains

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**An International Symposium on Orthogonality, Quadrature and Related Topics (OrthoQuad 2014), January 20-24, 2014, Puerto de la Cruz, Tenerife, Spain**

**In memory of Prof. Pablo González Vera**

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