Adaptive T-spline Refinement for Isogeometric Analysis in Planar Geometries

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http://www.dca.iusiani.ulpgc.es/proyecto2012-2014

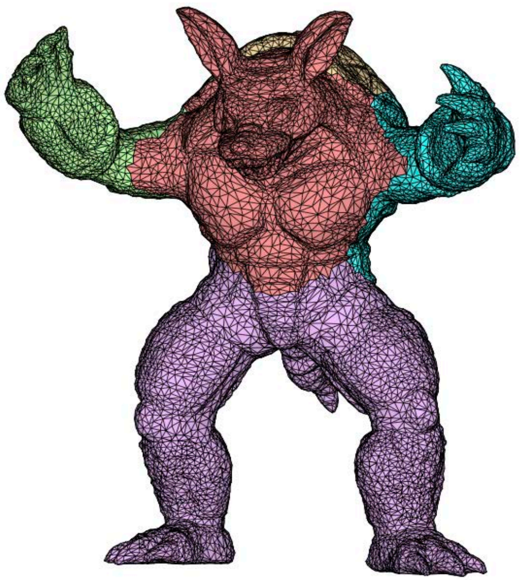
- **The Meccano Method (based on tetrahedral mesh optimization)**
  - **The initial algorithm for tetrahedral mesh generation**
  - **Volumetric parameterization of tetrahedral meshes**
  - **Application to isogeometric solid modeling and analysis**

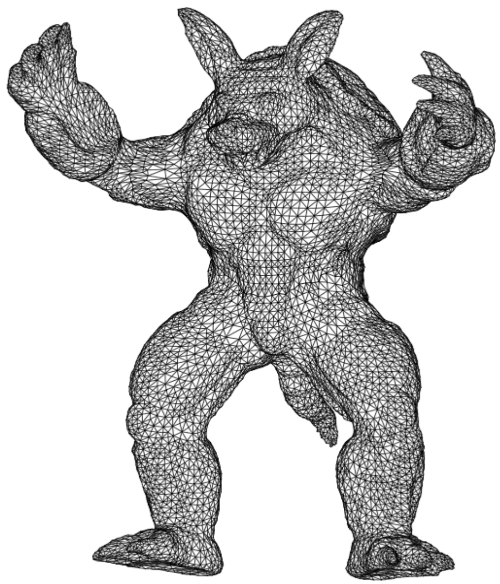
- **The Meccano Method (based on a new 2D T-mesh optimization)**
  - **The new algorithm for two-dimensional T-mesh generation**
  - **T-spline parameterization of 2D geometries**
  - **Geometries with several materials or holes**
  - **Application to isogeometric modeling and analysis**

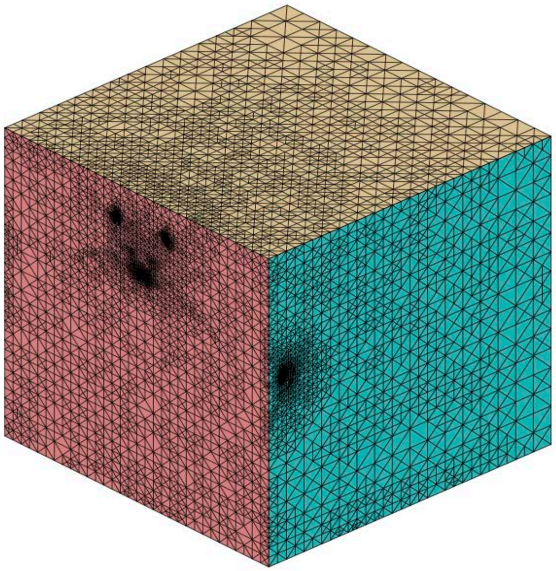
- **Comments and Future Research**

Meccano Method for Complex Solids

Volume parameterization based on SUS of tetrahedral meshes



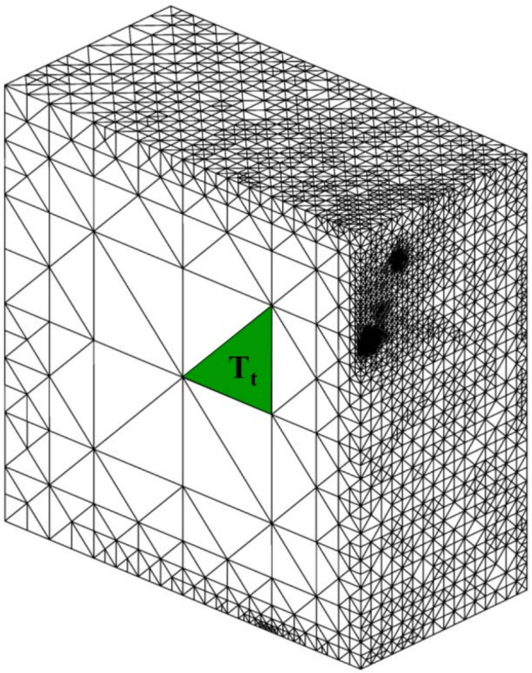




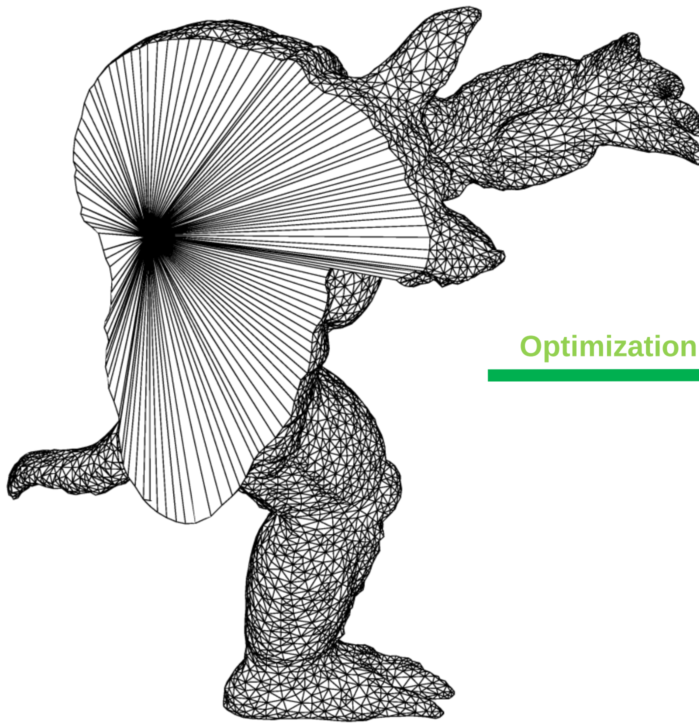
- Parameterization
- Refinement
- Untangling/Smoothing

Meccano Method for Complex Solids

Volume parameterization based on SUS of tetrahedral meshes

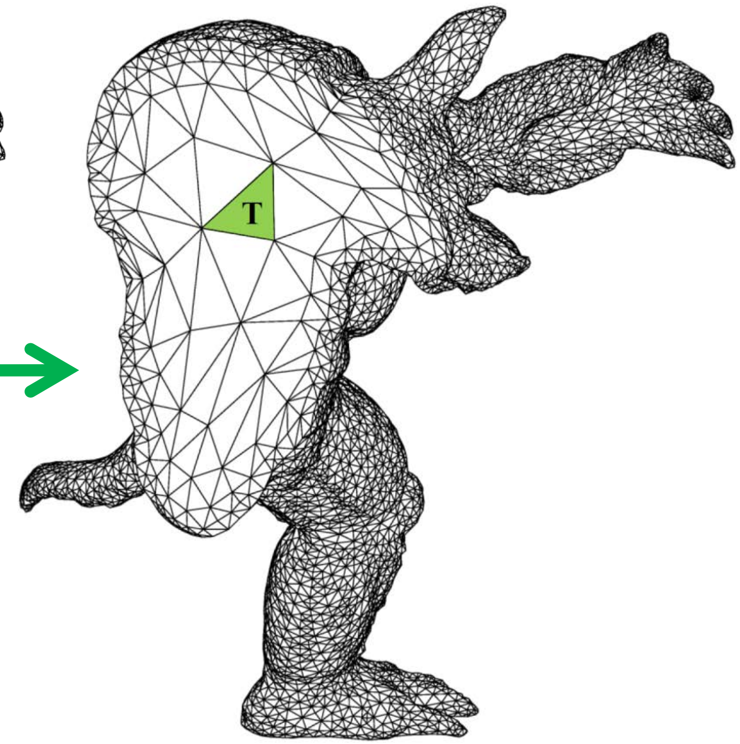


Parameter space
(meccano mesh)



Physical space
(tangled mesh)

Optimization →



Physical space
(optimized mesh)

Target Element T_t ← Optimization — **Physical Element T** (to get less distortion in the parameterization)

Meccano Method for Complex Solids

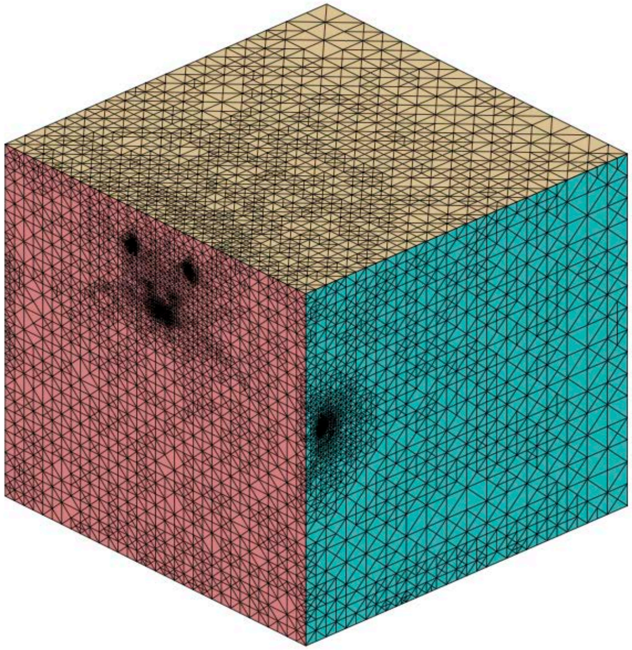
Algorithm steps

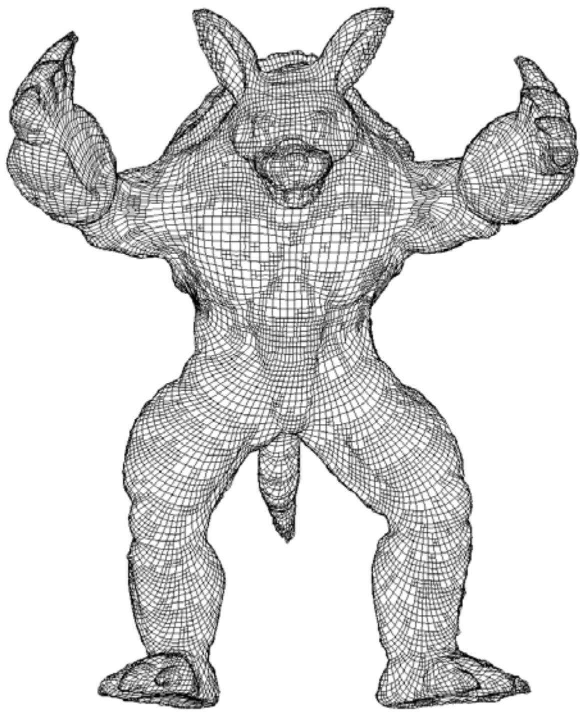
- Meccano construction
- Parameterization of the solid surface
  - Solid boundary partition
  - Floater's parameterization
- Coarse tetrahedral mesh of the meccano
  - Partition into hexahedra
  - Hexahedron subdivision into six tetrahedra
- Solid boundary approximation
  - Kossaczky's refinement
  - Distance evaluation
- Inner node relocation
  - Coons patches
- Simultaneous untangling and smoothing
  - SUS

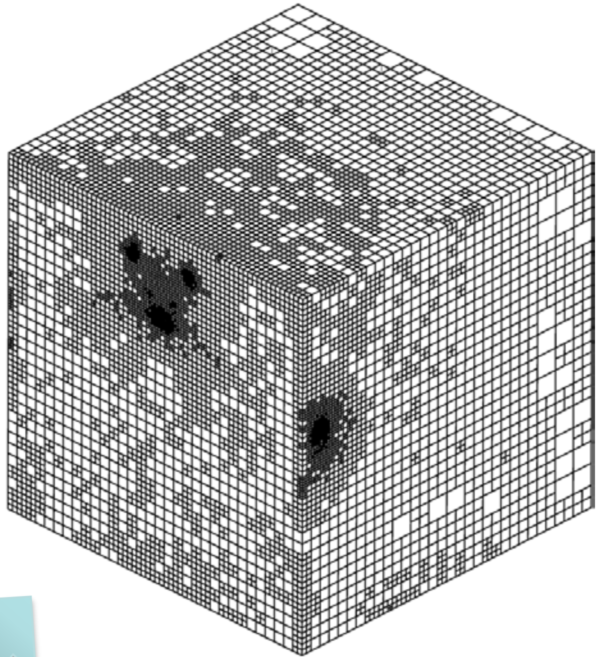
Meccano Method for Complex Solids

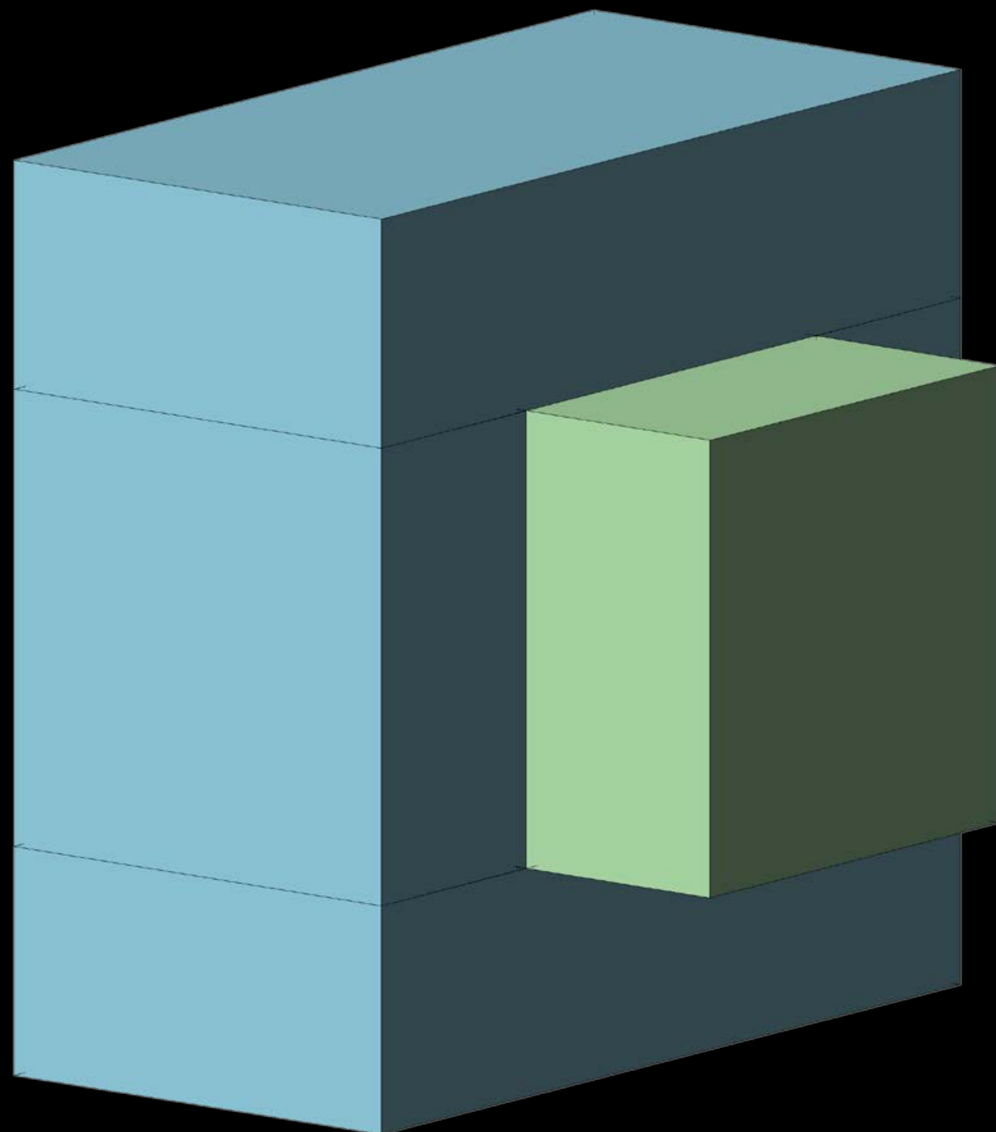
Volume parameterization based on SUS of tetrahedral meshes

- Octree subdivision

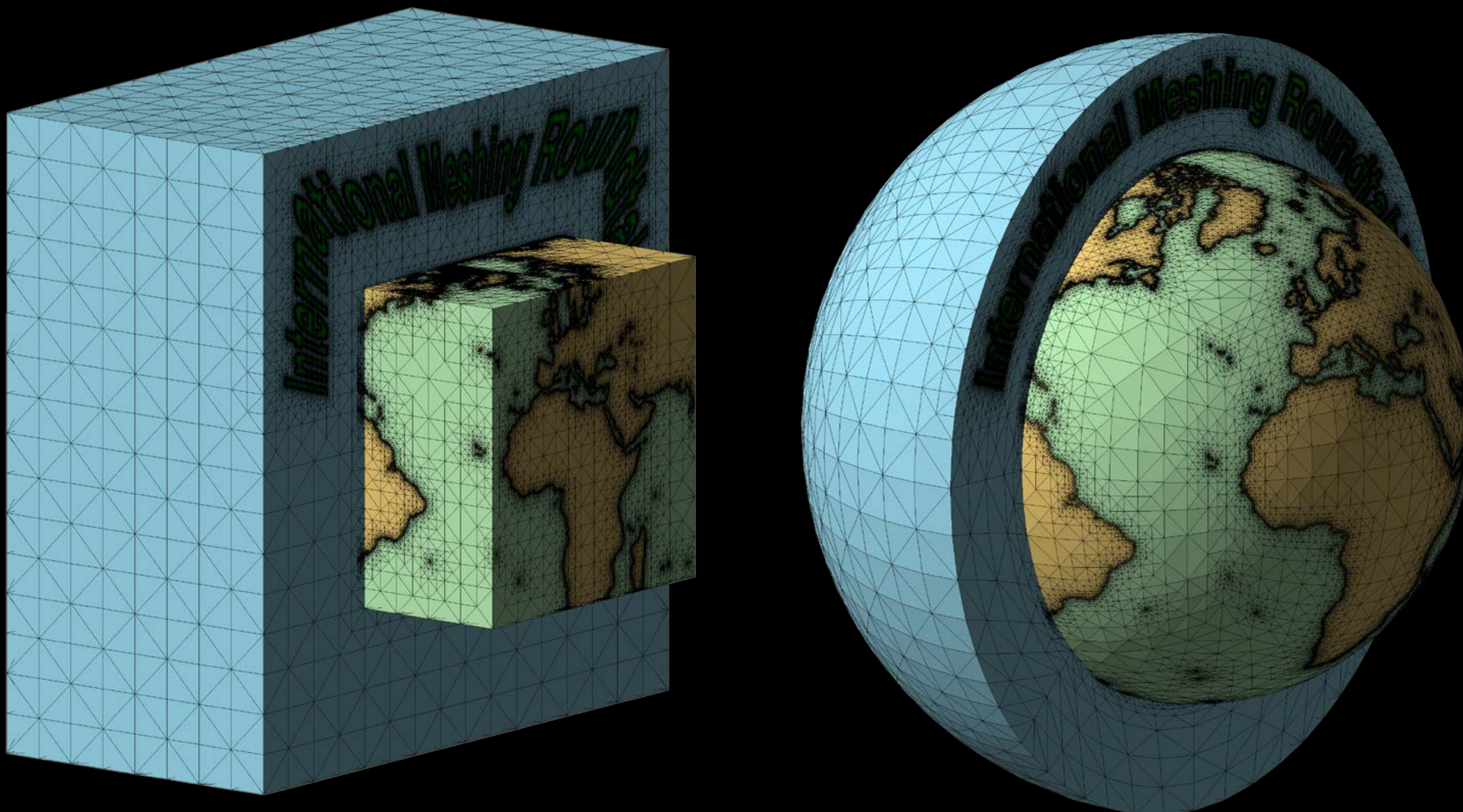


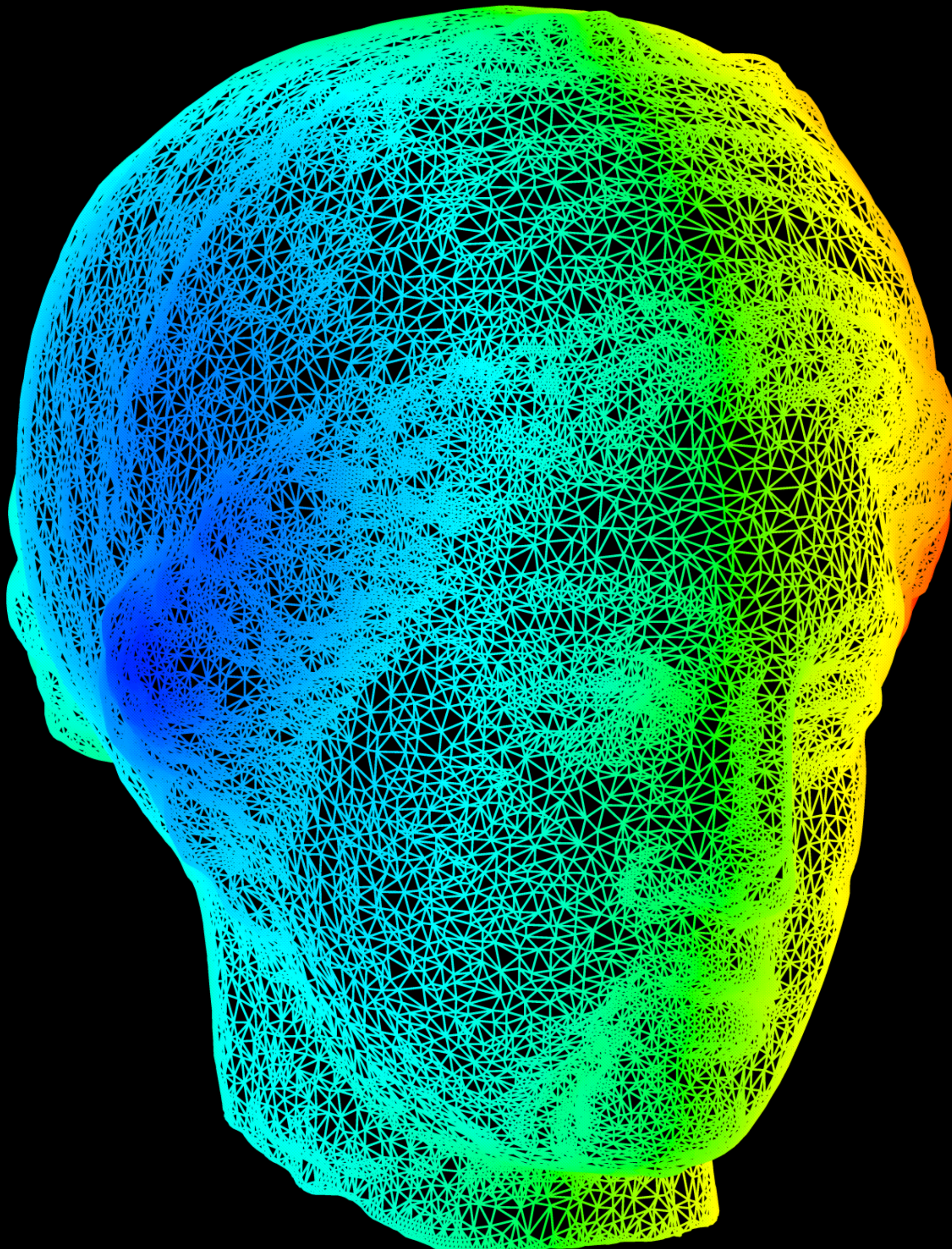






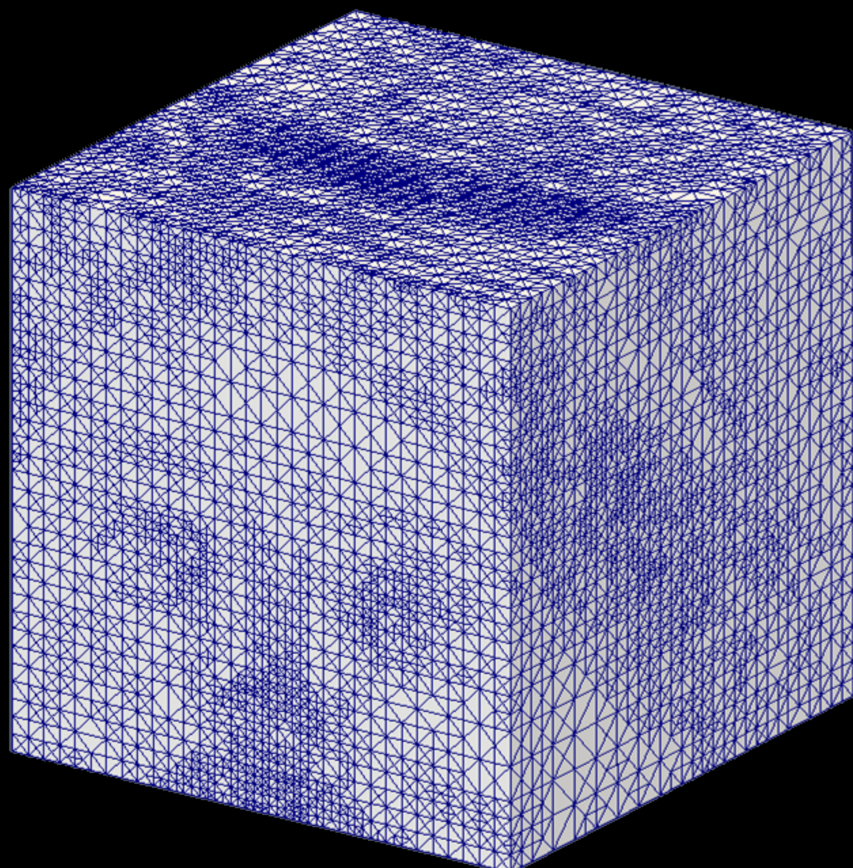
16th IMR (2007)

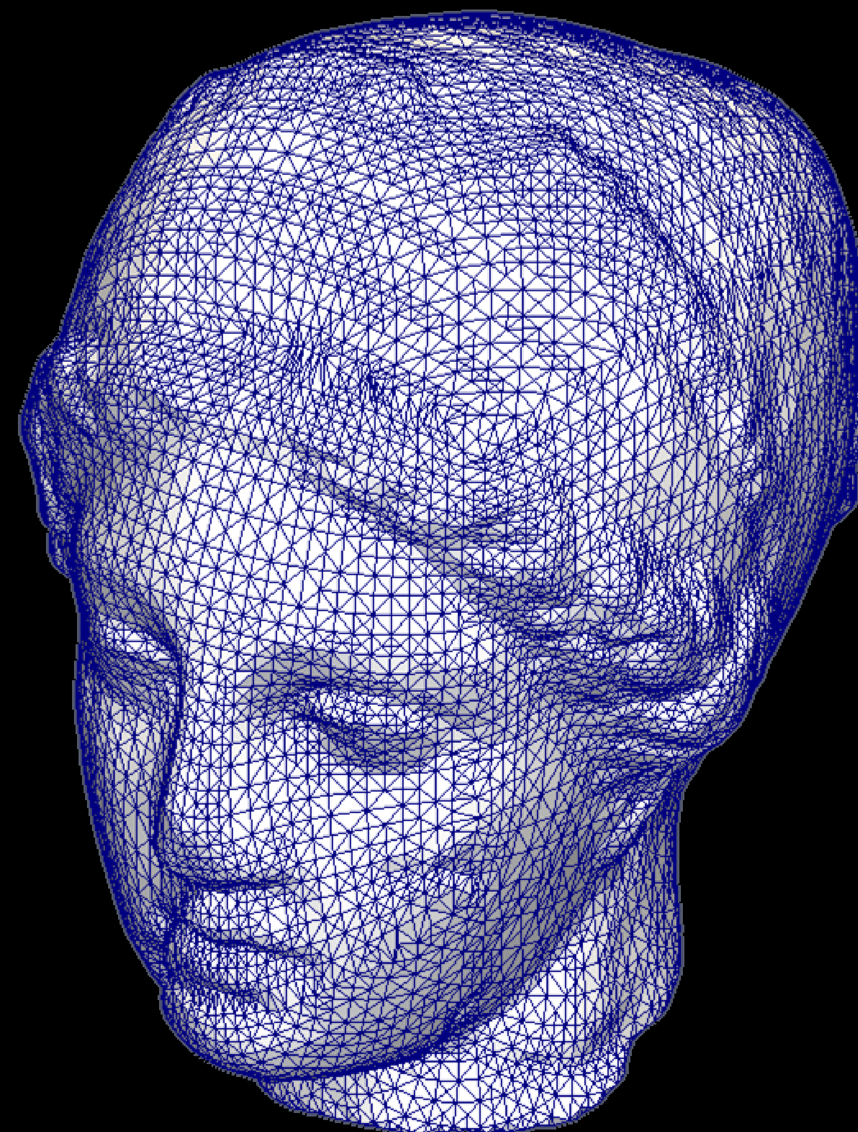




INPUT DATA: Surface Triangulation

http://www.cyberware.com/





Adaptive Isogeometric Refinement (EWC 2012)

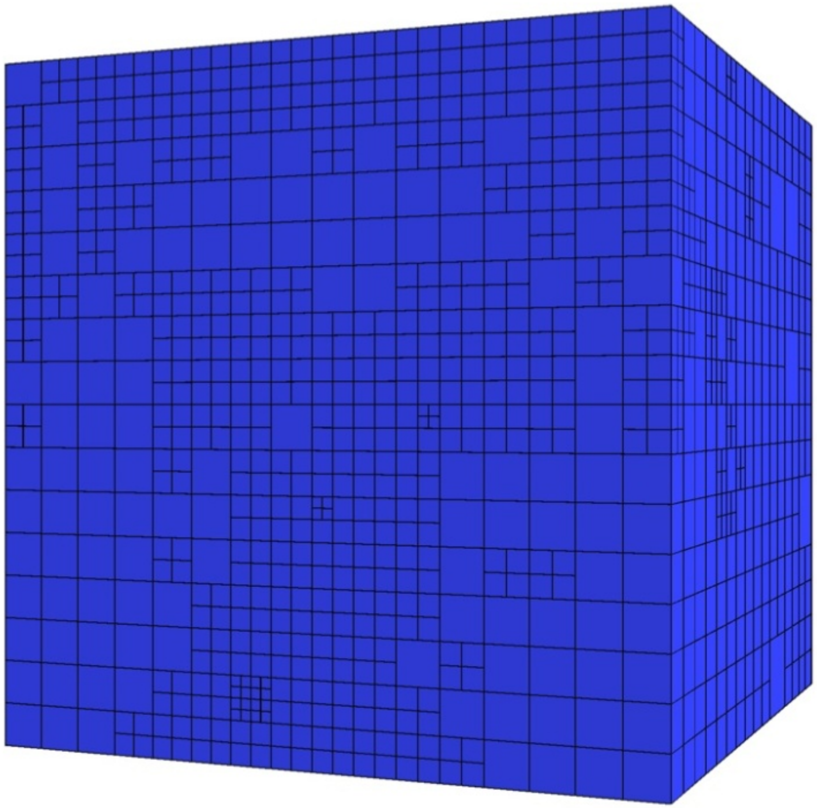
Application in Igea: Poisson problem with a central source

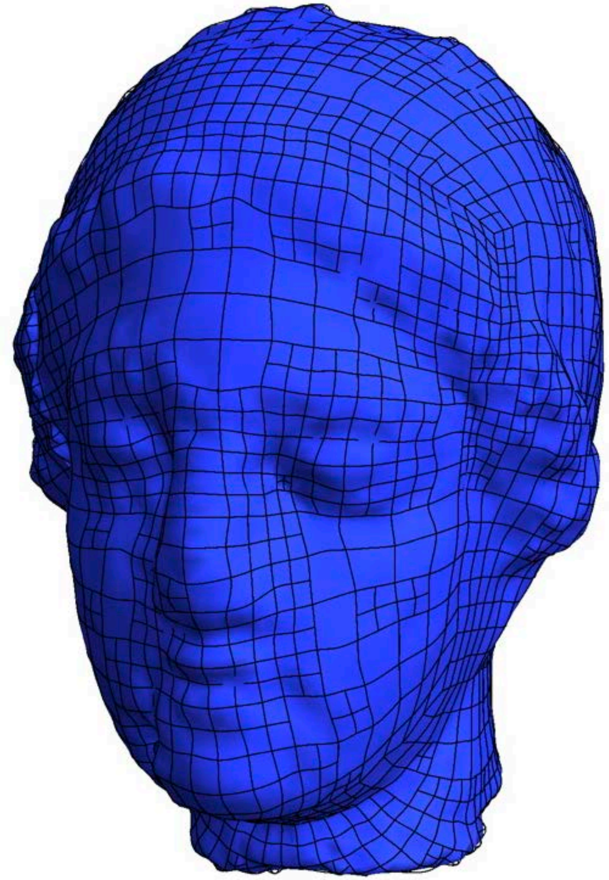
$$\Delta u = \frac{1}{25} e^{-\frac{(x^2+y^2+z^2)}{10}} \left(-15 + x^2 + y^2 + z^2\right) \quad \text{in } \Omega$$

$$u\big|_{\partial\Omega} = 0$$

Exact solution:

$$u \approx e^{-\frac{(x^2+y^2+z^2)}{10}}$$



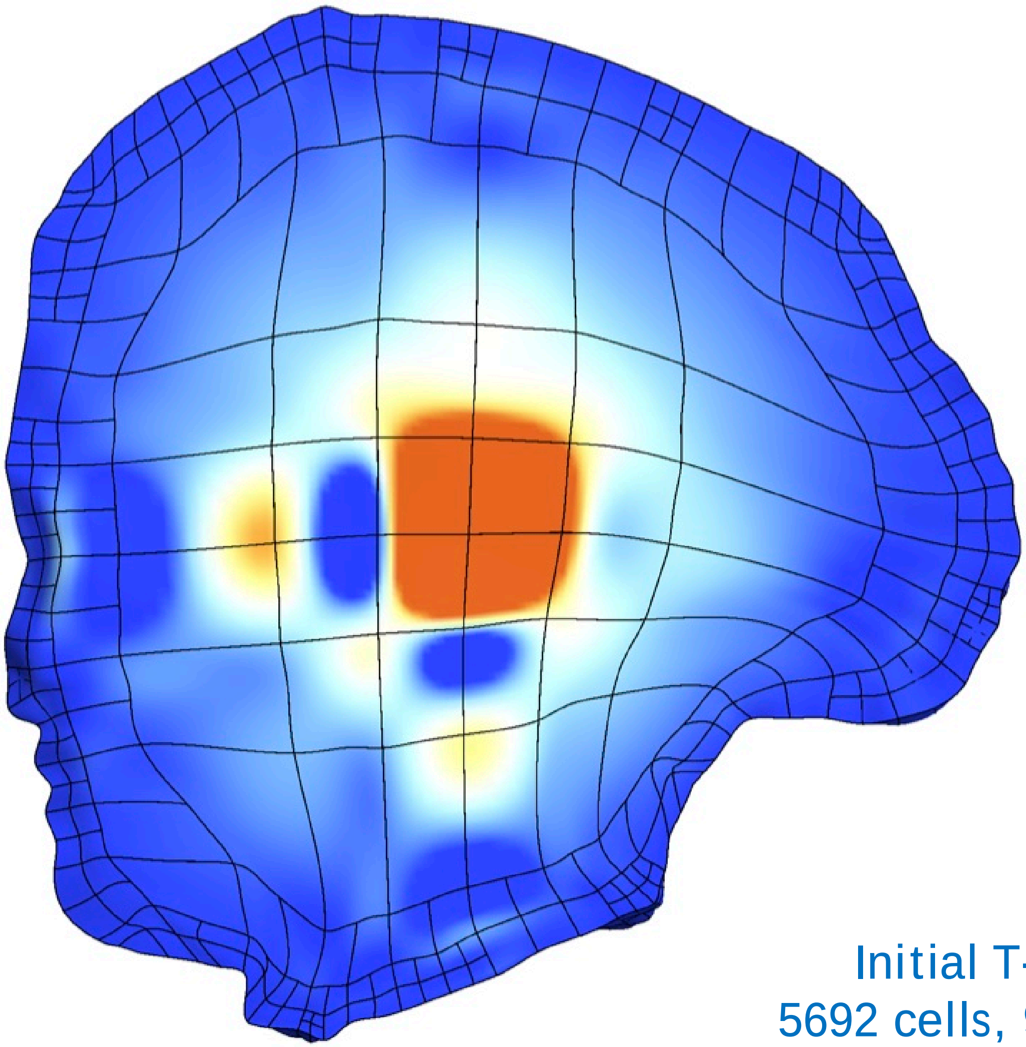


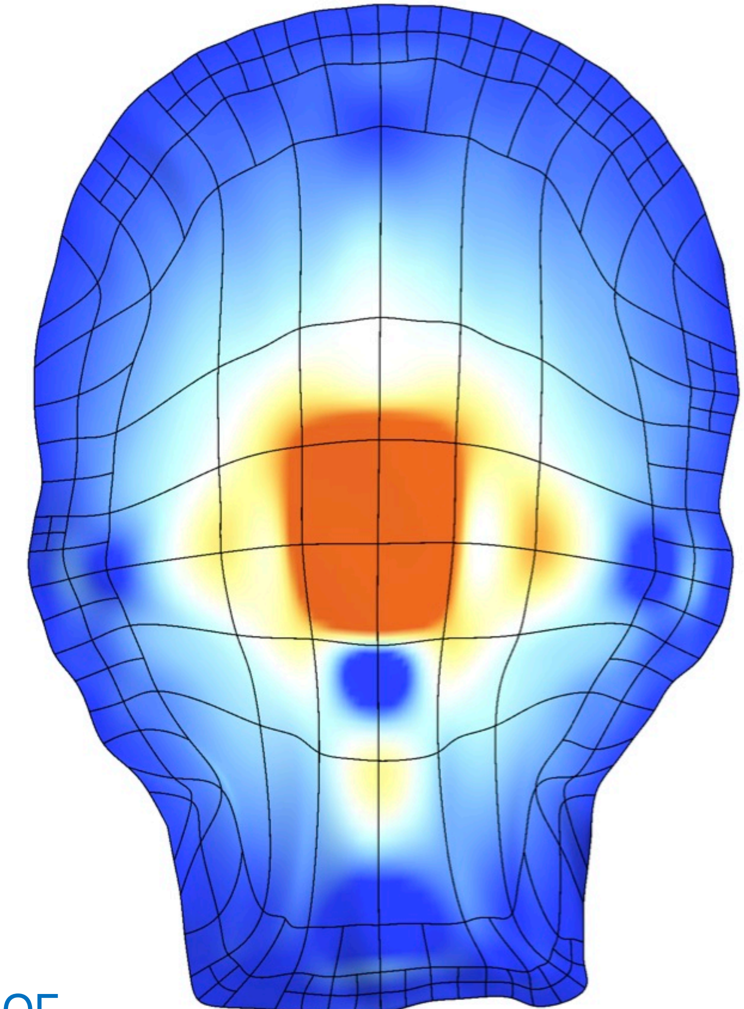
$\mathbf{S}(\xi)$

T-mesh → T-spline

Adaptive Isogeometric Refinement (EWC 2012)

Igea: T-spline of Numerical Solution



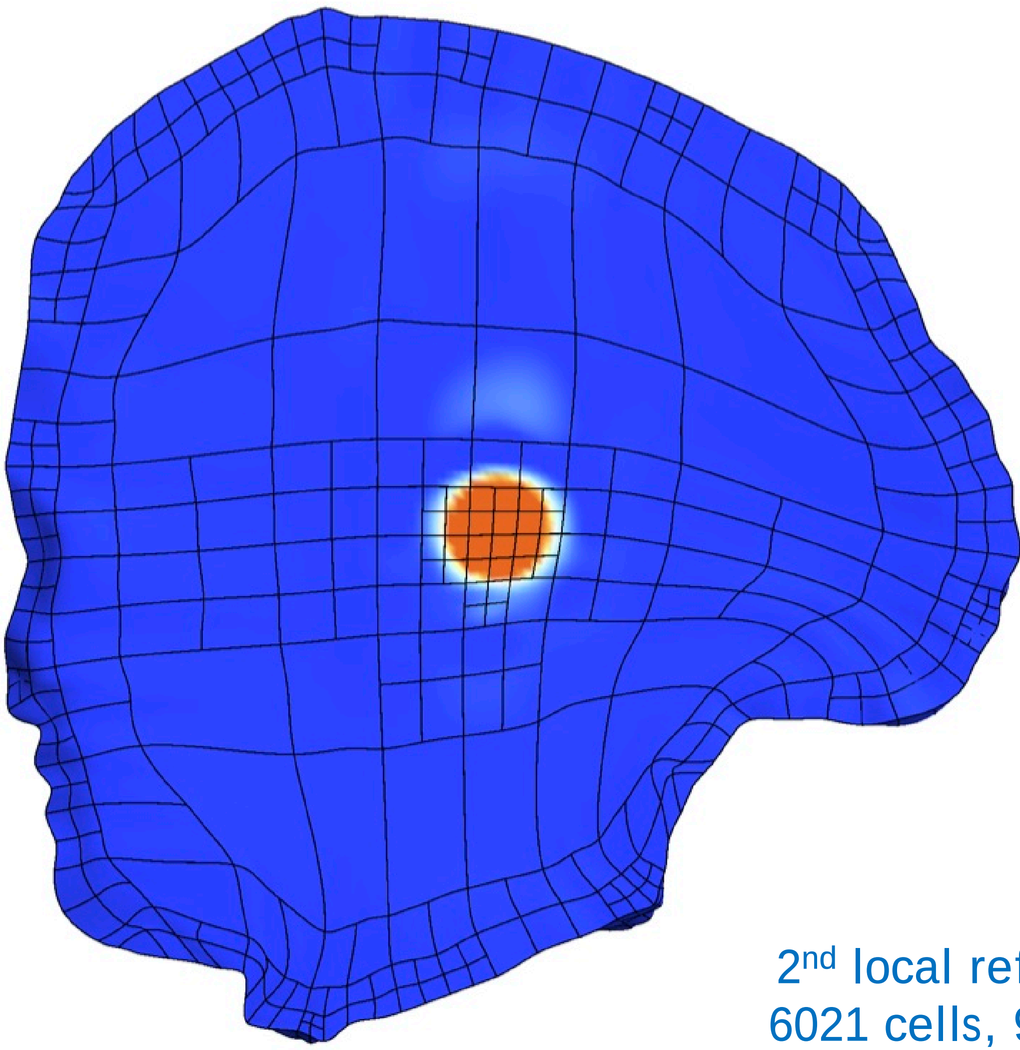


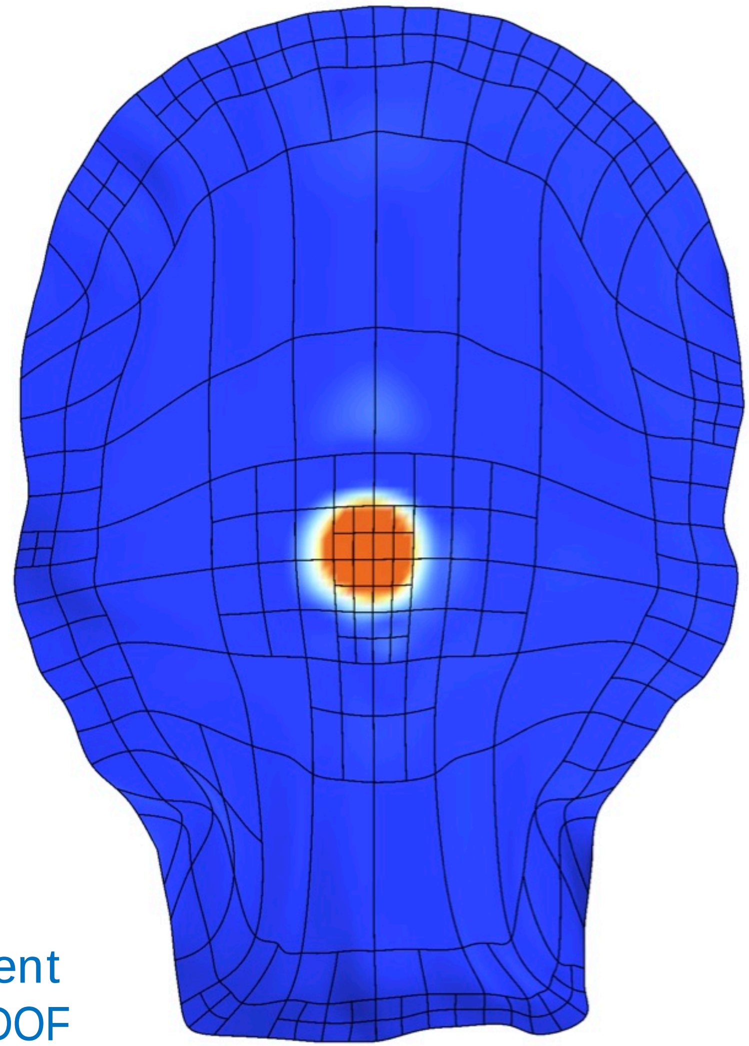
Initial T-mesh
5692 cells, 9304 DOF

Adaptive Isogeometric Refinement (EWC 2012)

Igea: T-spline of Numerical Solution

Error indicator : $\eta\left(\Omega_e\right)^2=\int_{\Omega_e} h^2\left(f+\Delta u_h\right)^2 \, d\Omega$



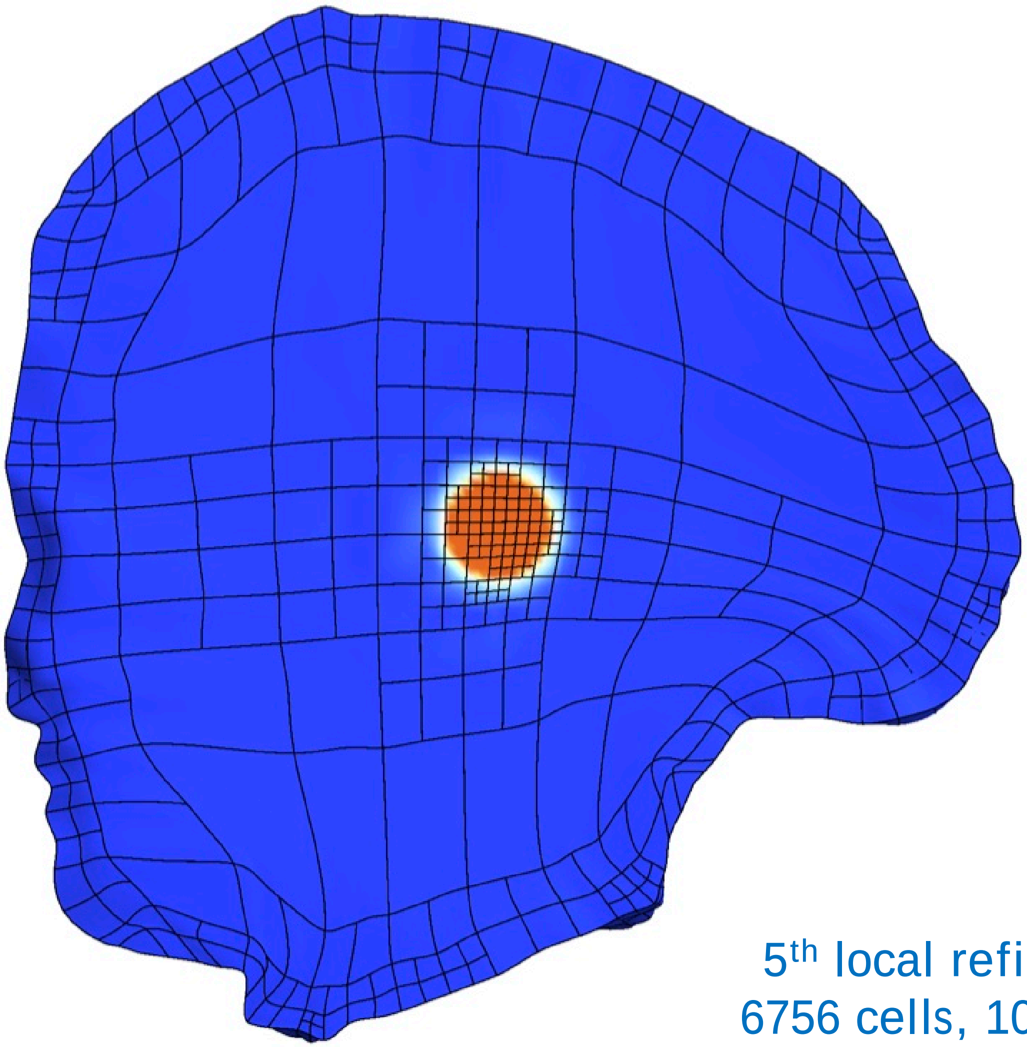


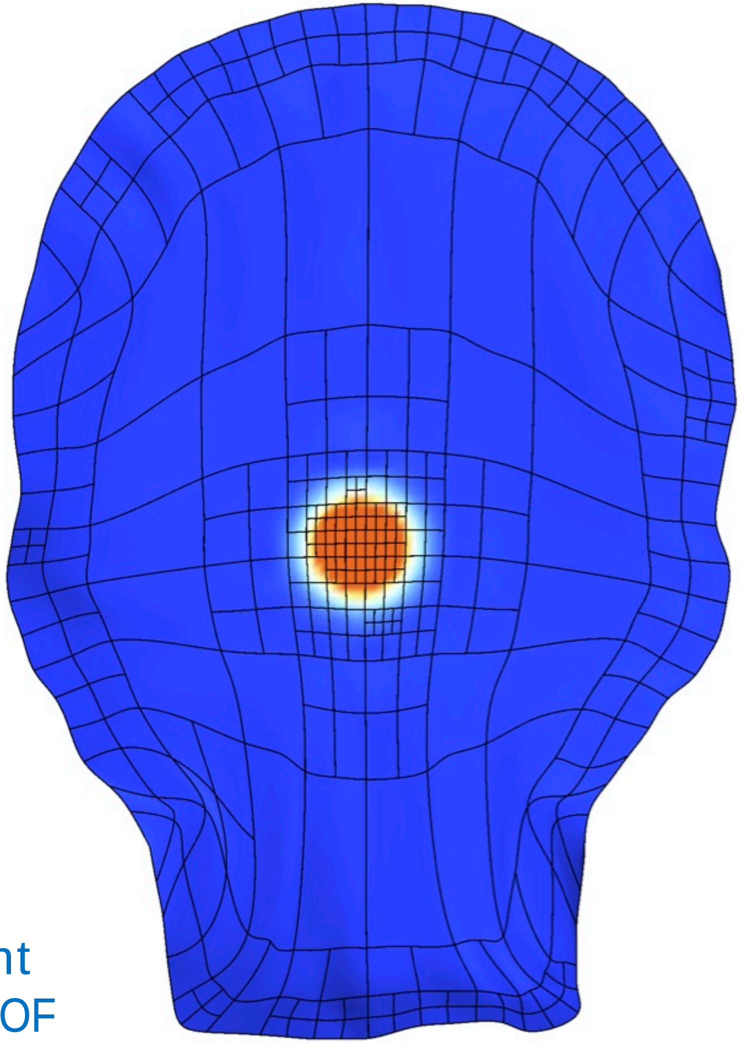
2nd local refinement
6021 cells, 9807 DOF

Adaptive Isogeometric Refinement (EWC 2012)

Igea: T-spline of Numerical Solution

Error indicator : $\eta\left(\Omega_e\right)^2 = \int_{\Omega_e} h^2 \left(f + \Delta u_h\right)^2 \, d\Omega$





5^{th} local refinement
6756 cells, 10838 DOF

1. The key of the meccano method is the simultaneous untangling and smoothing (SUS) procedure.

2. The quality of the T-spline mapping (i.e., positive Jacobian, good uniformity and orthogonality of the isoparametric curves) depends on the quality of the T-mesh in the physical space. We have to fix a quality metric for this mapping.

3. In order to simplify the procedure and to get less distortion in the volume parameterization, it should be interesting to **directly apply the meccano method on T-meshes instead of tetrahedral meshes**.

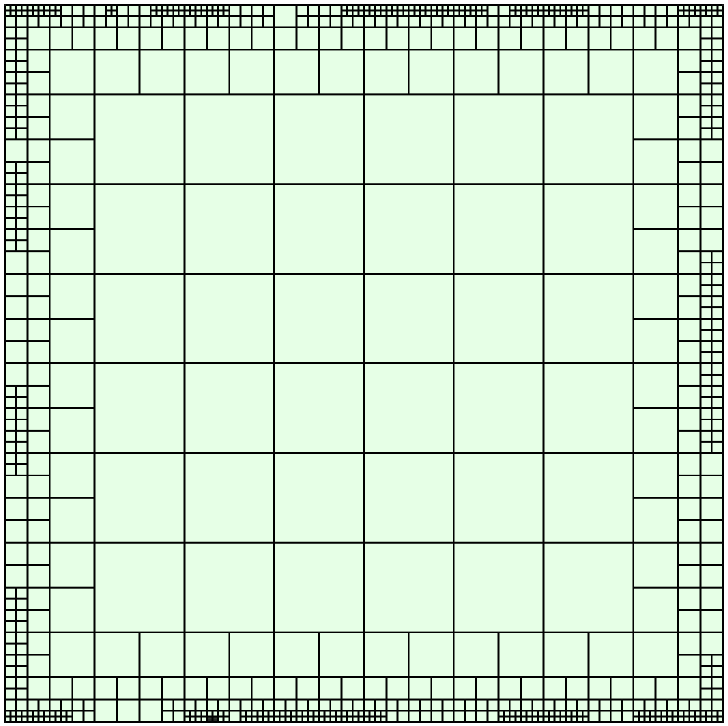
4. We have started analysing the problem in 2-D.

The Meccano Method on T-meshes in 2-D

Input data: Boundary representation of the object

Objective: Construction of a high quality T-spline parameterization

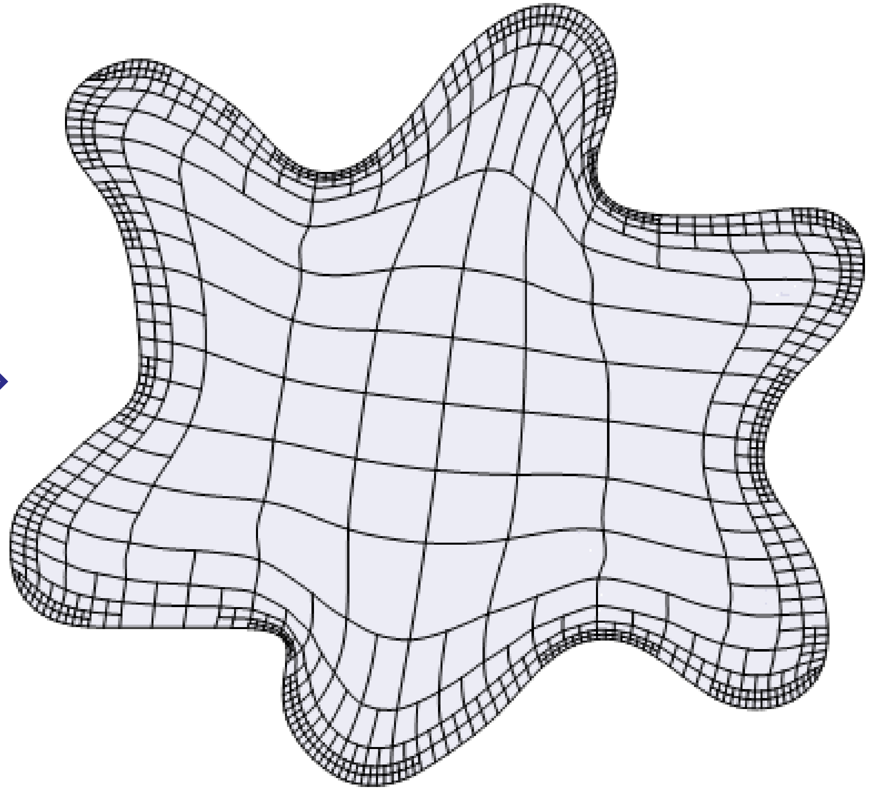
T-mesh



Parameter space

$$\mathbf{S}(\boldsymbol{\xi})$$

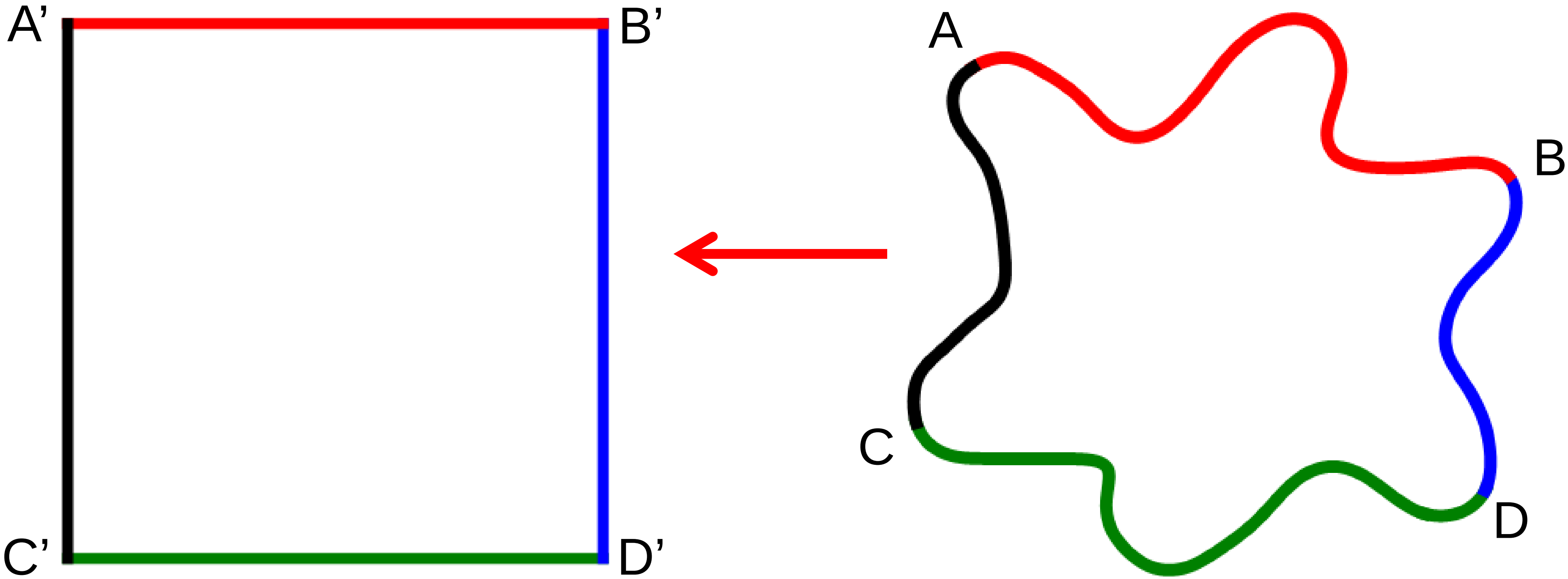
T-spline mesh



Physical space

The Meccano Method on T-meshes in 2-D

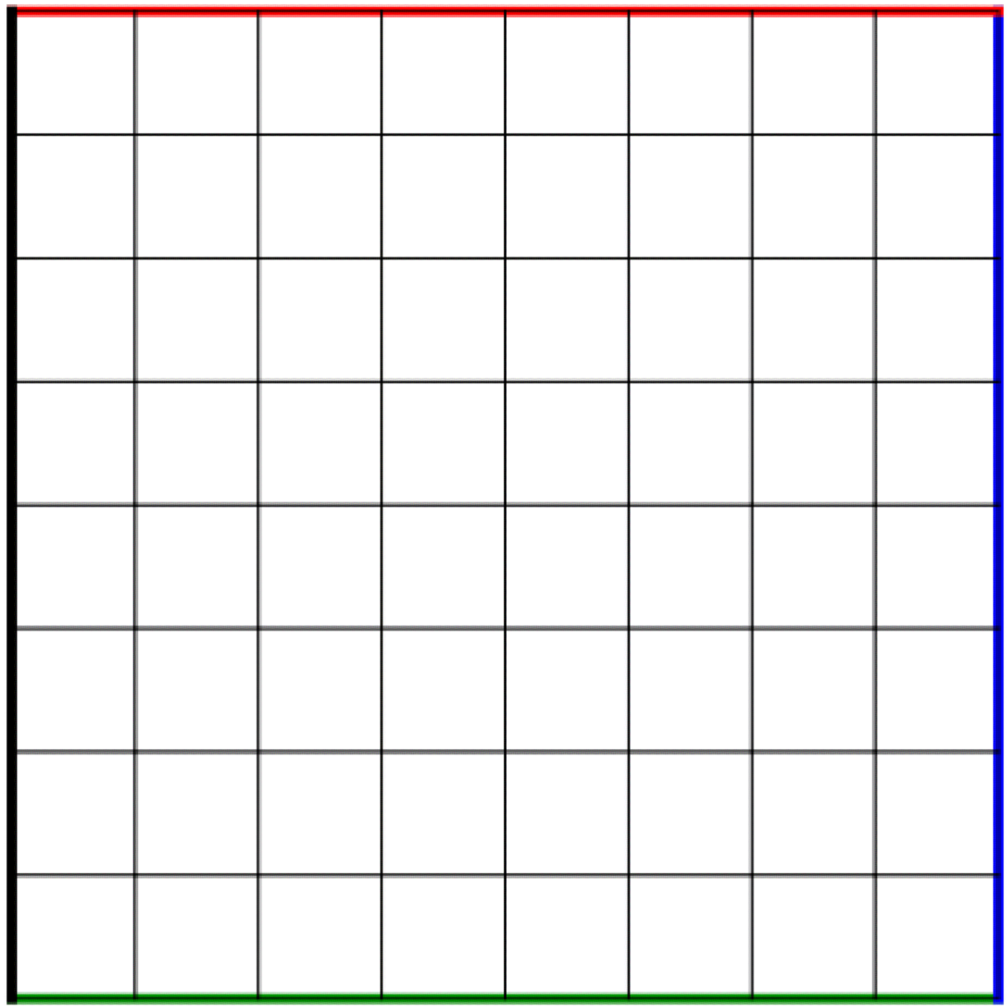
Step 1: Input boundary (image, polyline, curve, etc.) and boundary mapping



- Select four points (A, B, C, D) of the input boundary
- Boundary parameterization via chord-length

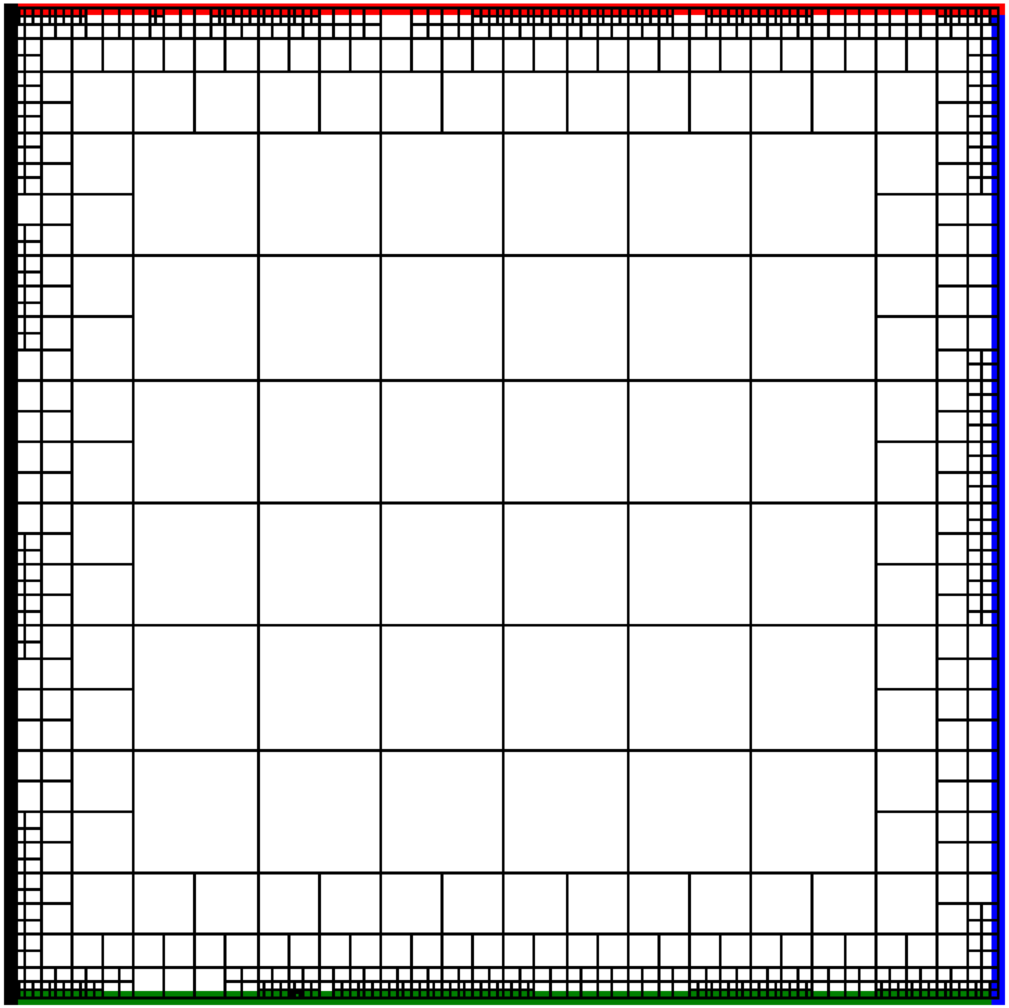
The Meccano Method on T-meshes in 2-D

Step 2: Coarse quadrilateral mesh of the meccano (parameter space)



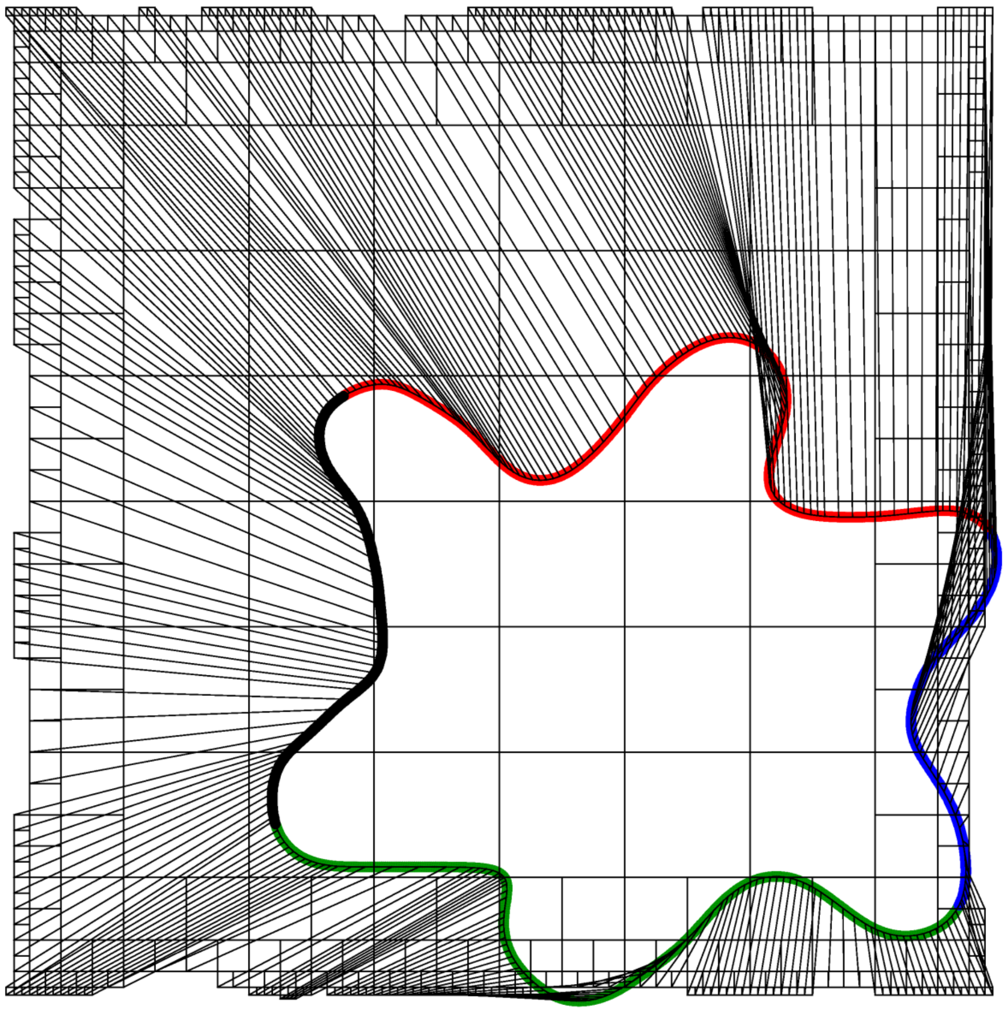
The Meccano Method on T-meshes in 2-D

Step 3: Refine mesh with quadtree subdivisions to approach the boundary



The Meccano Method on T-meshes in 2-D

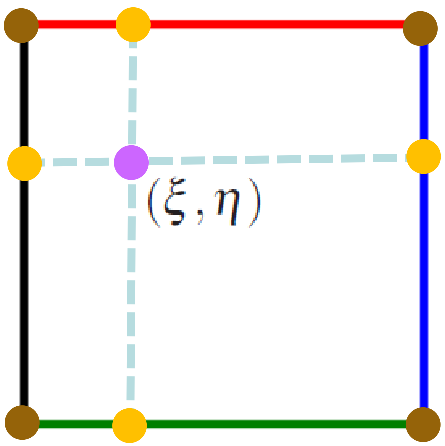
Step 4: Move the meccano boundary nodes to the object boundary

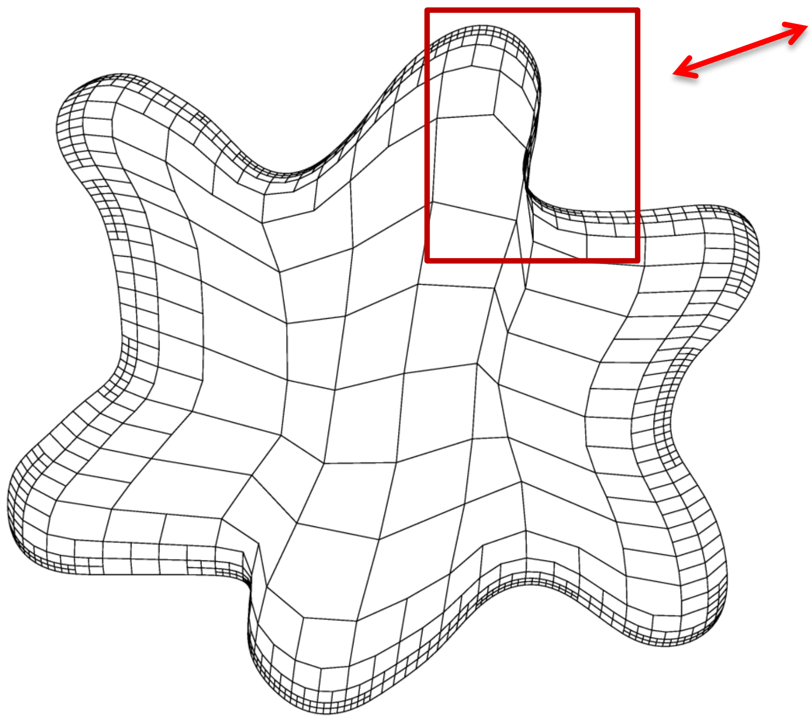


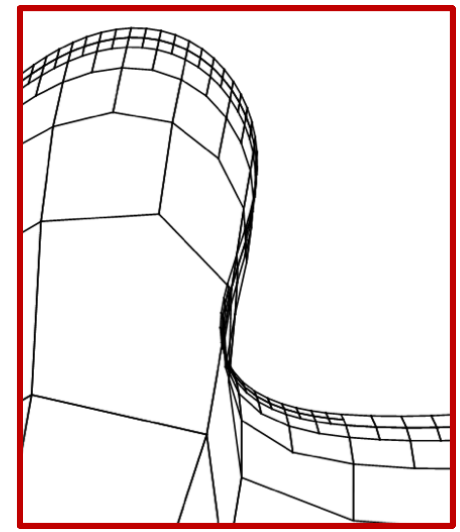
The Meccano Method on T-meshes in 2-D

Step 5: Inner node relocation with Coons patch to facilitate the optimization

$$\begin{aligned}\mathbf{x}(\xi,\eta) = &(1-\xi)\mathbf{x}(0,\eta)+\xi\mathbf{x}(1,\eta)\\ &+(1-\eta)\mathbf{x}(\xi,0)+\eta\mathbf{x}(\xi,1)\\ &-\begin{bmatrix}1-\xi & \xi\end{bmatrix}\begin{bmatrix}\mathbf{x}(0,0) & \mathbf{x}(0,1)\\ \mathbf{x}(1,0) & \mathbf{x}(1,1)\end{bmatrix}\begin{bmatrix}1-\eta\\ \eta\end{bmatrix}\end{aligned}$$



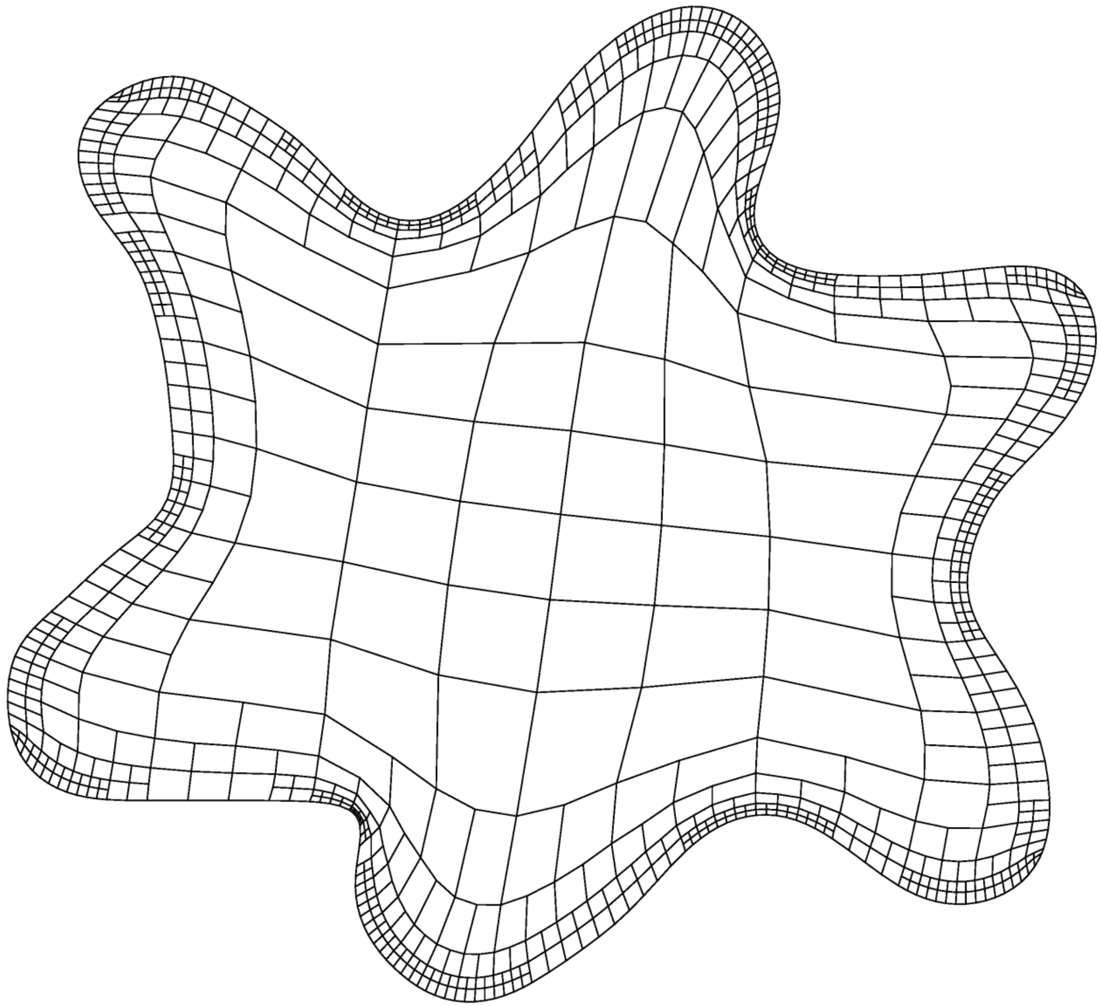




Mesh folder

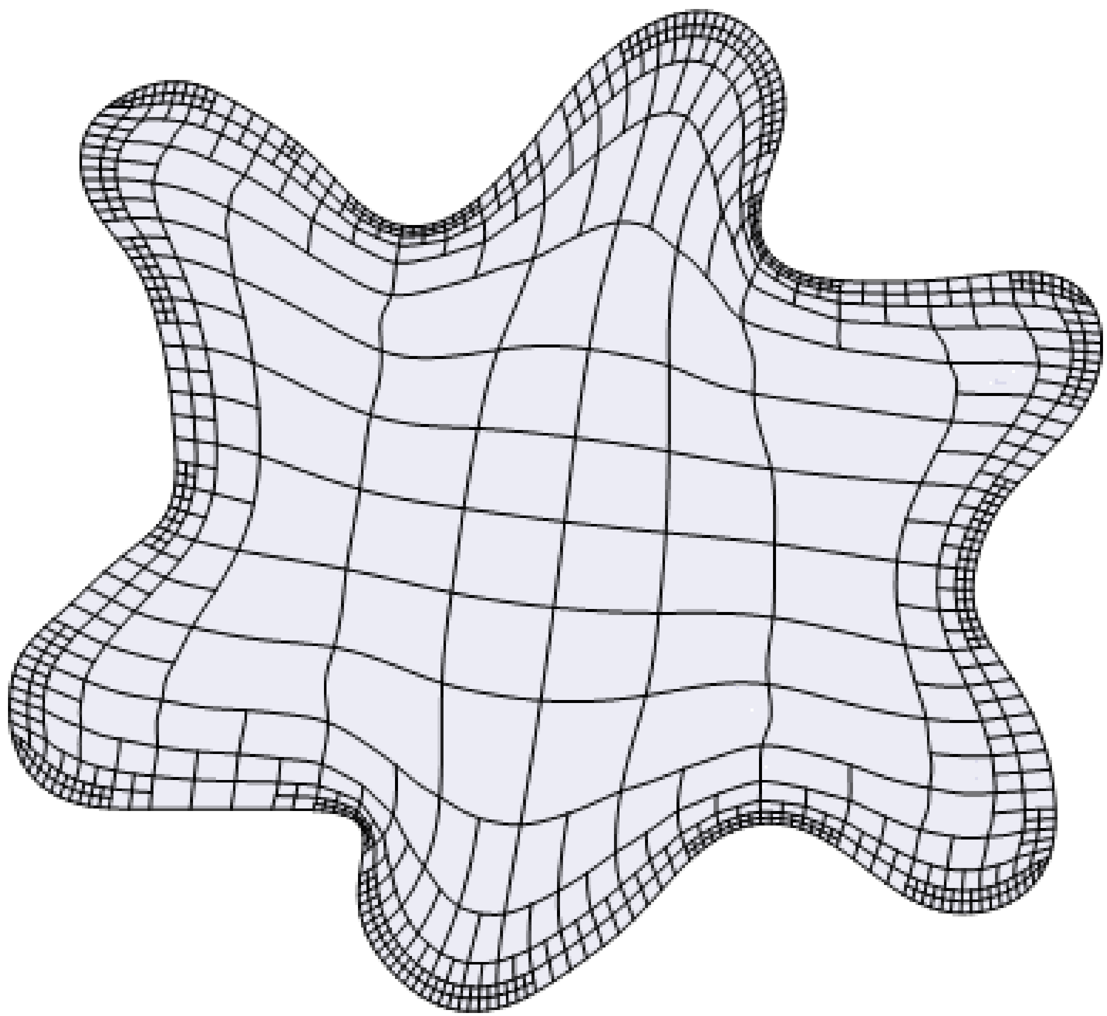
The Meccano Method on T-meshes in 2-D

Step 6: Simultaneous Untangling and Smoothing (SUS) of the T-mesh



The Meccano Method on T-meshes in 2-D

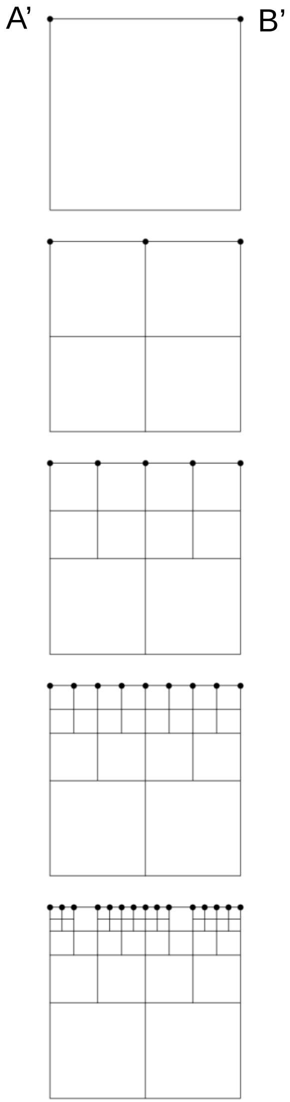
Step 7: T-spline representation of the spot



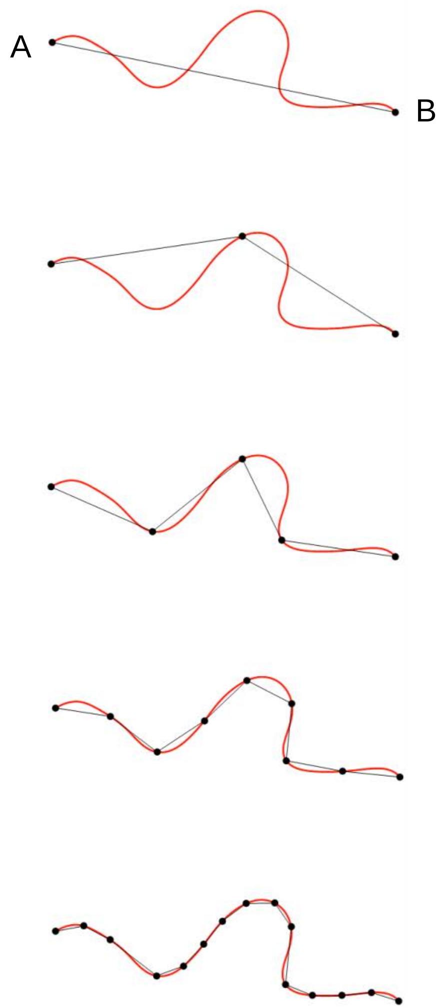
Boundary Approach in 2-D

Input data: Boundary polyline approximation (red color line)

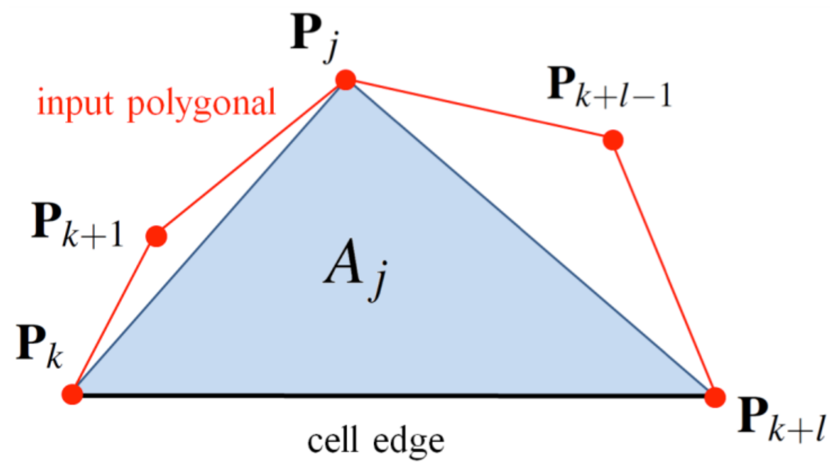
T-mesh adaptation in parameter space



Boundary approximation in physical space



Boundary edge refinement criterion:



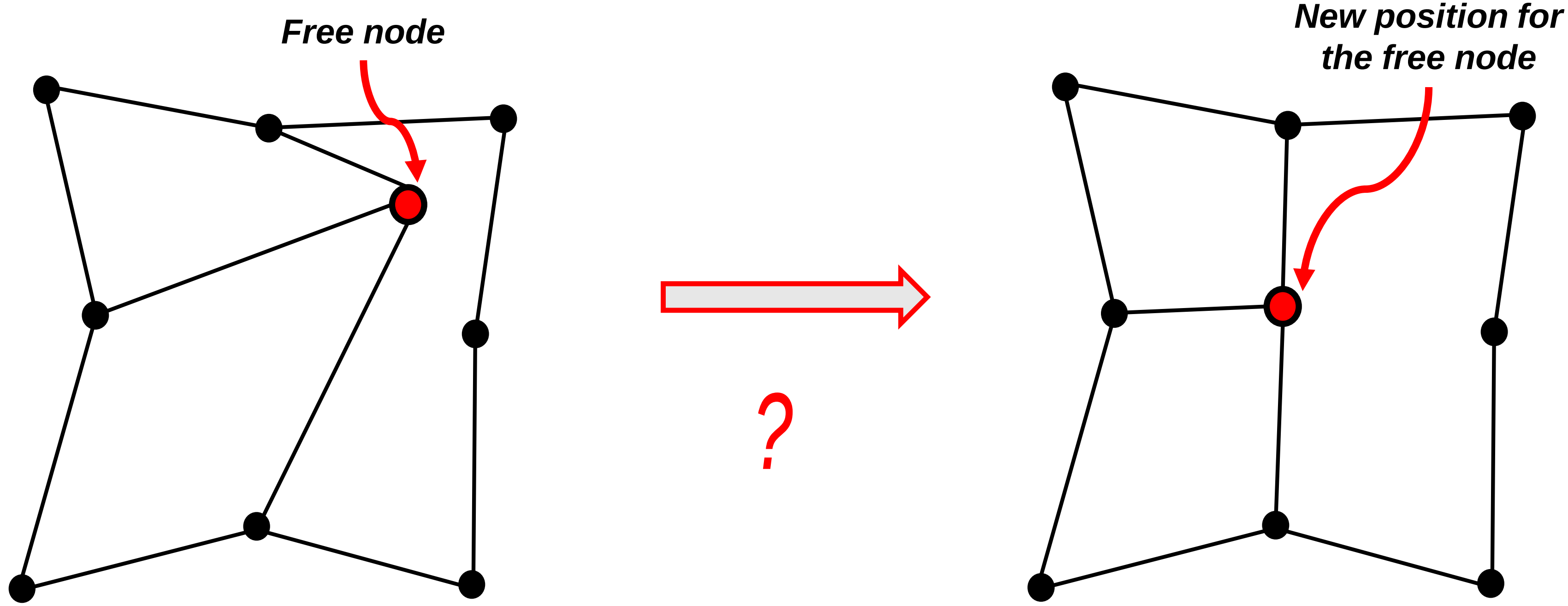
$$\exists j:\ A_j > \varepsilon \Rightarrow \text{refine}$$

$$A_j = \text{Area}(\mathbf{P}_k, \mathbf{P}_j, \mathbf{P}_{k+l})$$

$$k+1 \leqslant j \leqslant k+l-1$$

Local optimization

***Objective:* Improve the quality of the local mesh by minimizing an objective function**



Local mesh

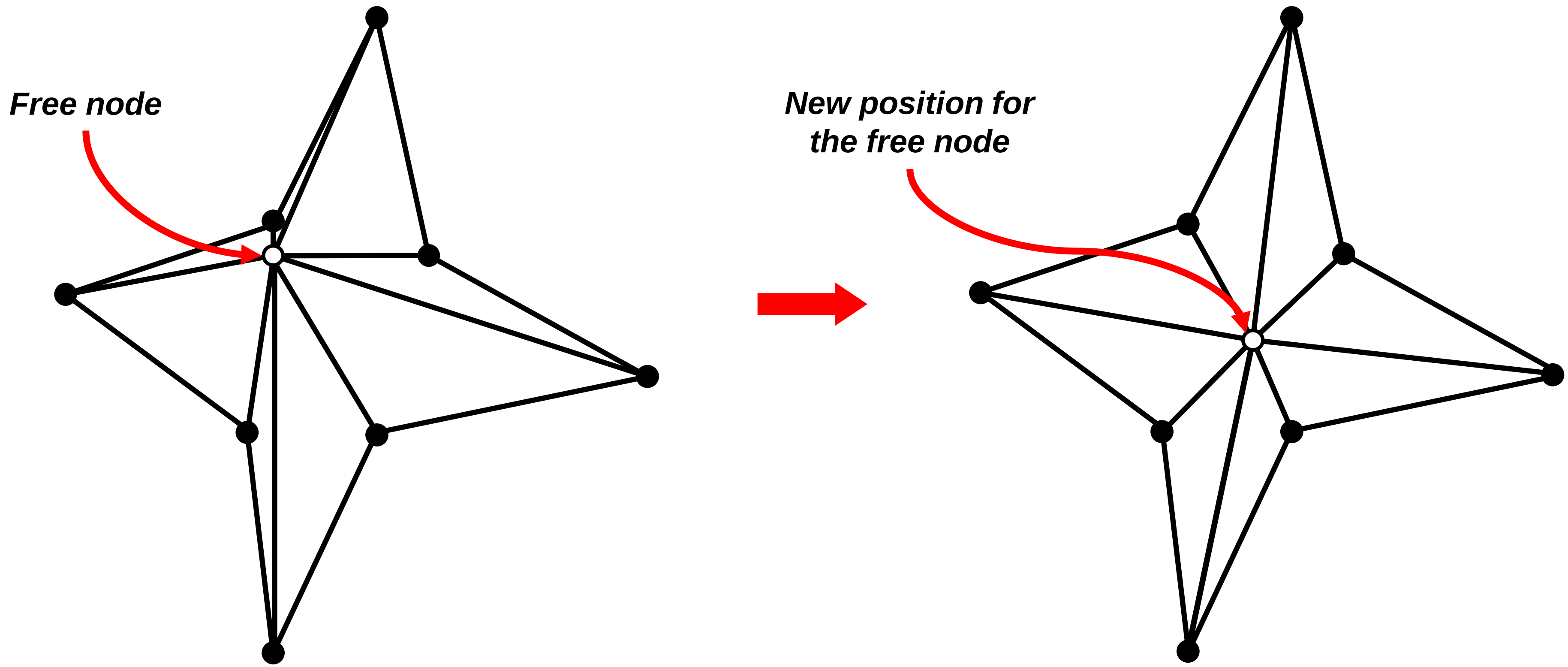
Optimized local mesh

Simultaneous Untangling and Smoothing

Case of plane triangulations (CMAME 2003)

Local optimization

***Objective:* Improve the quality of the local mesh by minimizing an objective function**

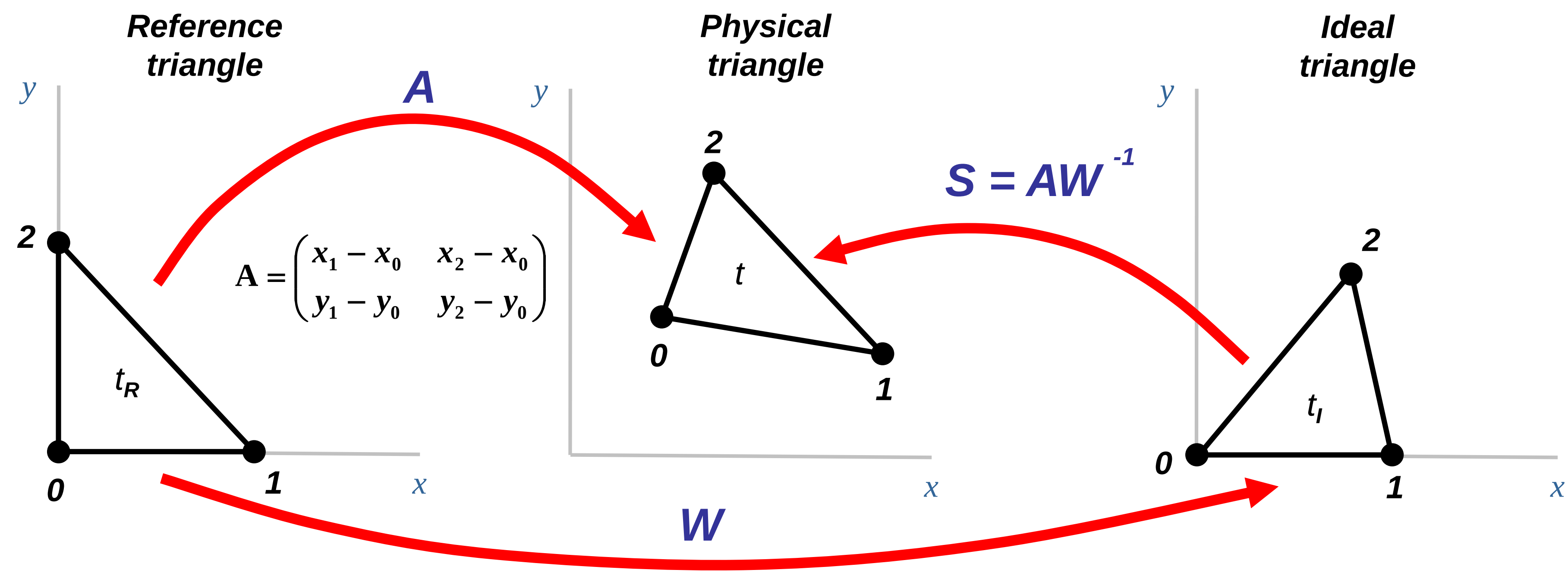


Local mesh   *Optimized local mesh*

Simultaneous Untangling and Smoothing (CMAME 2003)

Weighted Jacobian Matrix on a Plane

SIANI



$t_I \xrightarrow{S} t$ $S = AW^{-1}$: **Weighted Jacobian matrix**

An algebraic quality metric of t (mean ratio)

$$\boxed{\mathbf{q} = \frac{2\boldsymbol{\sigma}}{\|\mathbf{S}\|^2} = \frac{1}{\eta}}$$

where:

$$\|\mathbf{S}\| = \sqrt{\mathbf{tr}\left(\mathbf{S}^T\mathbf{S}\right)}$$

$$\boldsymbol{\sigma} = \det(\mathbf{S})$$

Simultaneous Untangling and Smoothing (CMAME 2003)

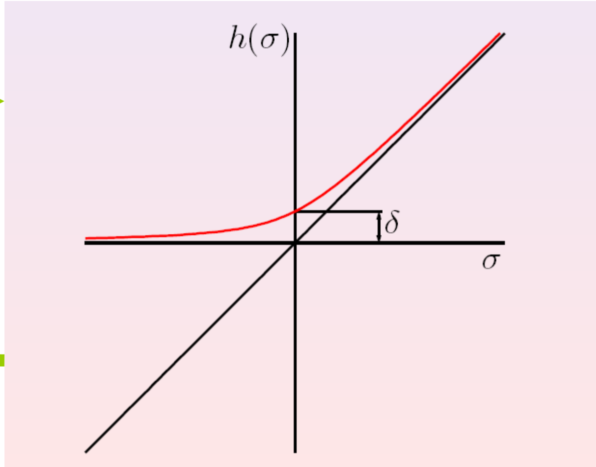
Local objective function for plane triangulations

SUS Code: Freely-available in http://www.dca.iusiani.ulpgc.es/proyecto2012-2014

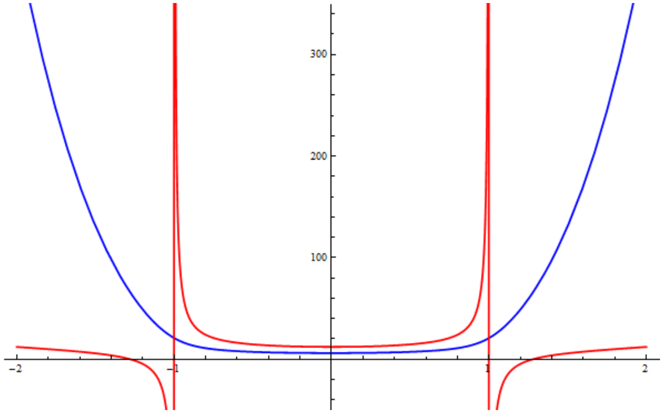
SIANI

Original function: $$K(\mathbf{x}) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2\,\sigma_m}$$

Modified function: $$K^*(\mathbf{x}) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2h(\sigma_m)}$$



$$h(\sigma) = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\delta^2}\right)$$

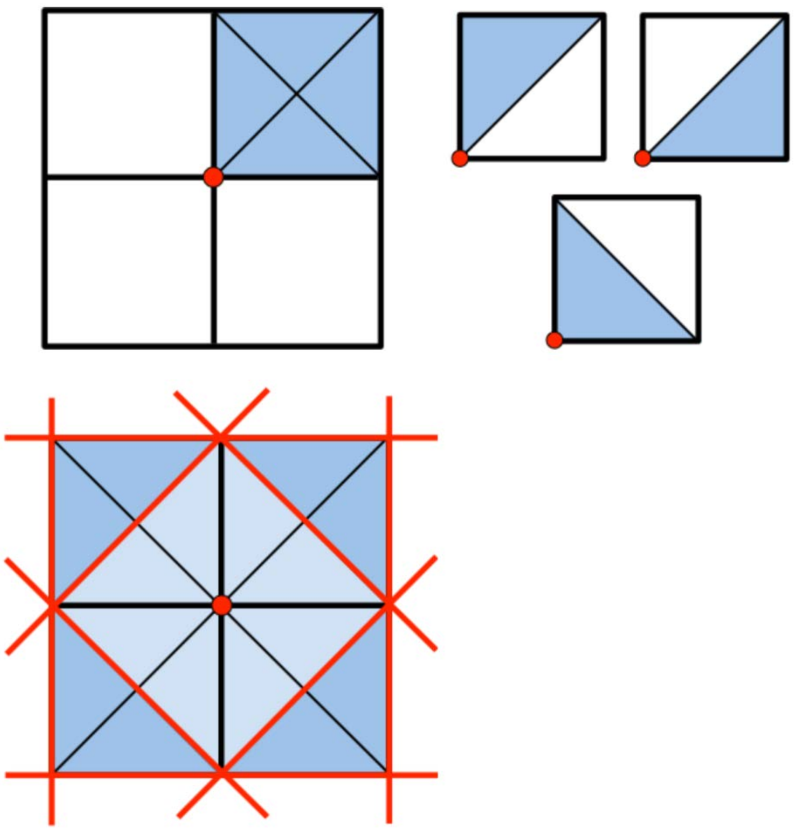


Modified function (blue) is regular in all R^2 and it approximates the same minimum that the original function (red). Moreover, it allows a simultaneous untangling and smoothing of triangular meshes

Simultaneous Untangling and Smoothing of T-meshes

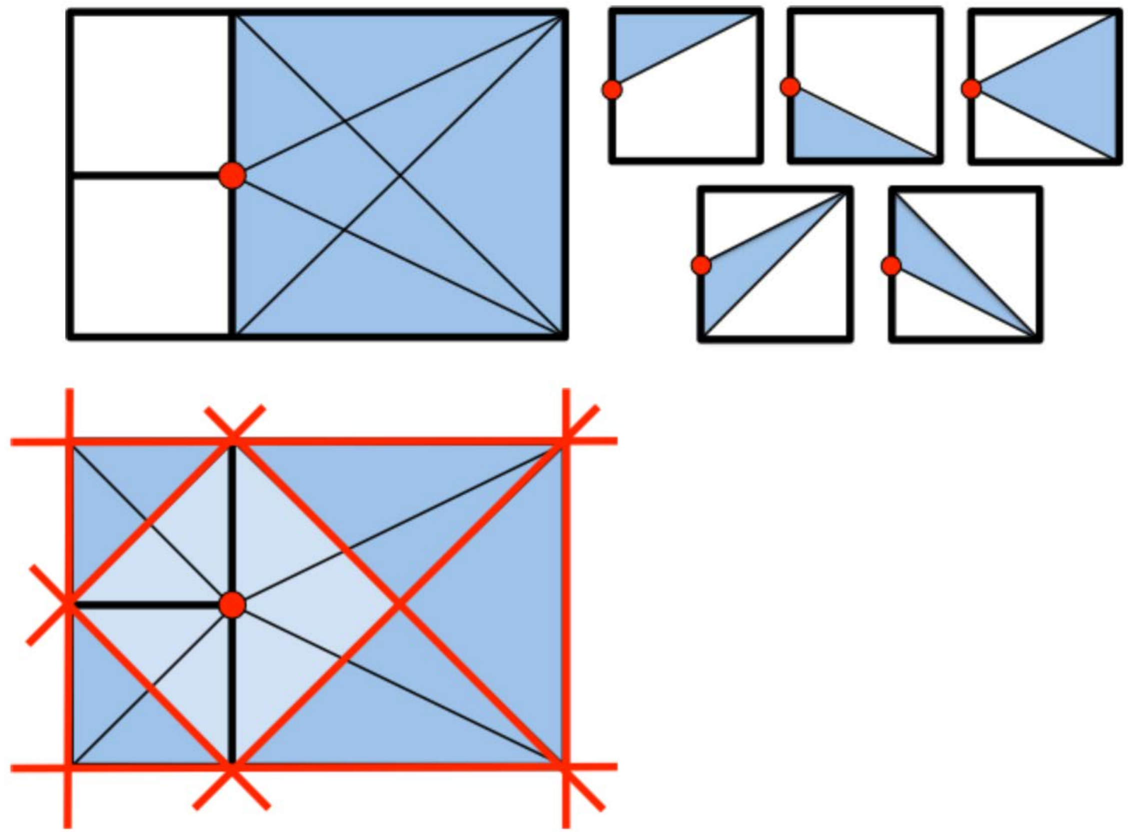
Triangle decomposition of the T-mesh cells

Case 1: Free node is a regular node



Barriers and feasible region for a regular node

Case 2: Free node is a hanging node



Barriers and feasible region for a hanging node

Simultaneous Untangling and Smoothing of T-meshes

Optimization is guided by the parametric T-mesh

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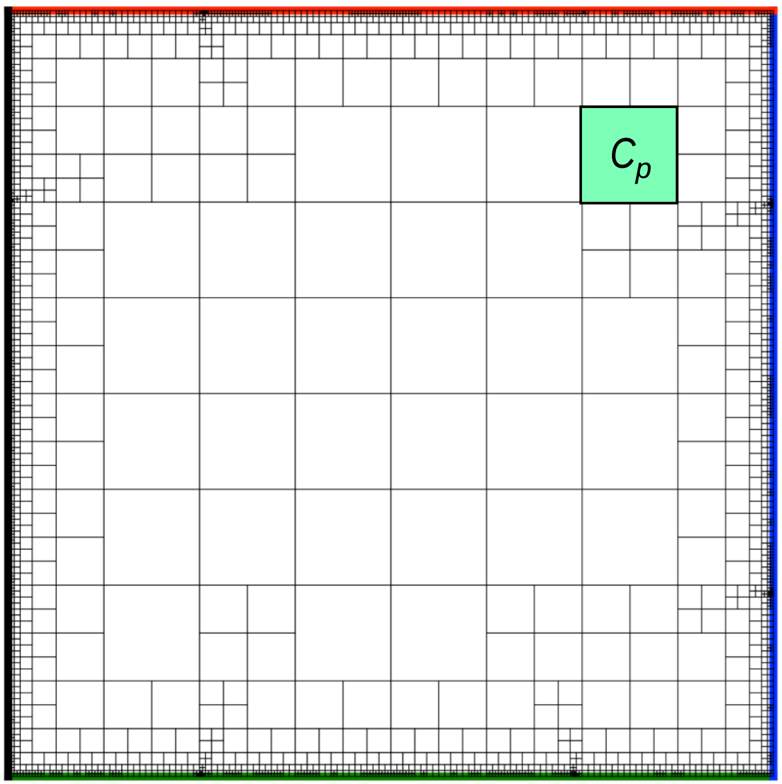
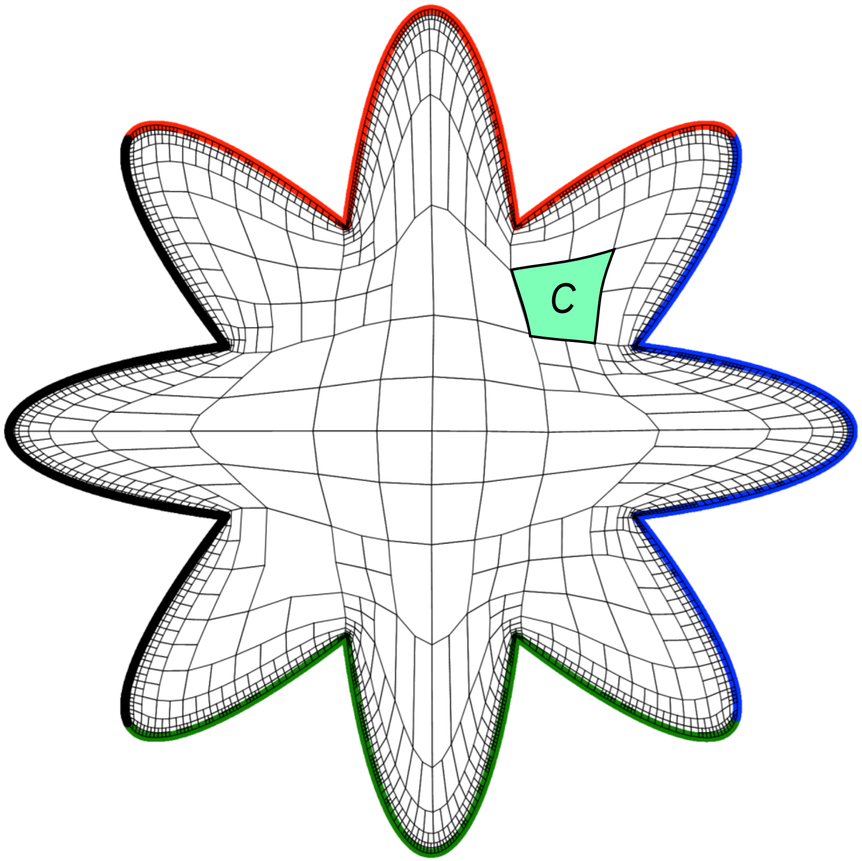
Physical cell C must be as similar as possible to the counterpart in the parametric space C_p



Parametric T-mesh

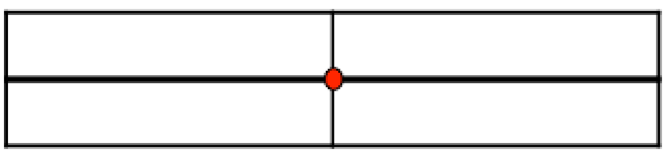


Optimized physical T-mesh

Simultaneous Untangling and Smoothing of T-meshes

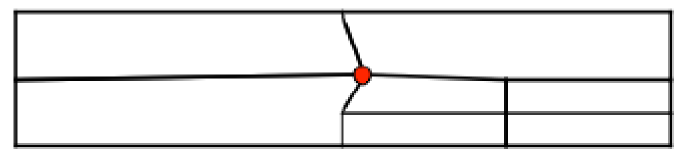
Problems appears with the non-weighted objective function K*

*Satisfactory result for a conformal submesh using K**



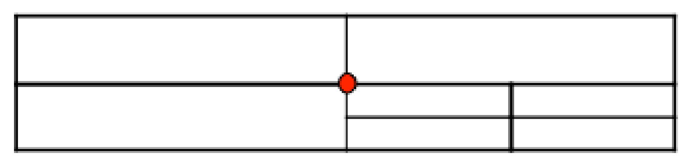
(a)

*Not satisfactory result for a non-conformal submesh using K**



(b)

Desirable result: Orthogonal mesh for a non-conformal case ***(weighted objective function*** K_{τ}^{*}***)***

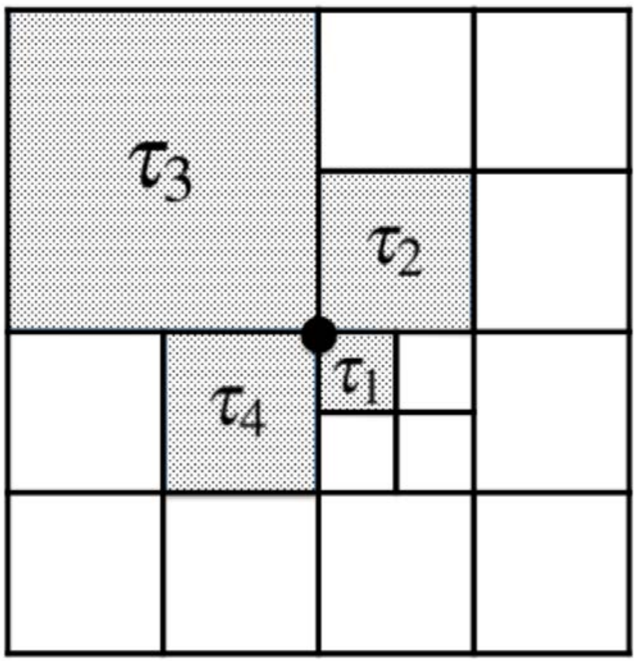


(c)

Simultaneous Untangling and Smoothing of T-meshes

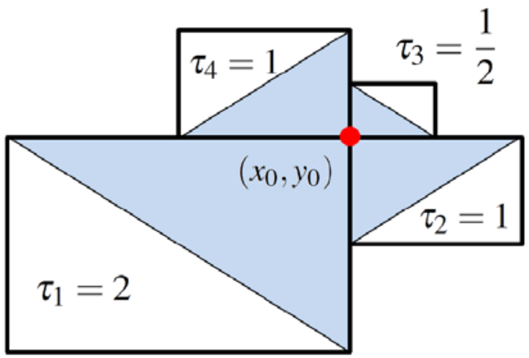
Solution by using weighted objective functions (regular node)

$$K_{\tau}^{*}(\mathbf{x}) = \tau_1 \sum_{m=1}^{3} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_2 \sum_{m=4}^{6} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_3 \sum_{m=7}^{9} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_4 \sum_{m=10}^{12} \frac{\|S_m\|^2}{2h(\sigma_m)}$$

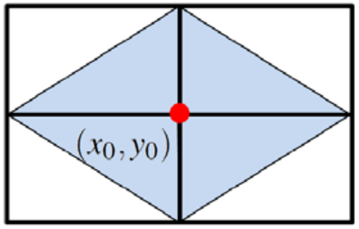


All possible weights for balanced quadtrees:

$$\Rightarrow \tau_1 = 1,\ \tau_2 = \tau_4 = 2 \text{ y } \tau_3 = 4$$



Non-conformal mesh

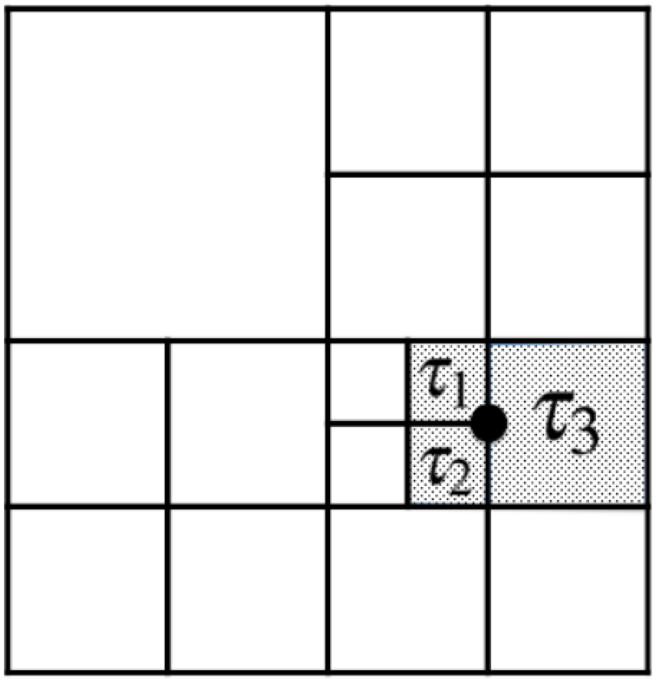


Conformal mesh

Simultaneous Untangling and Smoothing of T-meshes

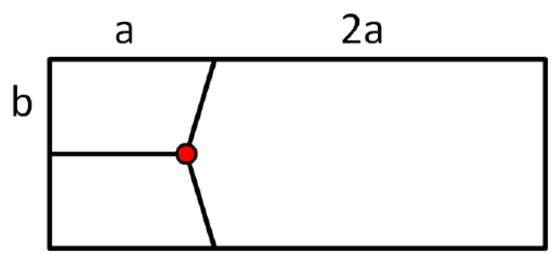
Solution by using weighted objective functions (hanging node)

$$K_{\tau}^{*}(\mathbf{x}) = \tau_1 \sum_{m=1}^{3} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_2 \sum_{m=4}^{6} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_3 \sum_{m=7}^{11} \frac{\|S_m\|^2}{2h(\sigma_m)}$$

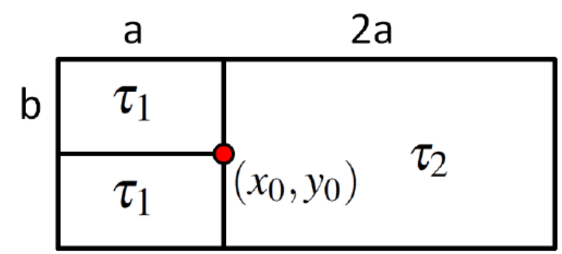


All possible weights for balanced quadtrees:

$$\tau_1 = \tau_2 = 1 \text{ y } \tau_3 = \frac{8}{5}$$



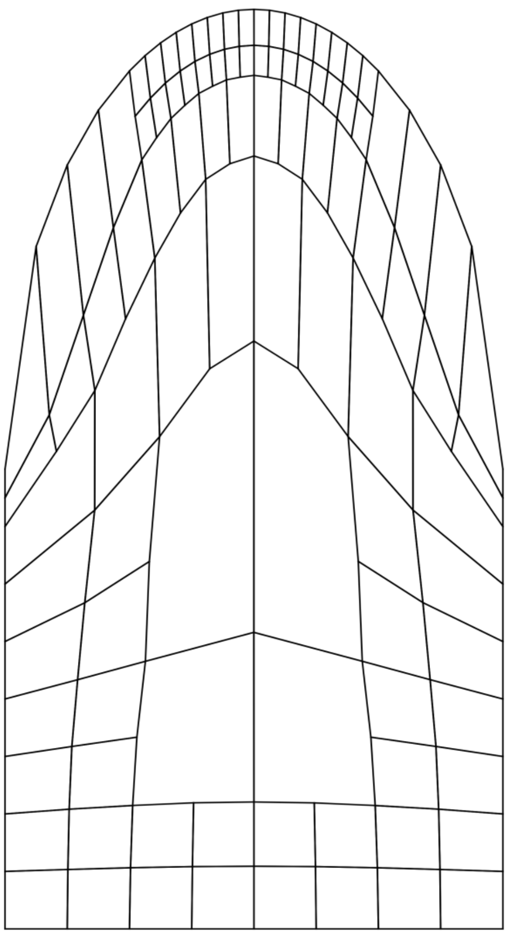
Optimized mesh
without weights



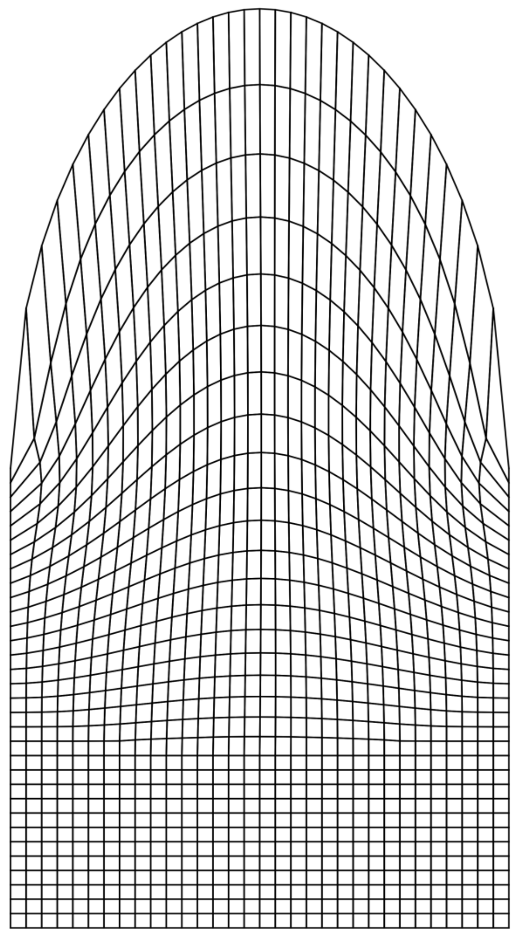
Optimized mesh
with weights

Objective Functions Comparison

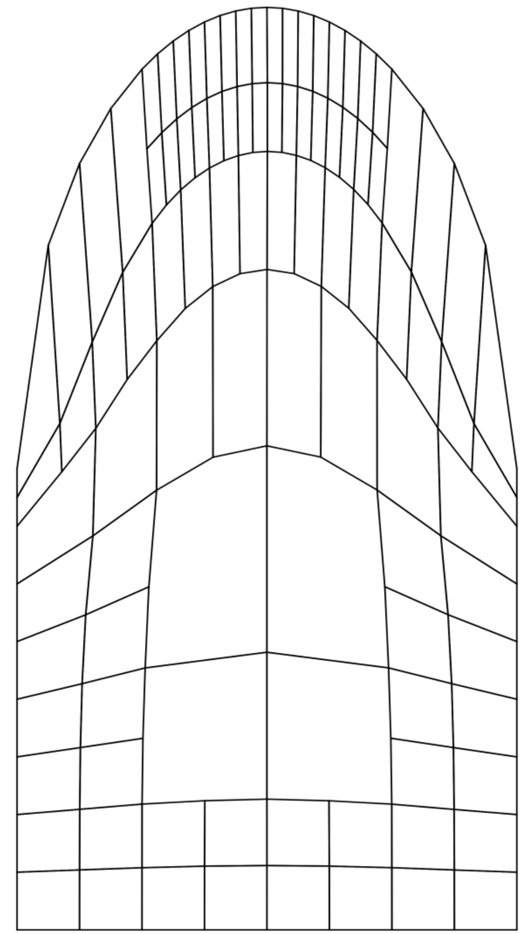
k^* (without weights) versus $k^*_{\square}$ (with weights)



T-mesh optimized with K^*
(without weights)



Uniform mesh optimized with K^*

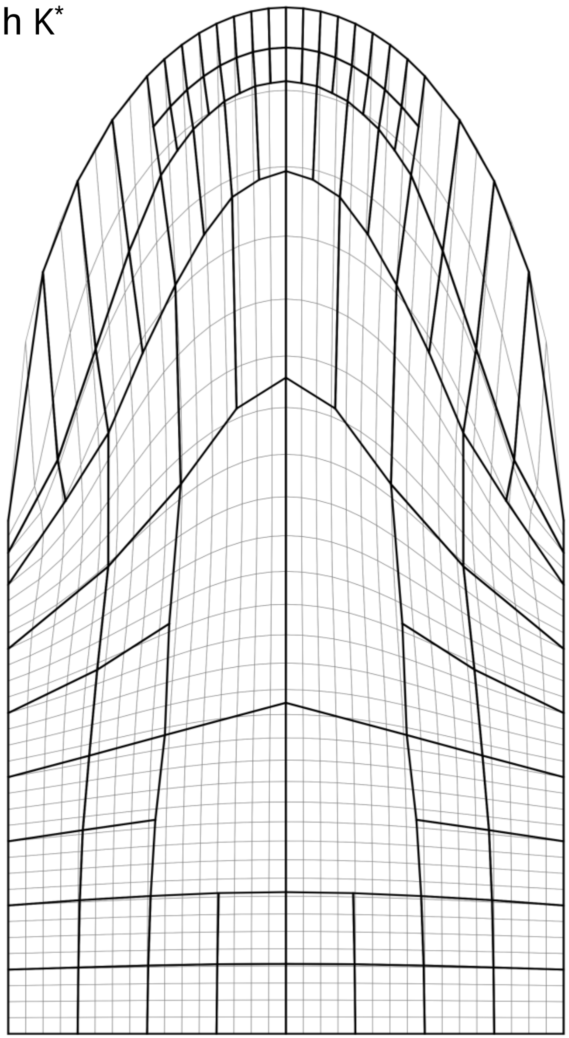


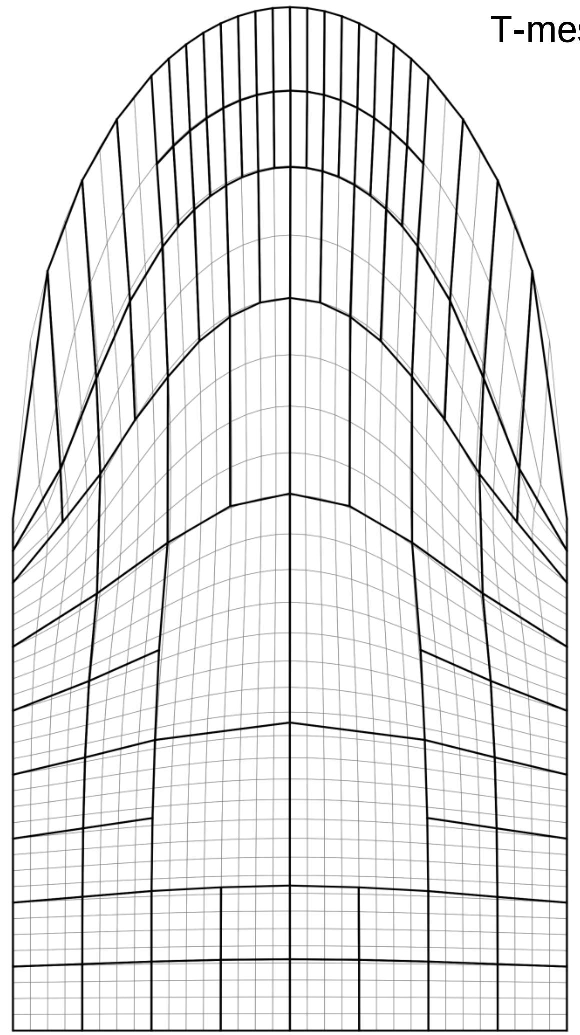
T-mesh optimized with $K^*_{\square}$
(with weights)
Close to uniform mesh positions

Objective Functions Comparison

k^* (without weights) versus $k^*_{\square}$ (with weights)

T-mesh optimized with K^*
(without weights)





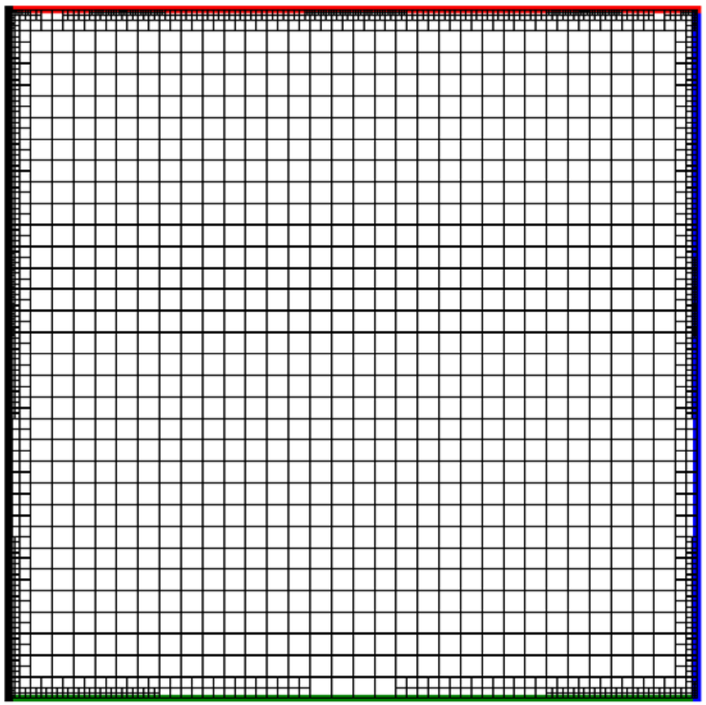
T-mesh optimized with $K^*_{\square}$
(with weights)

Simultaneous Untangling and Smoothing of T-meshes

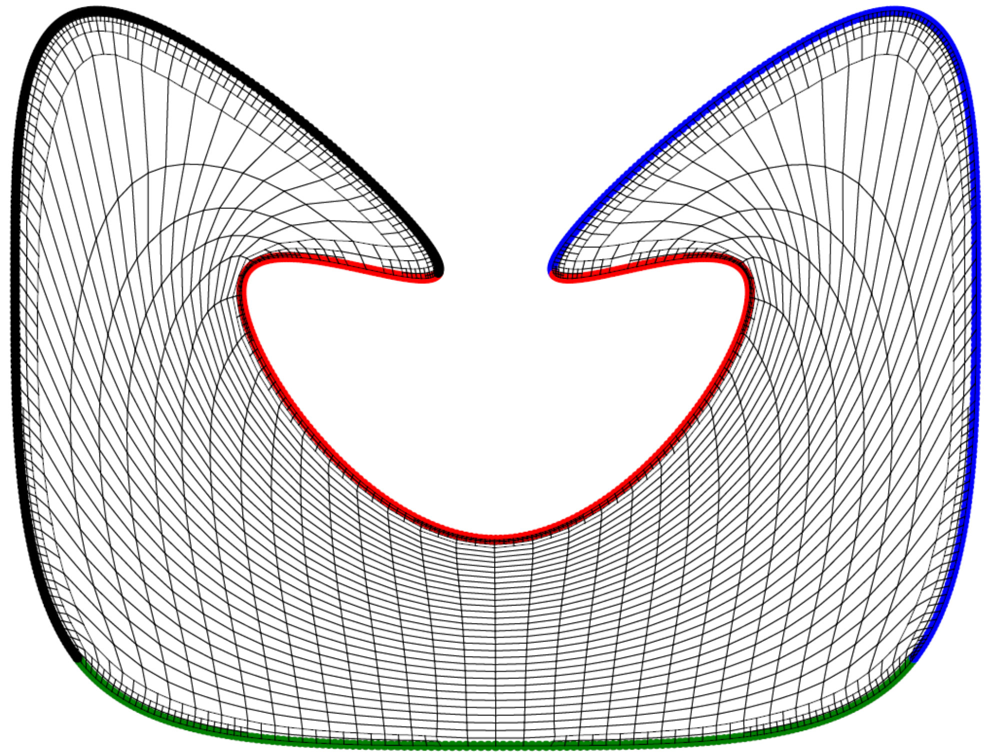
T-mesh transformation along the SUS process: Example

T-mesh

T-spline



$$\mathbf{S}(\boldsymbol{\xi}) \Longrightarrow$$

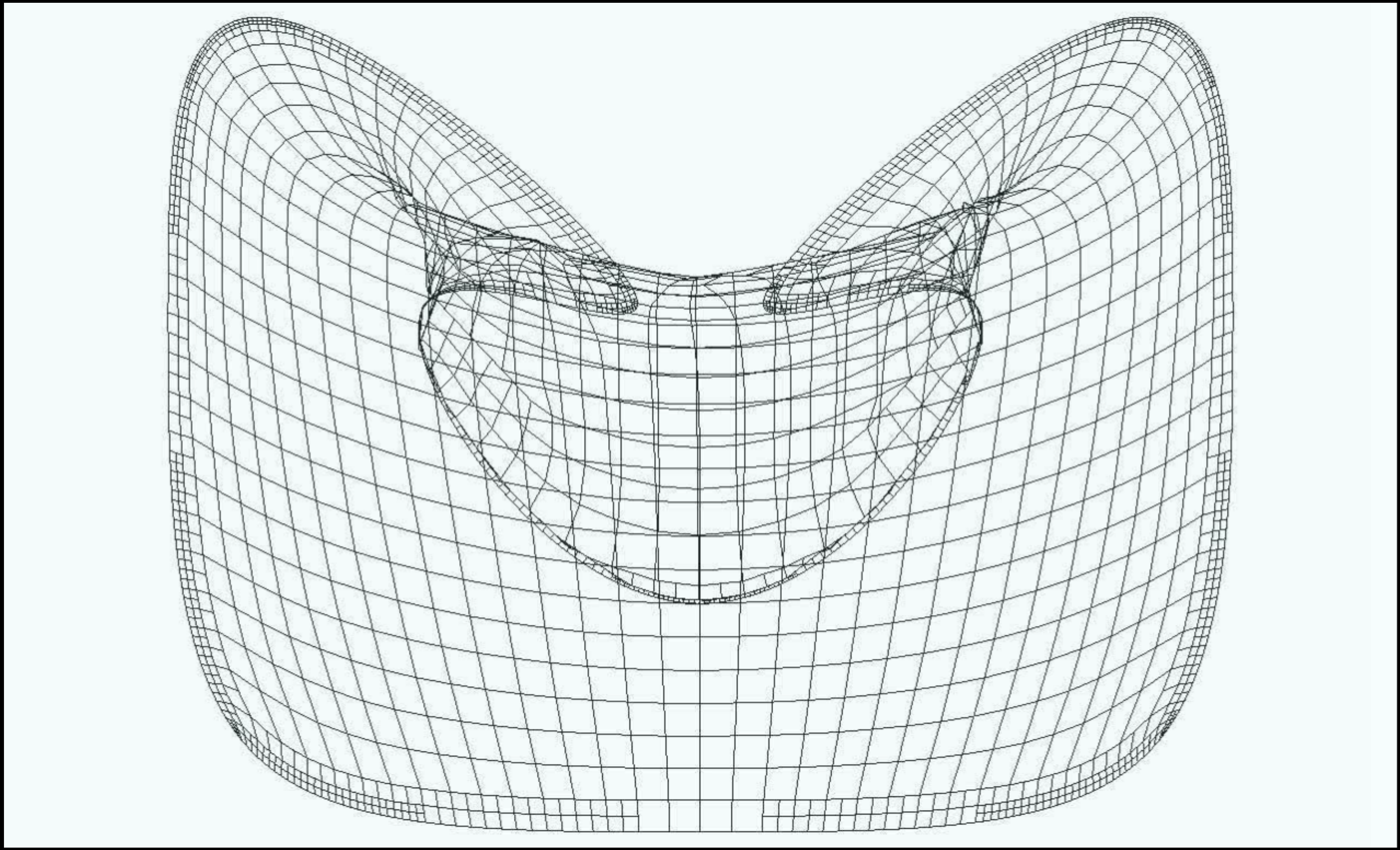


Parameter space

Physical space

Simultaneous Untangling and Smoothing of T-meshes

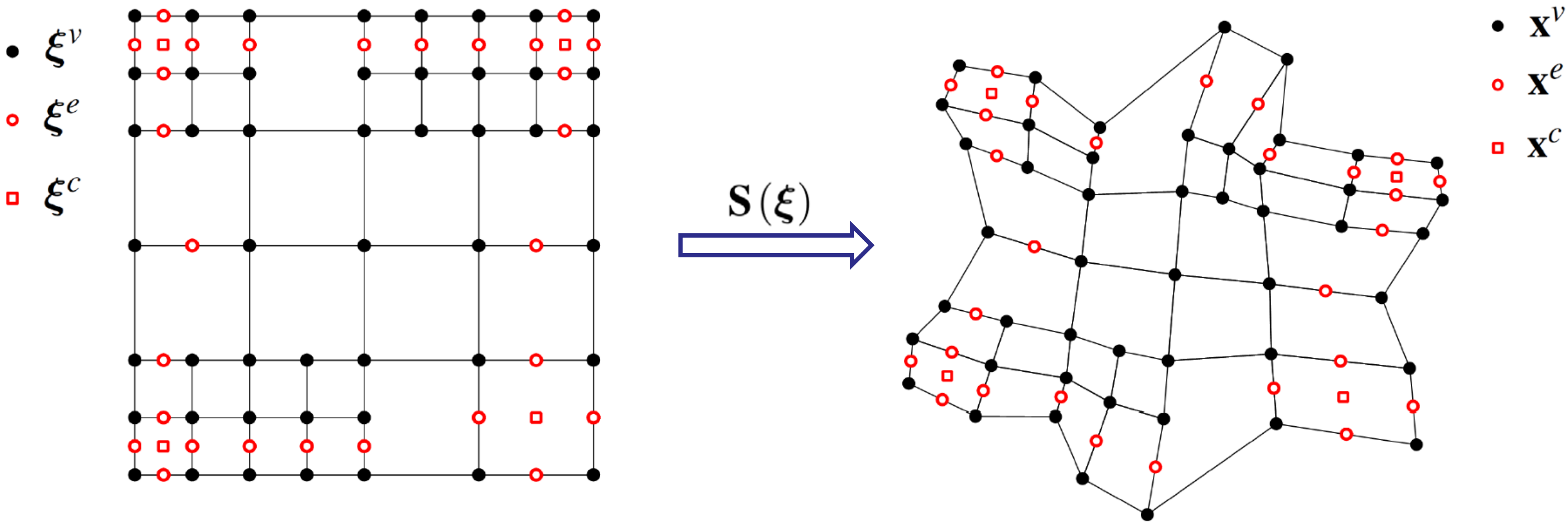
T-mesh transformation along the SUS process: Video



T-spline Parameterization

Determination of control points by imposing interpolation conditions

$$\mathbf{S}(\boldsymbol{\xi}) = \sum_{\alpha \in A} \mathbf{P}_\alpha R_\alpha(\boldsymbol{\xi})$$



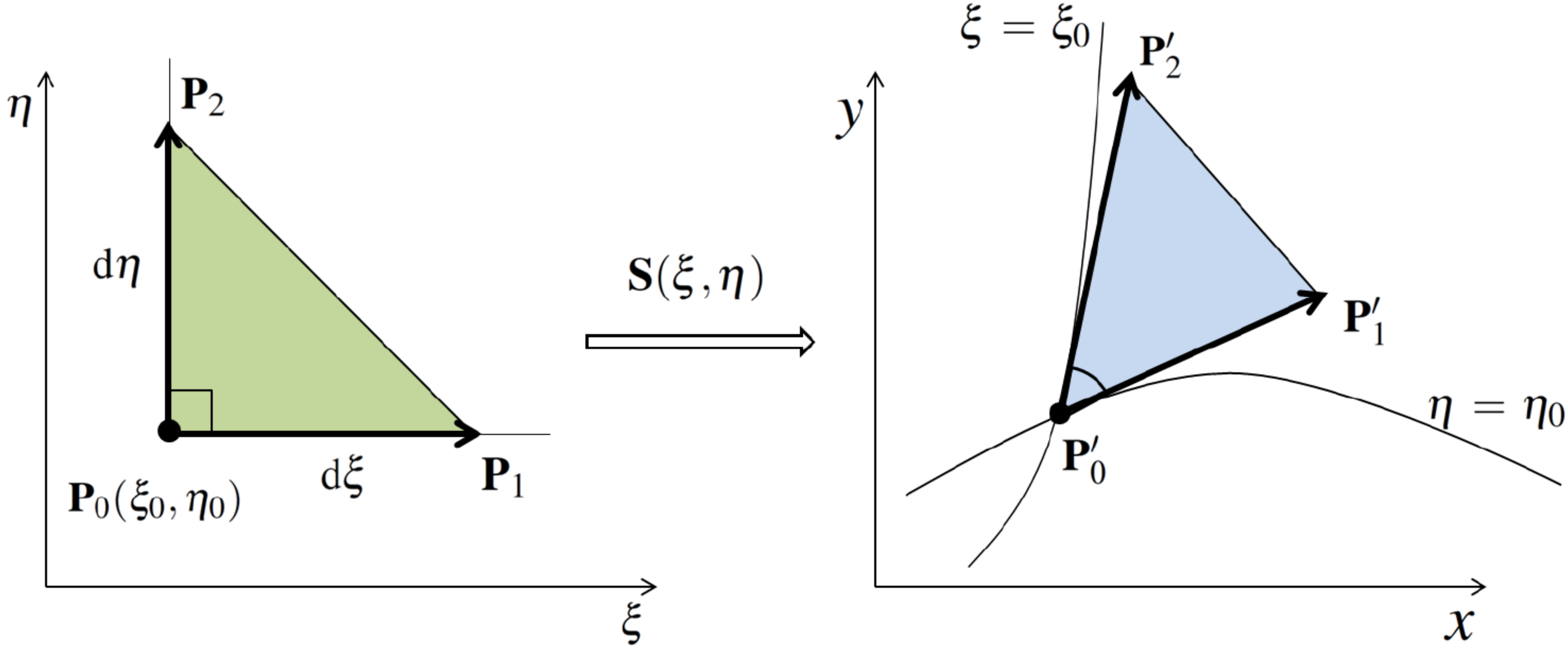
Mean Ratio Jacobian $J_r(\xi)$ of Parametric Transformation

A quality metric of the T-spline mapping at any point P_0

SIANI

$$-1 \leq J_r(\xi) = \frac{2\det(J)}{\|J\|^2} \leq 1$$

where J is the jacobian matrix of the T-spline mapping $\mathbf{S}$

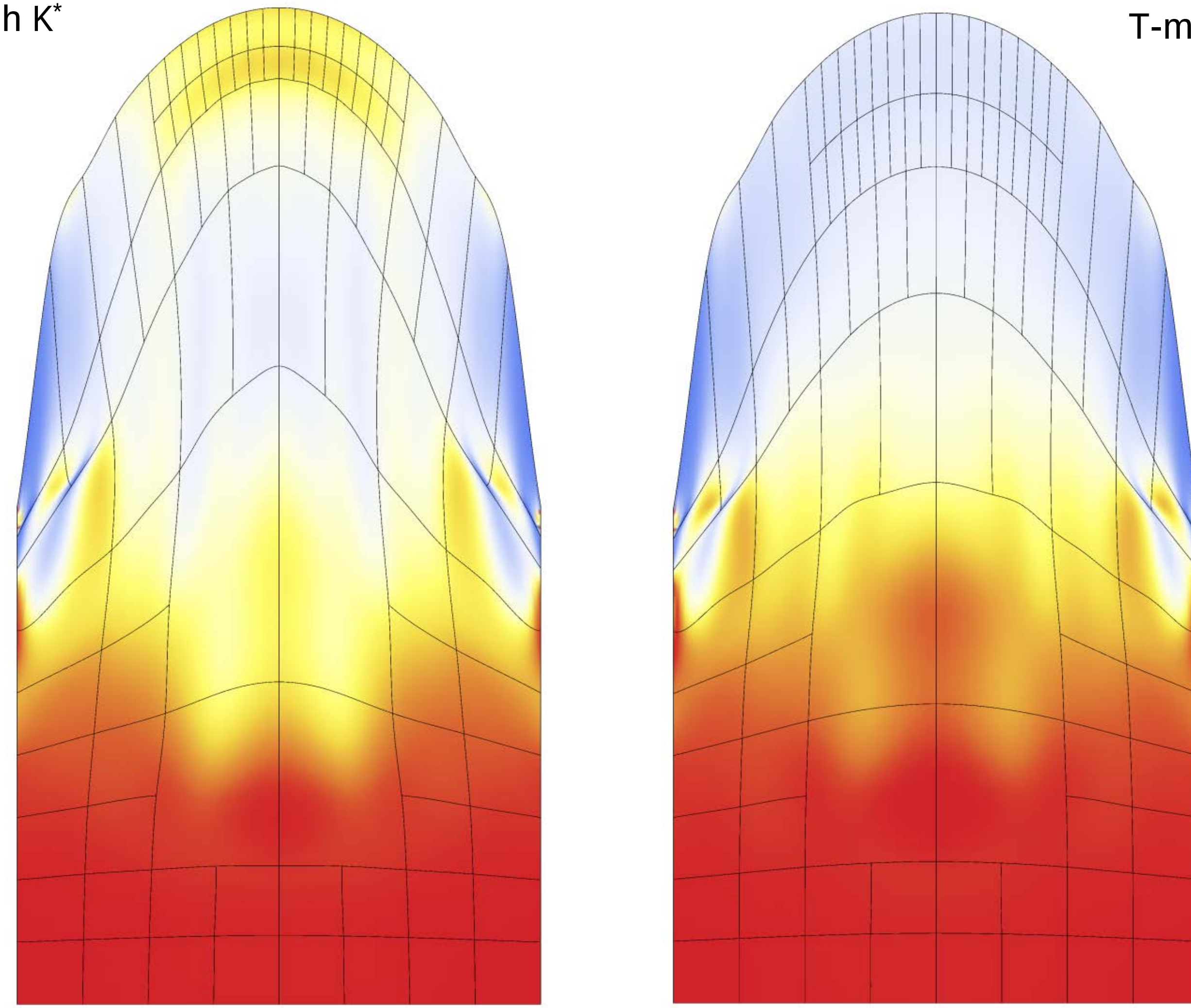


Objective Functions Comparison

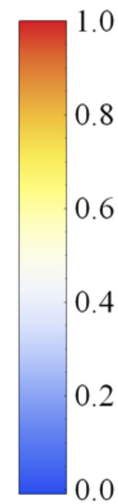
k^* (without weights) versus $k^*_{\square}$ (with weights)

Mean ratio Jacobian

T-mesh optimized with K^* (without weights)

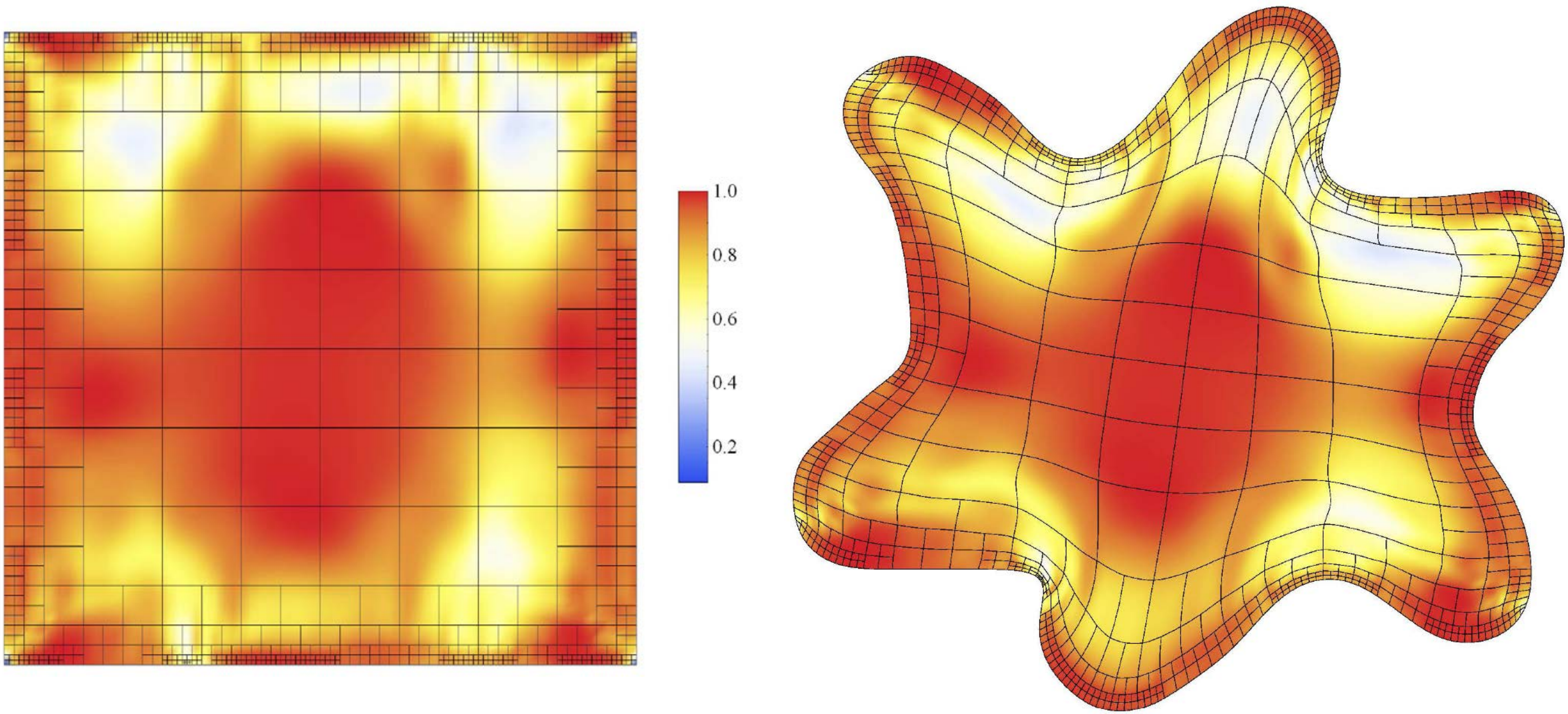


T-mesh optimized with $K^*_{\square}$ (with weights)



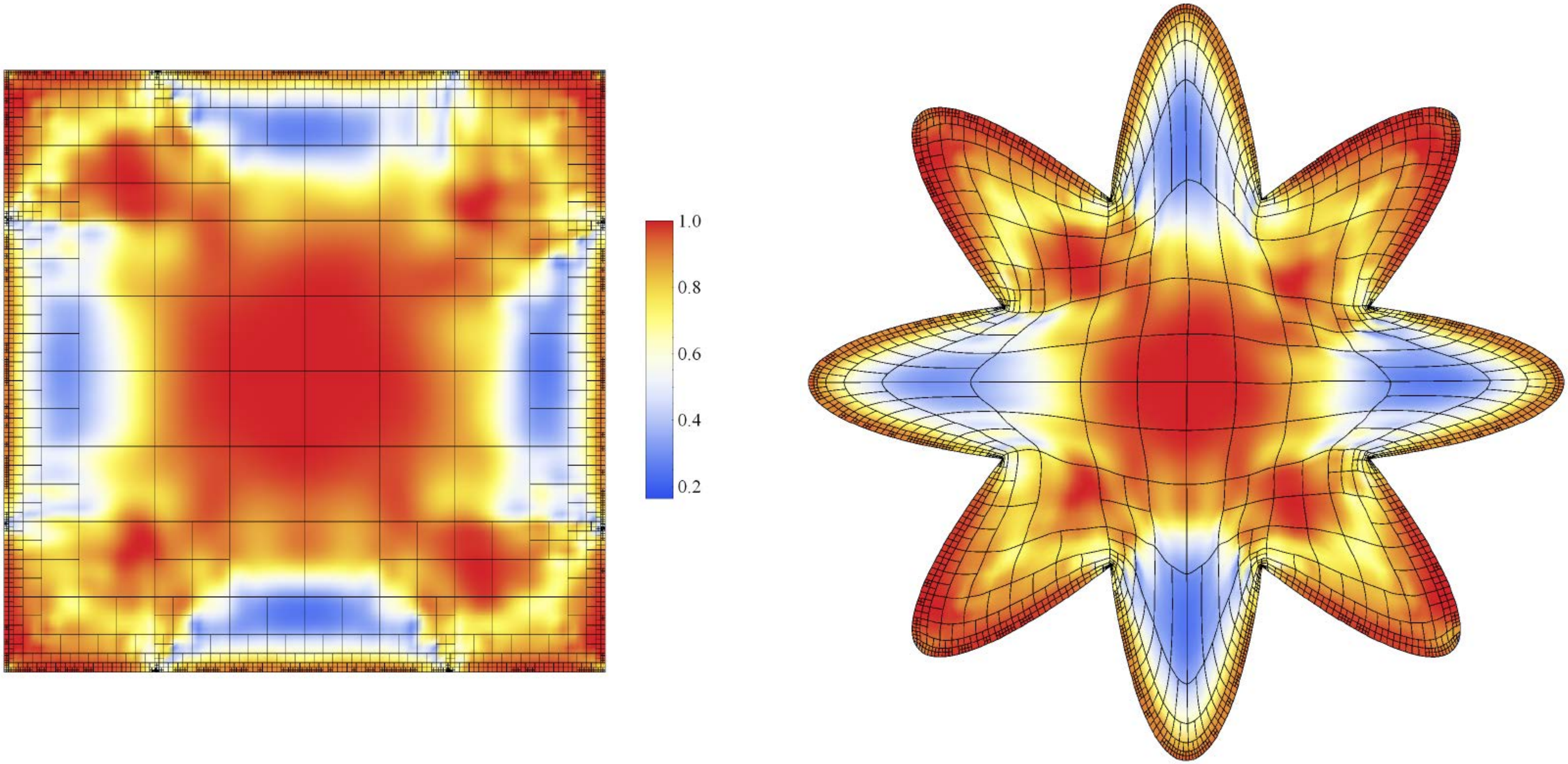
Applications: Isogeometric Modeling

The Spot (Mean Ratio Jacobian)



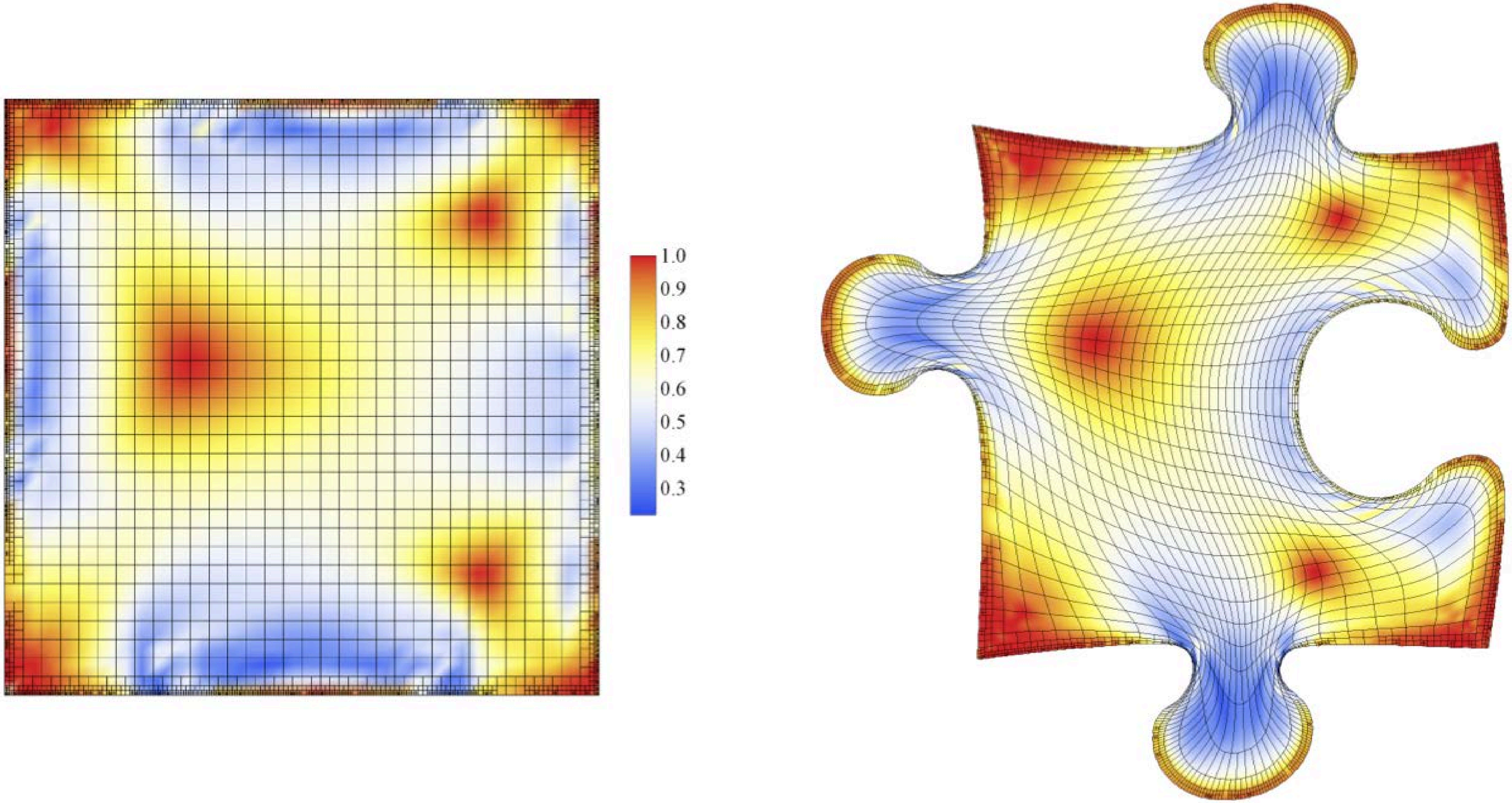
Applications: Isogeometric Modeling

The Flower (Mean Ratio Jacobian)



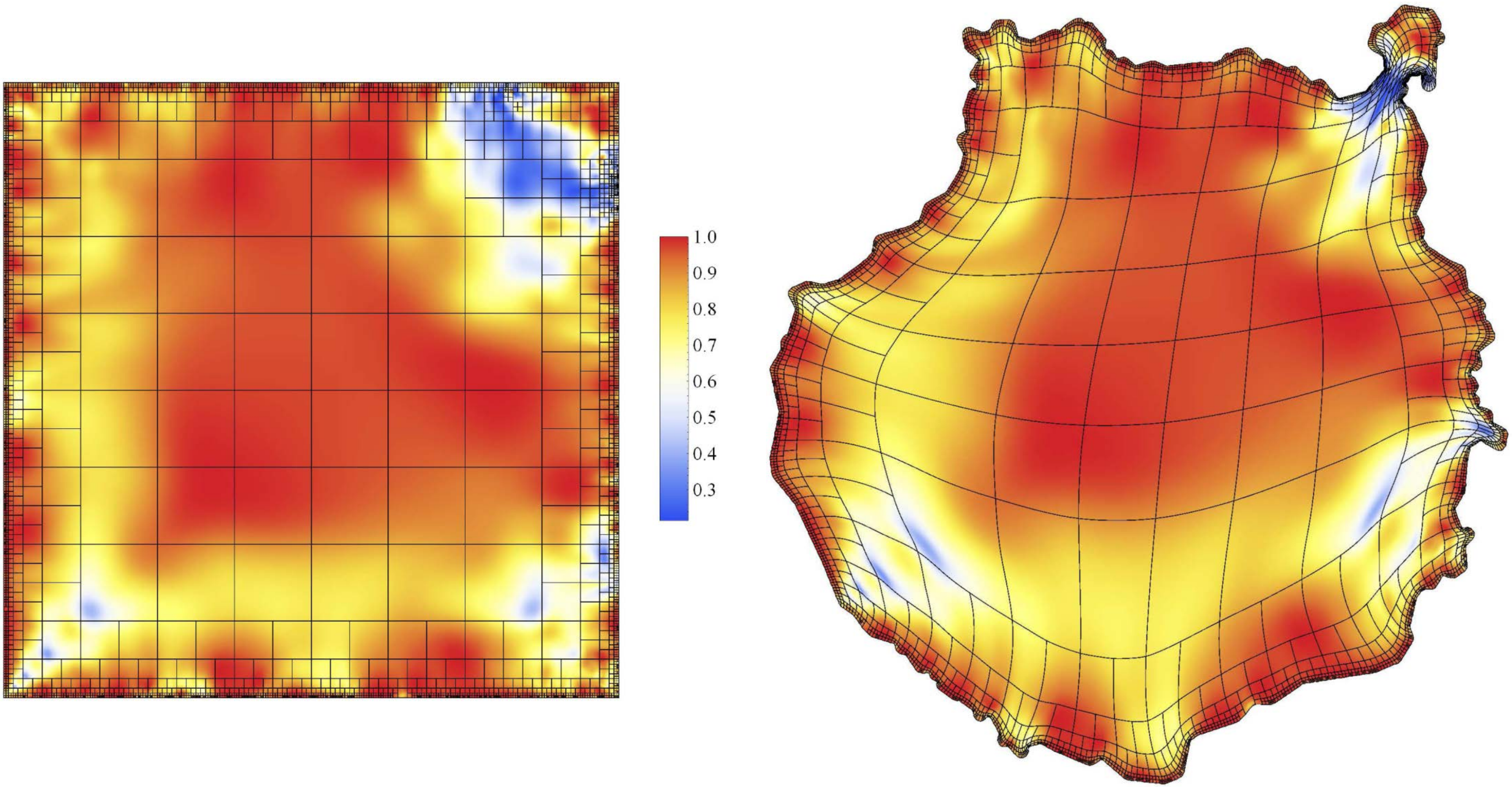
Applications: Isogeometric Modeling

Puzzle Piece (Mean Ratio Jacobian)



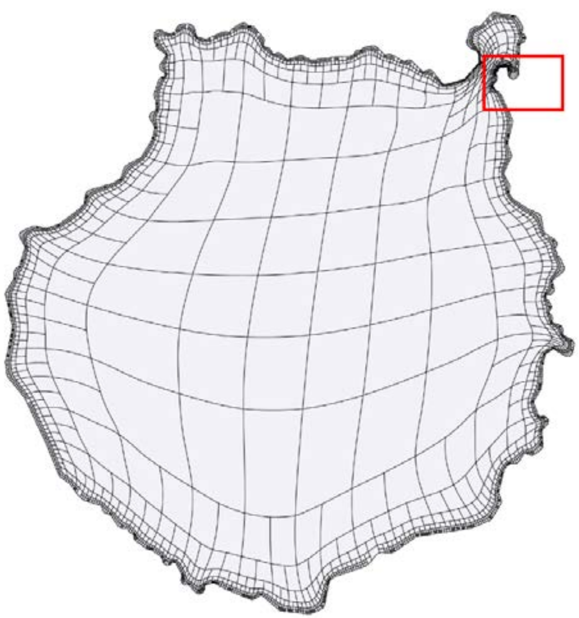
Applications: Isogeometric Modeling

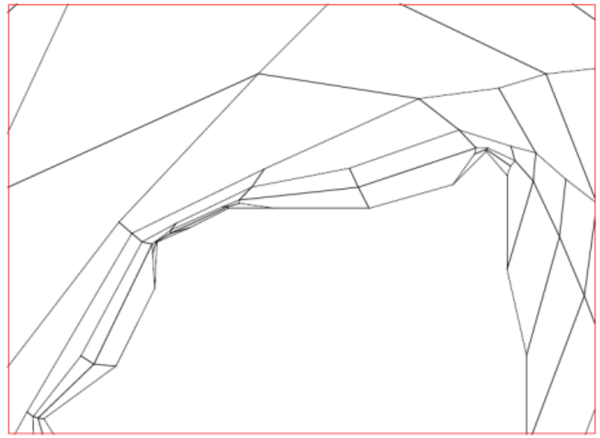
Gran Canaria Island (Mean Ratio Jacobian)



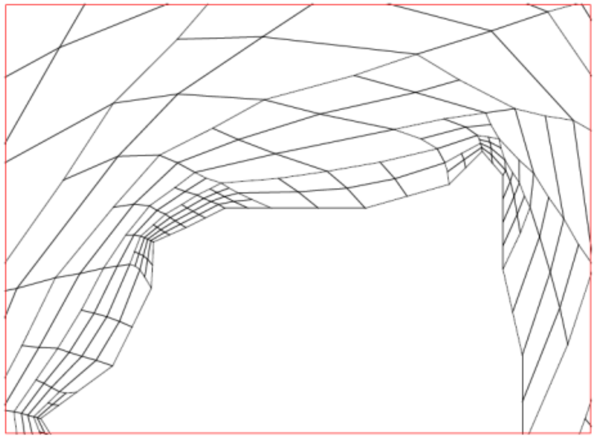
Applications: Isogeometric Modeling

Gran Canaria Island (adaptive refinement to improve the mean ratio Jacobian)

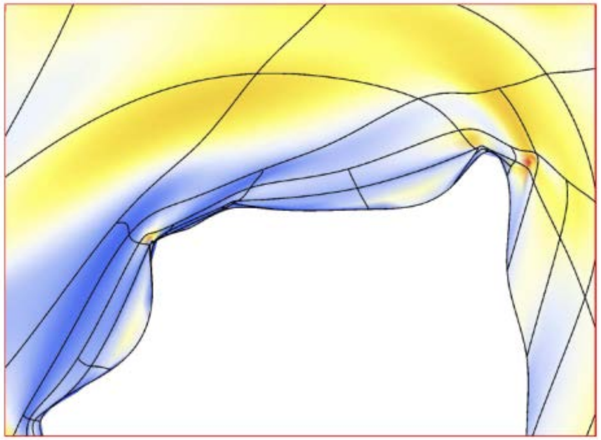




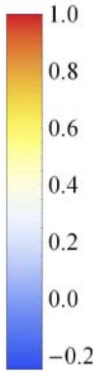
Initial T-mesh

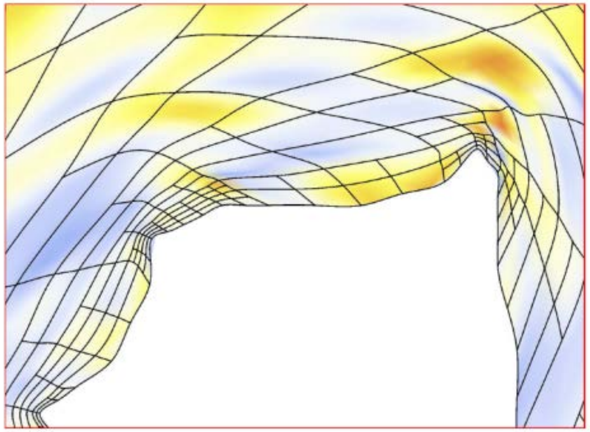


Refined T-mesh



Initial T-spline & Mean ratio Jacobian



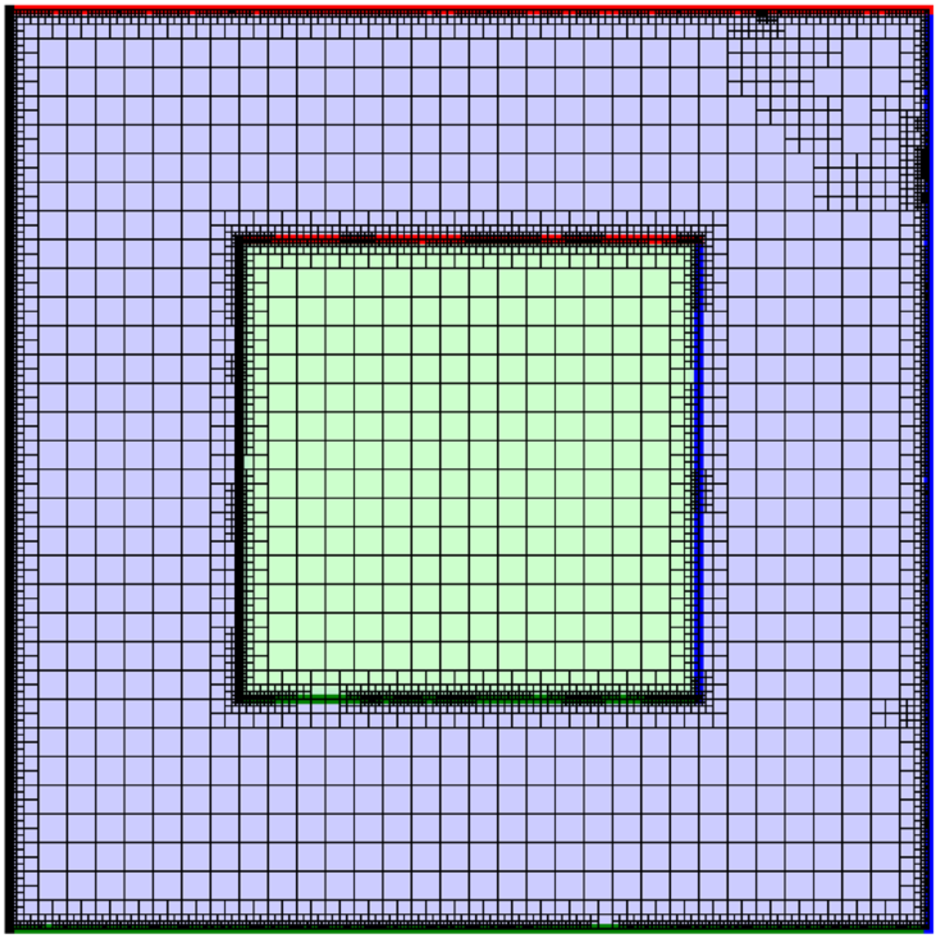


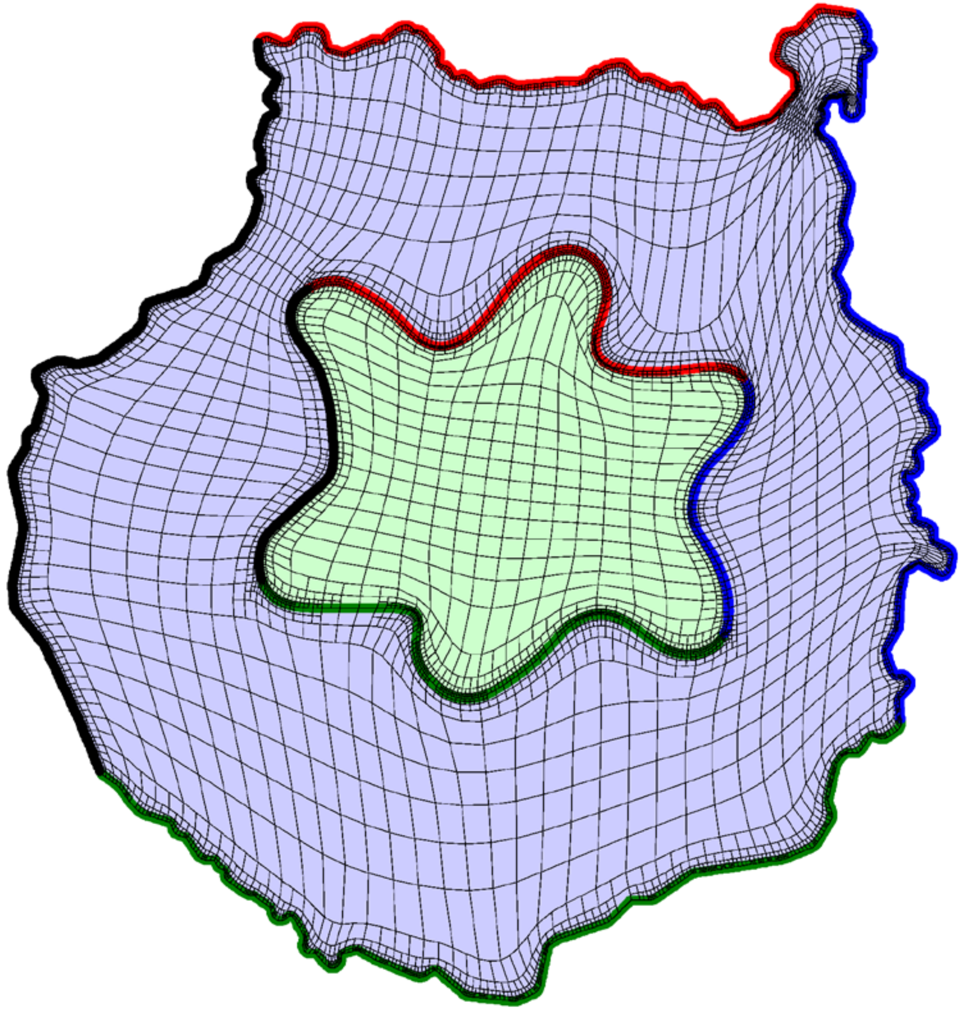
Refined T-spline & Mean ratio Jacobian

No negative Jacobian after refinement!

Applications: Isogeometric Modeling

Geometries with several materials



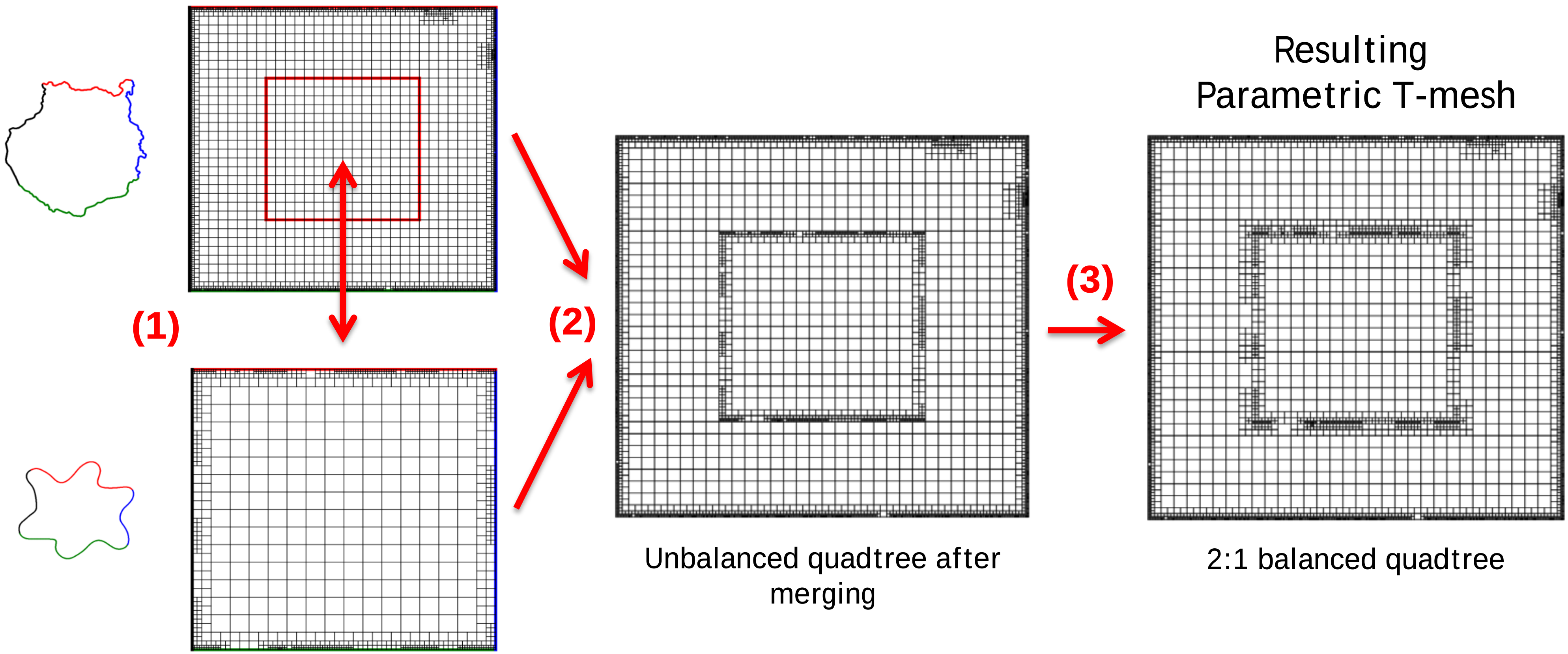


Applications: Isogeometric Modeling

Strategy for embedded planar geometries: Three steps

1. Individual quadtree approximation for each geometry (they can be processed in parallel)
2. Insert a quadtree in a region of another quadtree
3. Balance the resulting quadtree

Resulting Parametric T-mesh

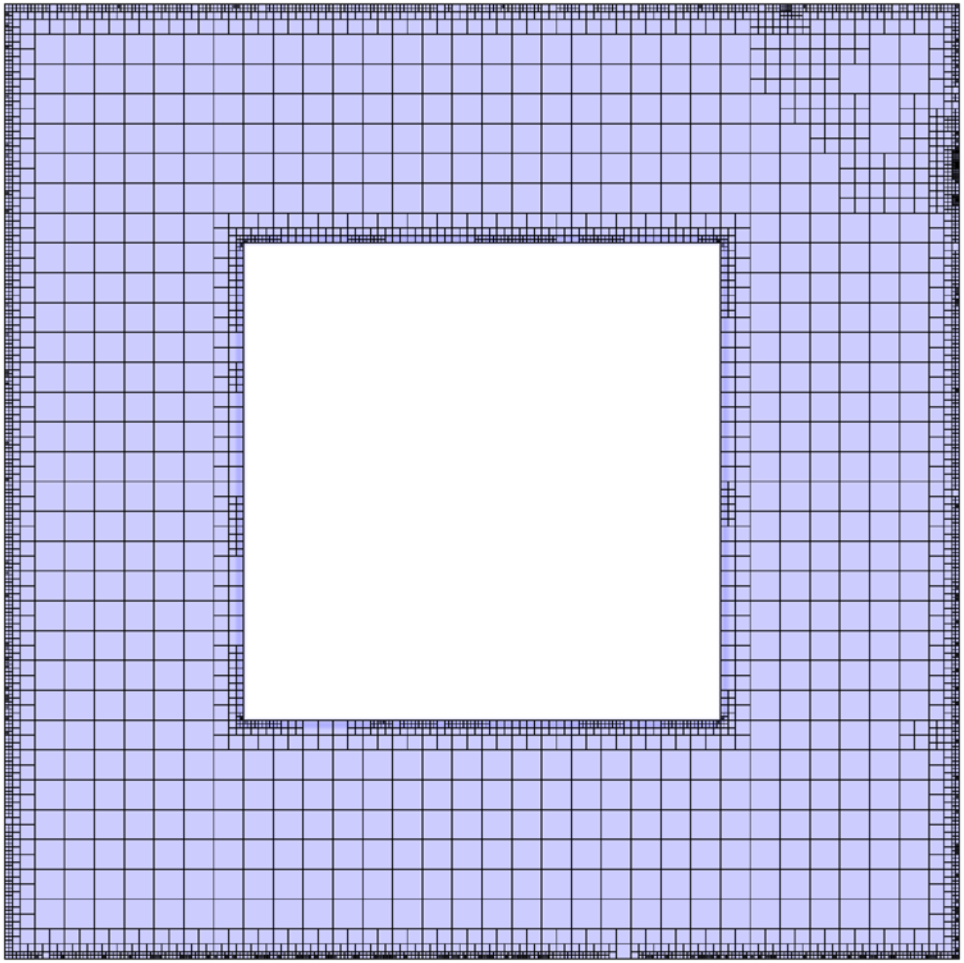


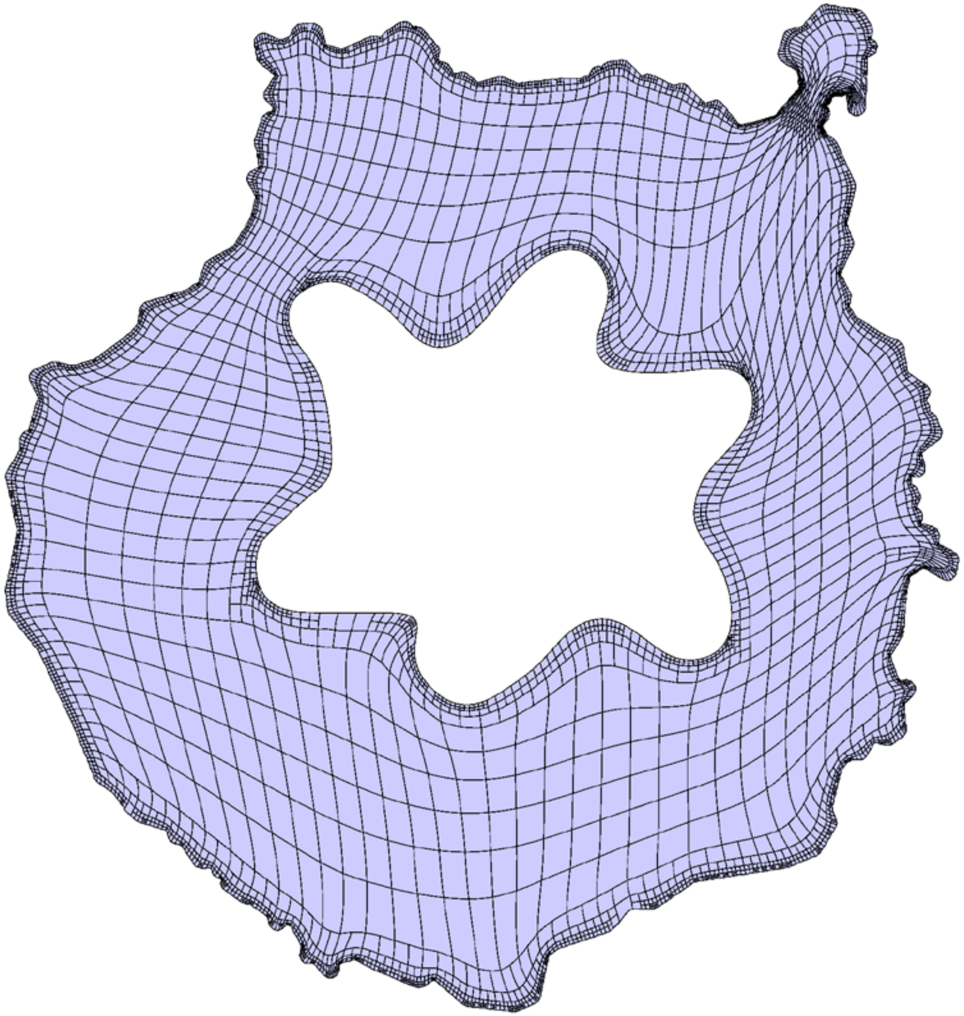
Unbalanced quadtree after merging

2:1 balanced quadtree

Applications: Isogeometric Modeling

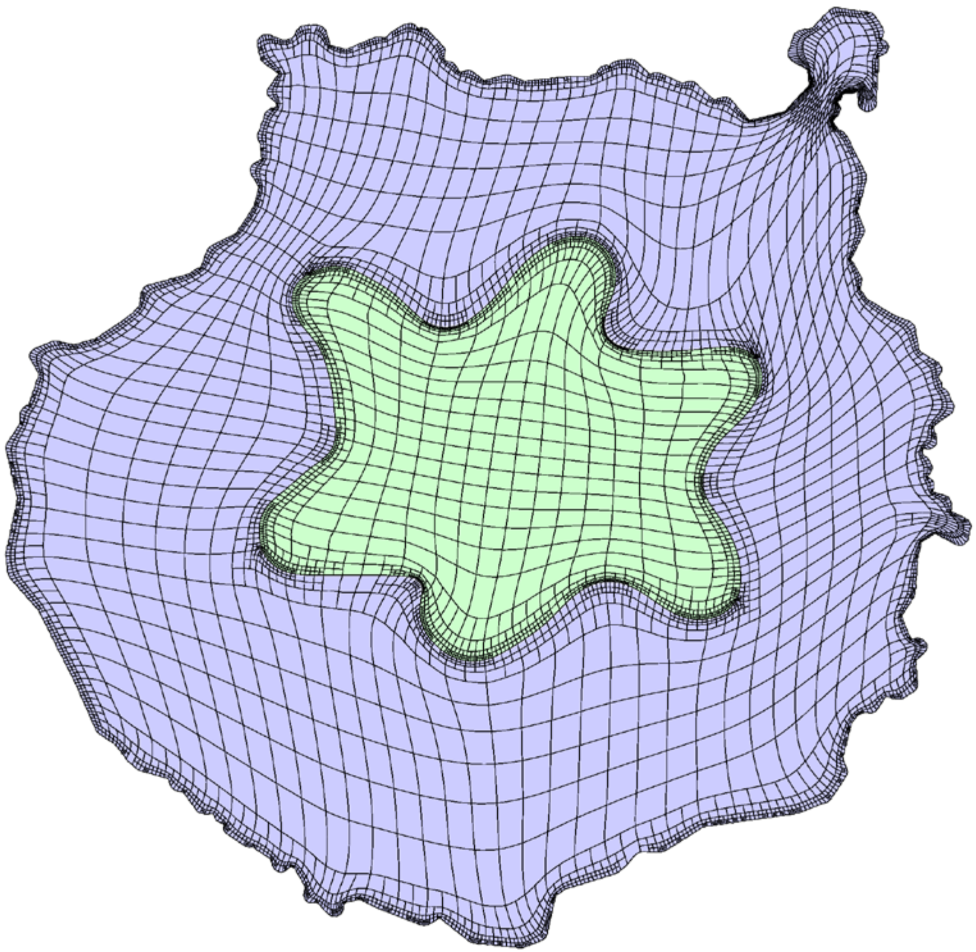
Geometries with holes



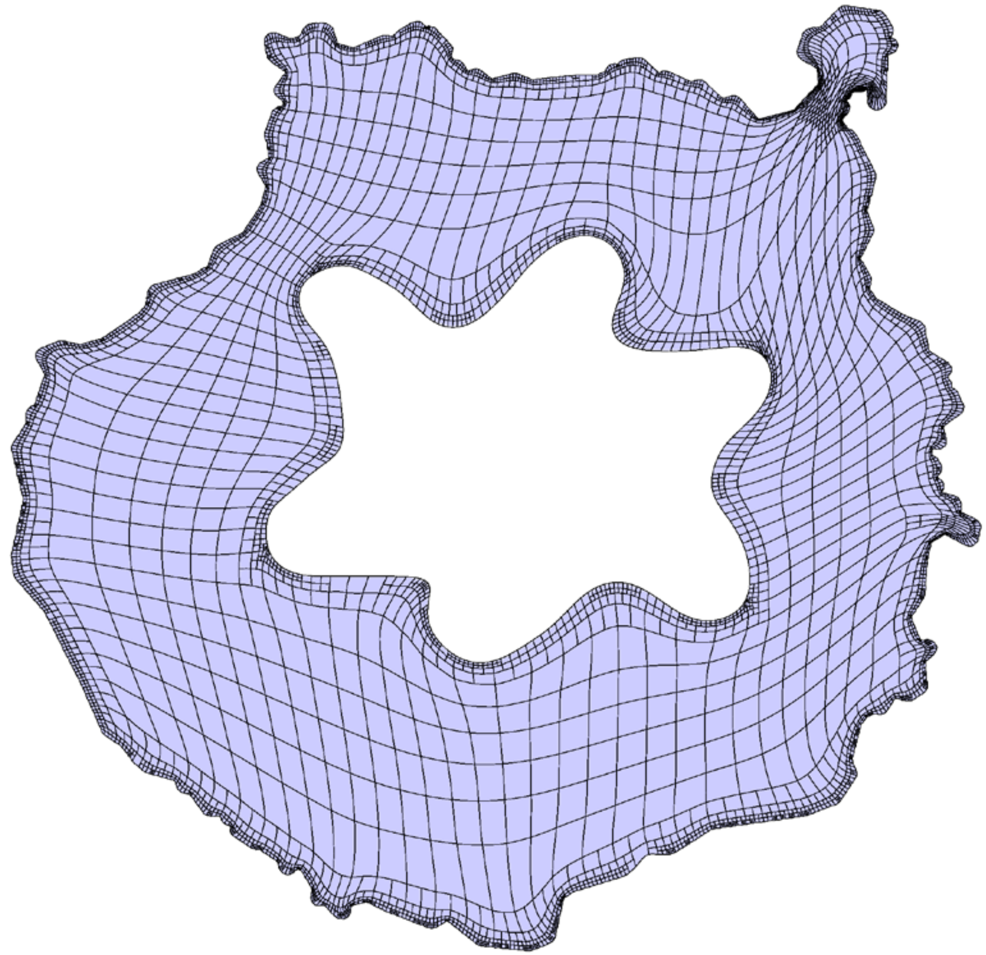


Applications: Isogeometric Modeling

Embedded planar geometries (T-spline representation in physical space)



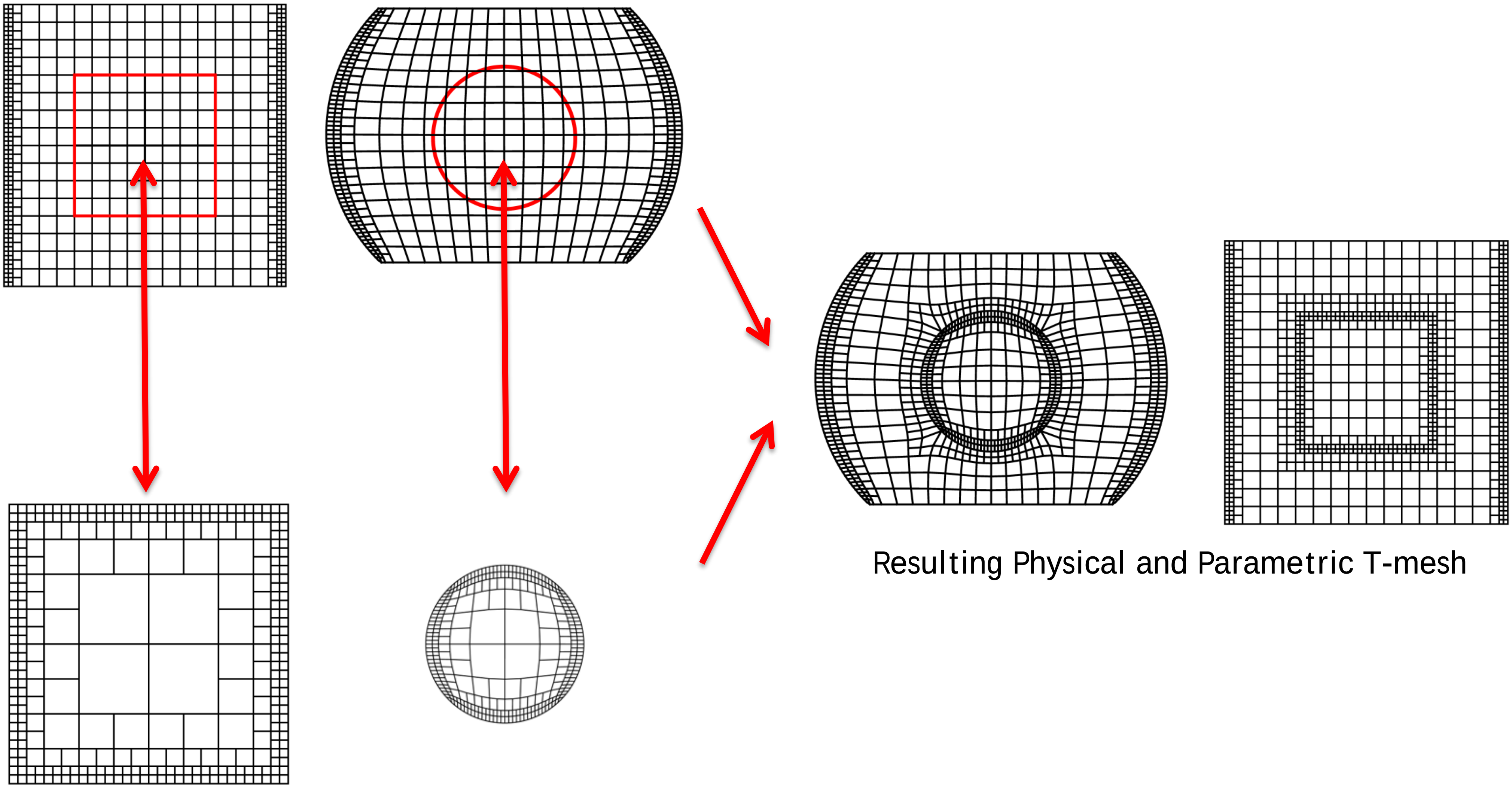
T-spline representation with different materials



T-spline representation with a hole

Applications: Isogeometric Modeling

Strategy for embedded planar geometries: Other example

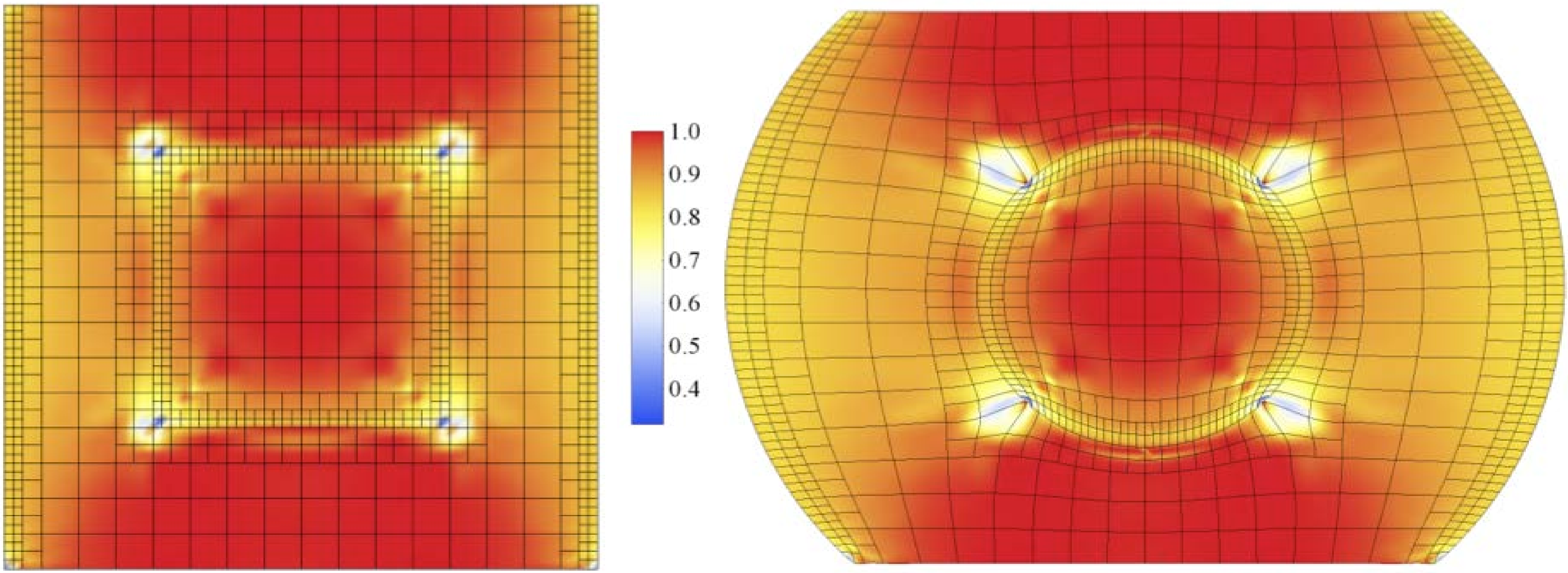


Resulting Physical and Parametric T-mesh

Applications: Isogeometric Modeling

Strategy for embedded planar geometries: Other example

Mean ratio Jacobian

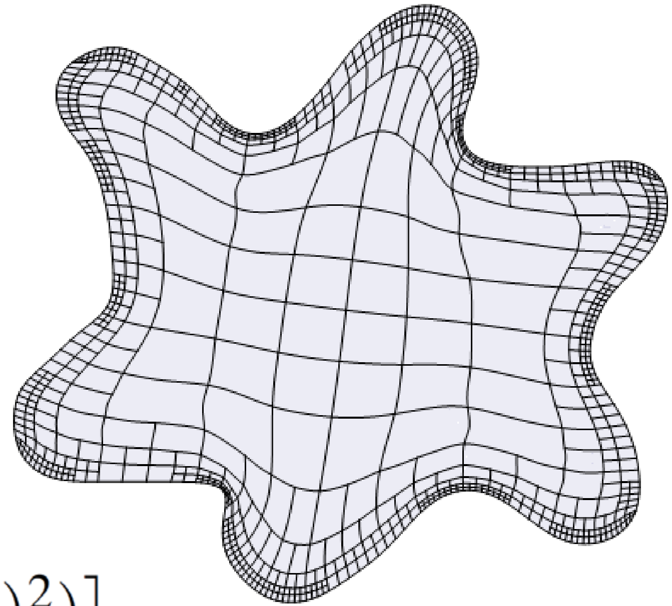


Local Nested Adaptive Refinement

Numerical solution of a Poisson problem

Concentrate source in relation to the initial mesh size

$$
\begin{aligned}
-\triangle u &= f \qquad \text{in } \Omega, \\
u &= g \qquad \text{on } \partial\Omega.
\end{aligned}
$$



Exact solution:

$$u(x,y) = \exp\left[-10^3((x-0.6)^2 + (y-0.35)^2)\right]$$

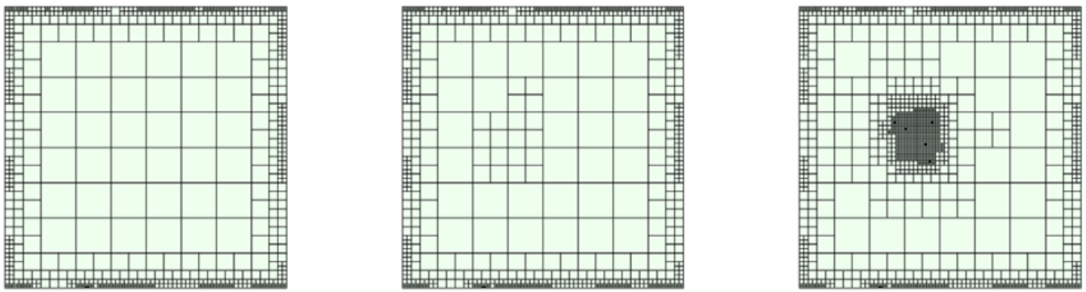
Residual-type error indicator:

$$\eta(\Omega_e)^2 = \|h(f+\Delta u_h)\|^2_{0,\Omega_e} = \int_{\Omega_e} h^2 (f+\Delta u_h)^2 \, \mathrm{d}\Omega$$

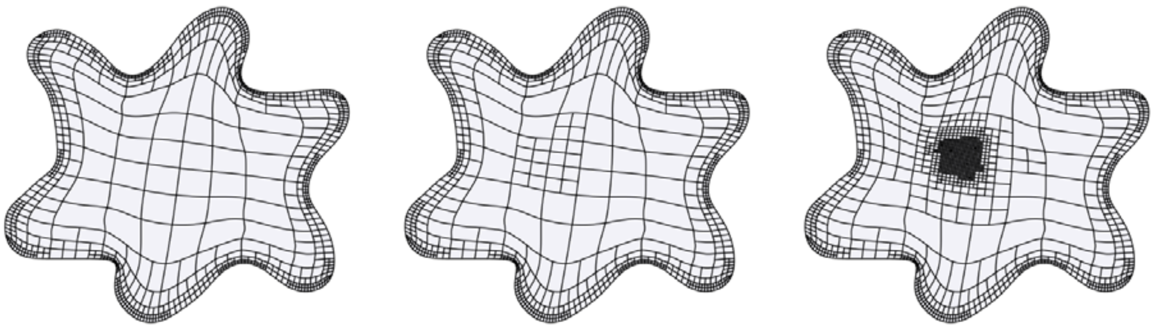
Local Nested Adaptive Refinement

Numerical solution of a Poisson problem

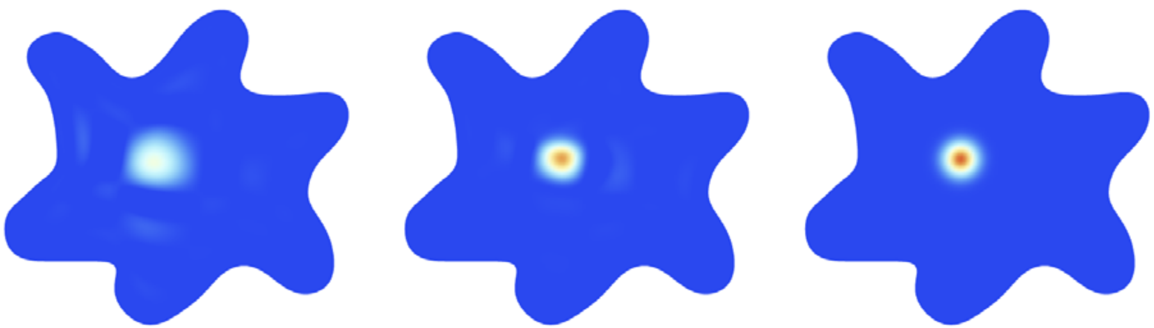
Error indicator : $\eta\left(\Omega_e\right)^2 = \int_{\Omega_e} h^2 \left(f + \Delta u_h\right)^2 \, d\Omega$



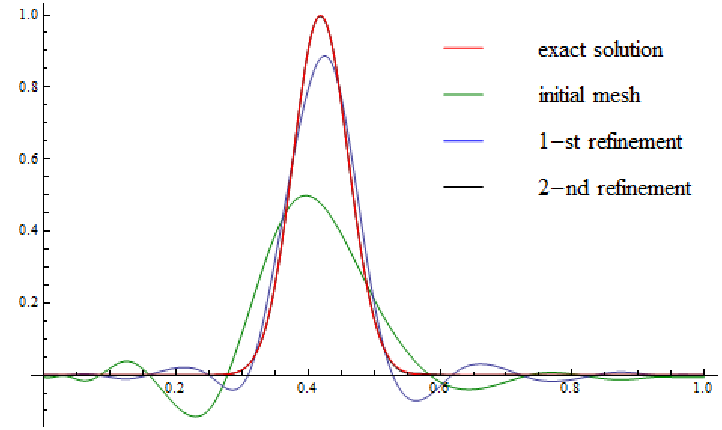
(a) 844 cells,1456 DOF (b) 859 cells, 1476 DOF (c) 1552 cells, 2233 DOF



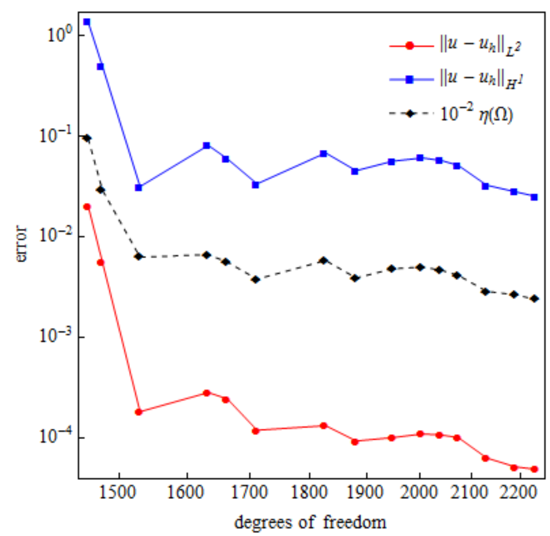
(d) Initial mesh (e) 1-st refinement (f) 14-th refinement



(g) Initial solution (h) 1-st refinement (i) 14-th refinement



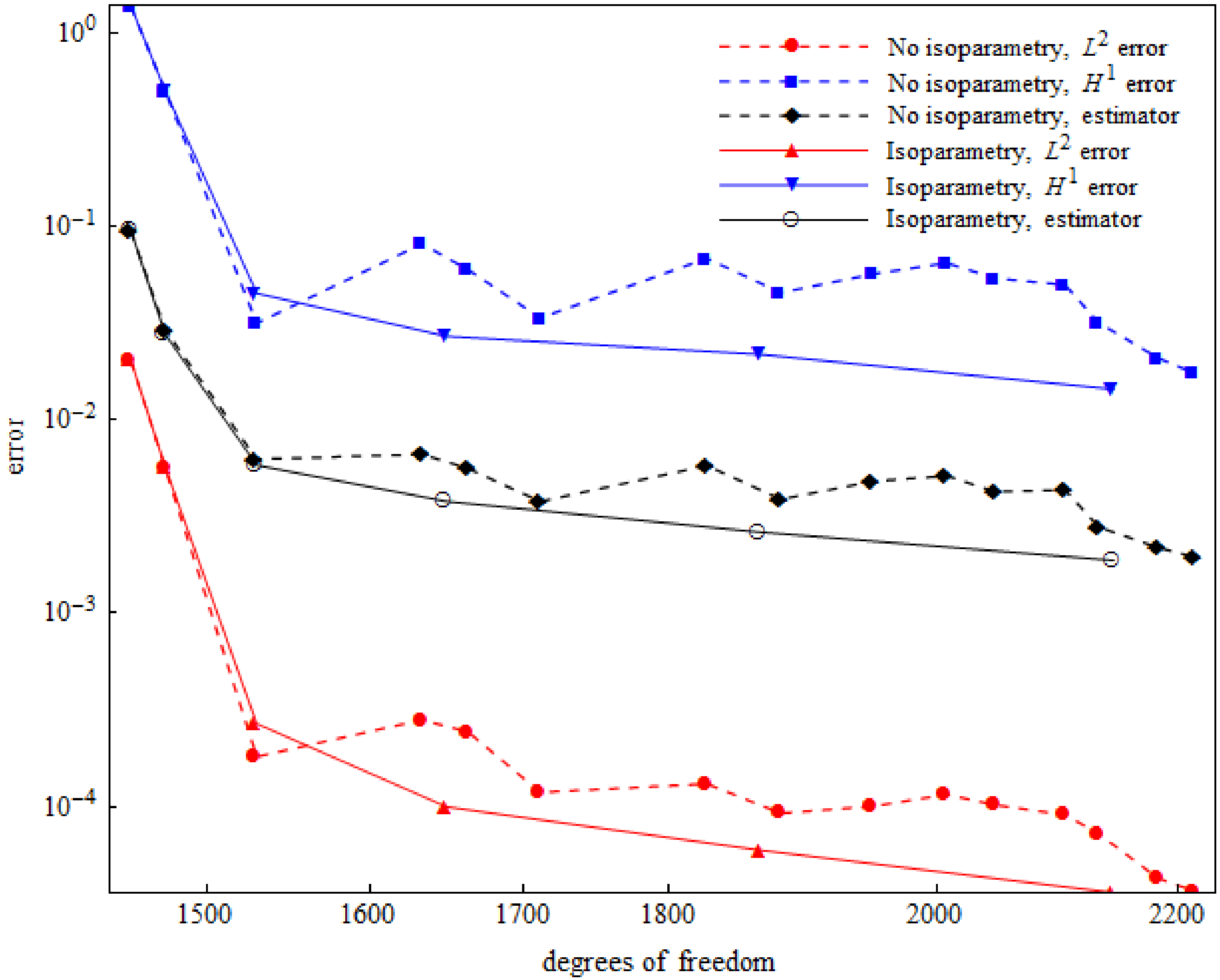
Numerical solution across a section



Convergence behavior

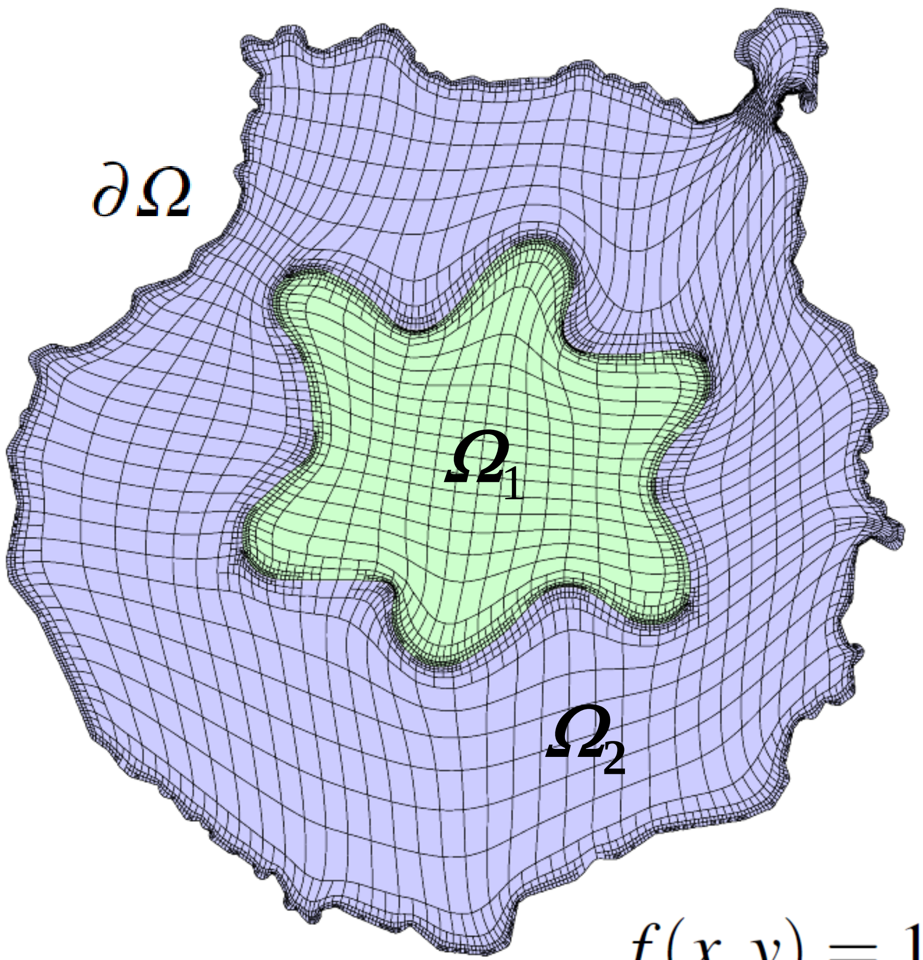
Local Nested Adaptive Refinement

Influence of isoparametry in the convergence behavior



Poisson Problem for a Domain with Two Materials

Statement of the problem



$$
\begin{aligned}
-\nabla(k(\mathbf{x})\nabla u) &= f \qquad \text{in } \Omega, \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

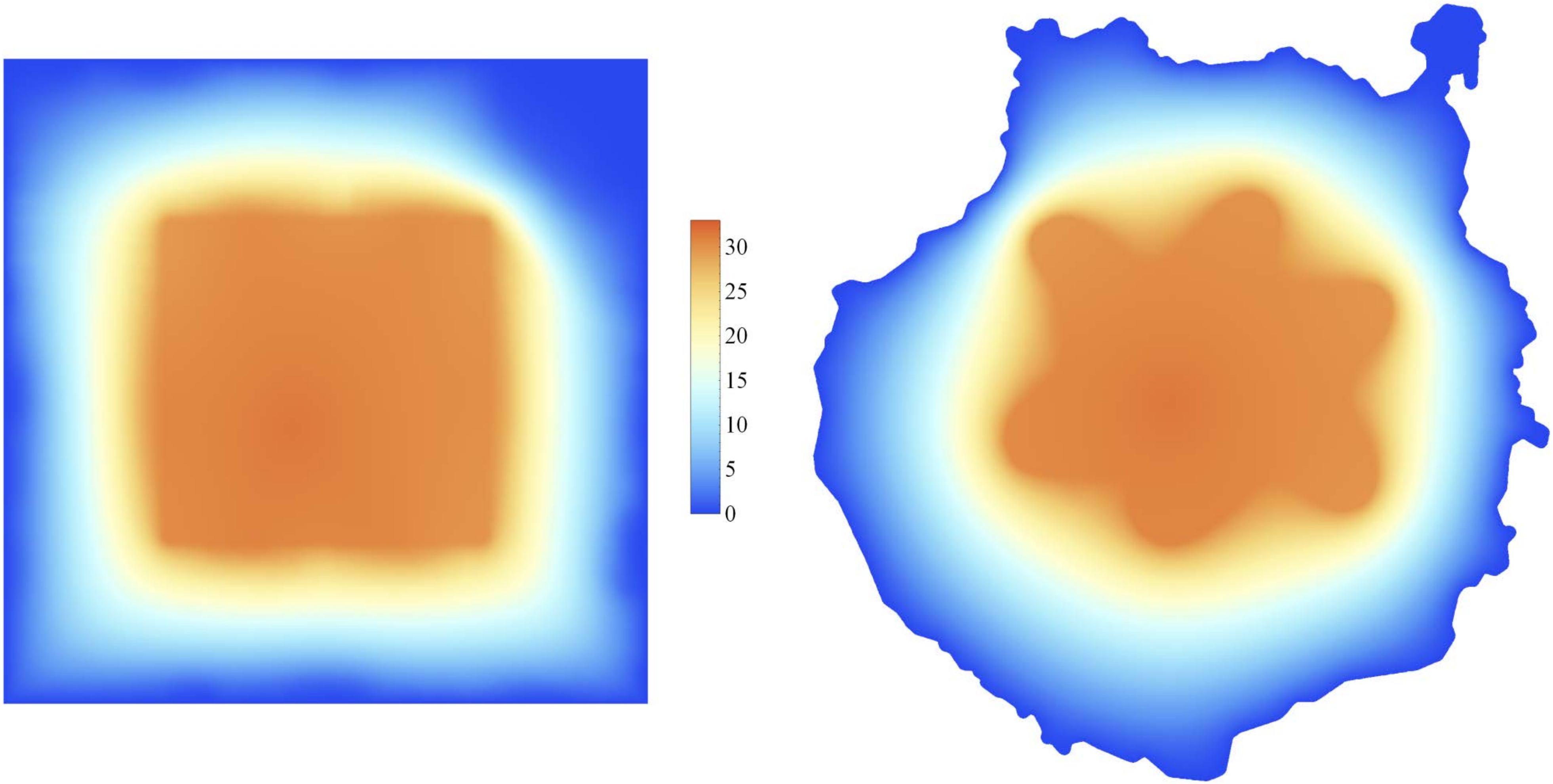
$$
k(\mathbf{x}) = \begin{cases} k_1 & \text{if } \mathbf{x} \in \Omega_1 \\ k_2 & \text{if } \mathbf{x} \in \Omega_2 \end{cases}
$$

$$
k_1 = 1 \quad >> \quad k_2 = 0.01
$$

$$
f(x,y) = 10^3 \exp\left[-10^3\left((x-0.5)^2 + (y-0.5)^2\right)\right]
$$

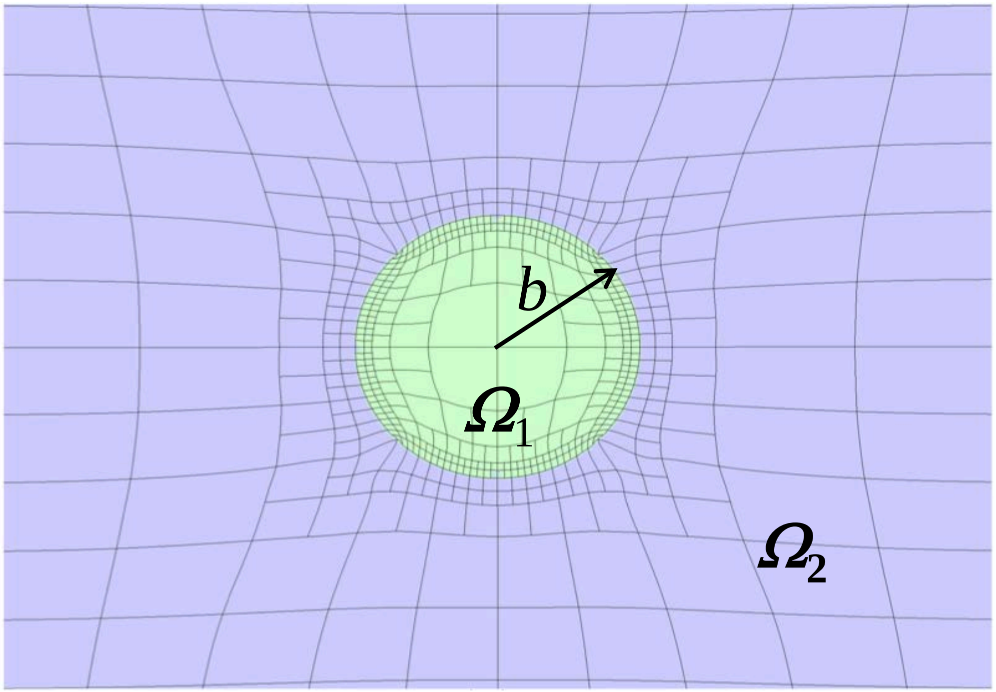
Poisson Problem for a Domain with Two Materials

Numerical solution in parametric and physical domain



Poisson Problem for a Domain with Two Materials

Dielectric cylinder in an uniform horizontal electric field E_0



T-spline detail

$$-\nabla(k(\mathbf{x})\nabla u) = 0 \qquad \text{in } \Omega,$$
$$u = g \qquad \text{on } \partial\Omega.$$

$$k(\mathbf{x}) = \begin{cases} \varepsilon_0 \varepsilon_r & \text{if } \mathbf{x} \in \Omega_1 \quad \rho < b \\ \varepsilon_0 & \text{if } \mathbf{x} \in \Omega_2 \quad \rho \geq b \end{cases}$$

The analytic solution in cylindrical coordinates (ρ, φ) is given by:

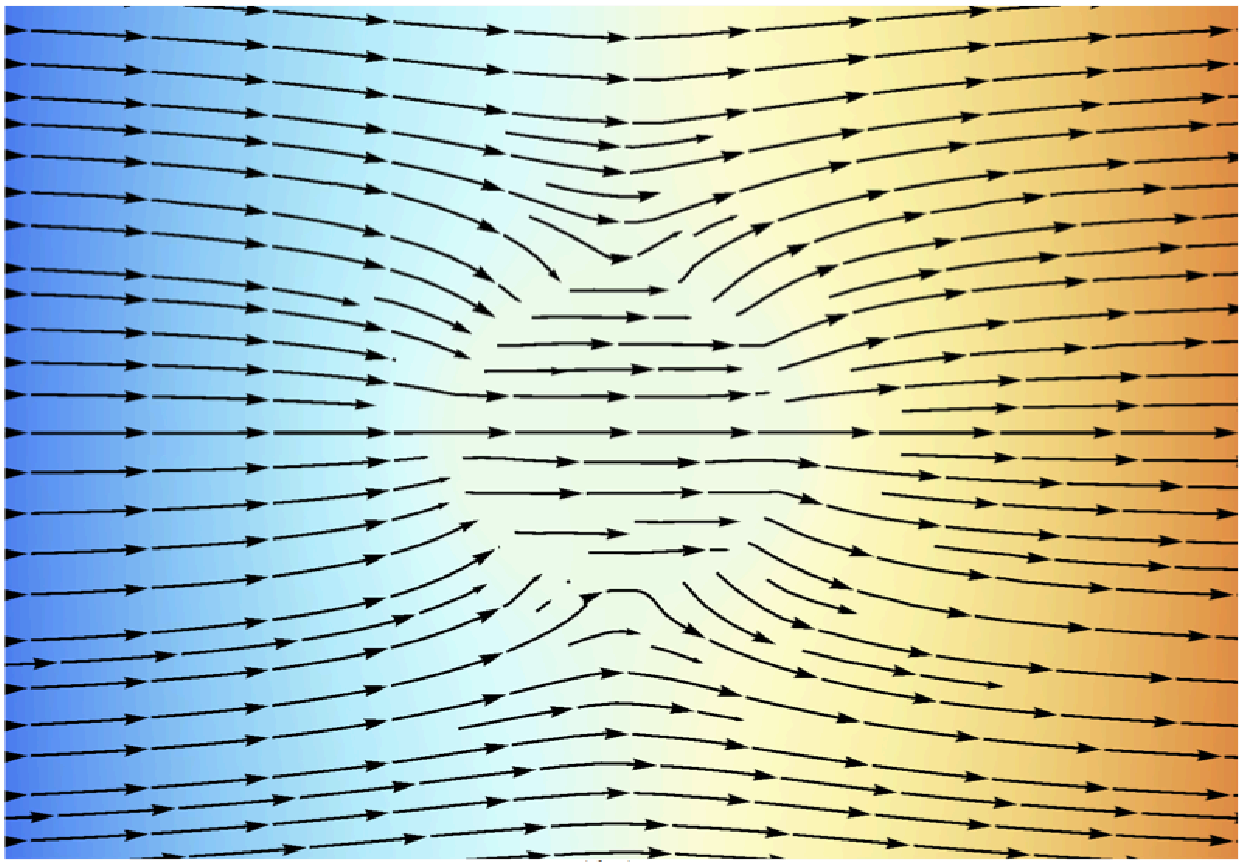
$$u_{\rho<b} = \frac{-2E_0 \rho \cos\varphi}{(\varepsilon_r + 1)}$$

$$u_{\rho \geq b} = E_0 \cos\varphi \left(-\rho + \frac{b^2(\varepsilon_r - 1)}{\rho(\varepsilon_r + 1)} \right)$$

Poisson Problem for a Domain with Two Materials

Dielectric cylinder in an uniform electric field

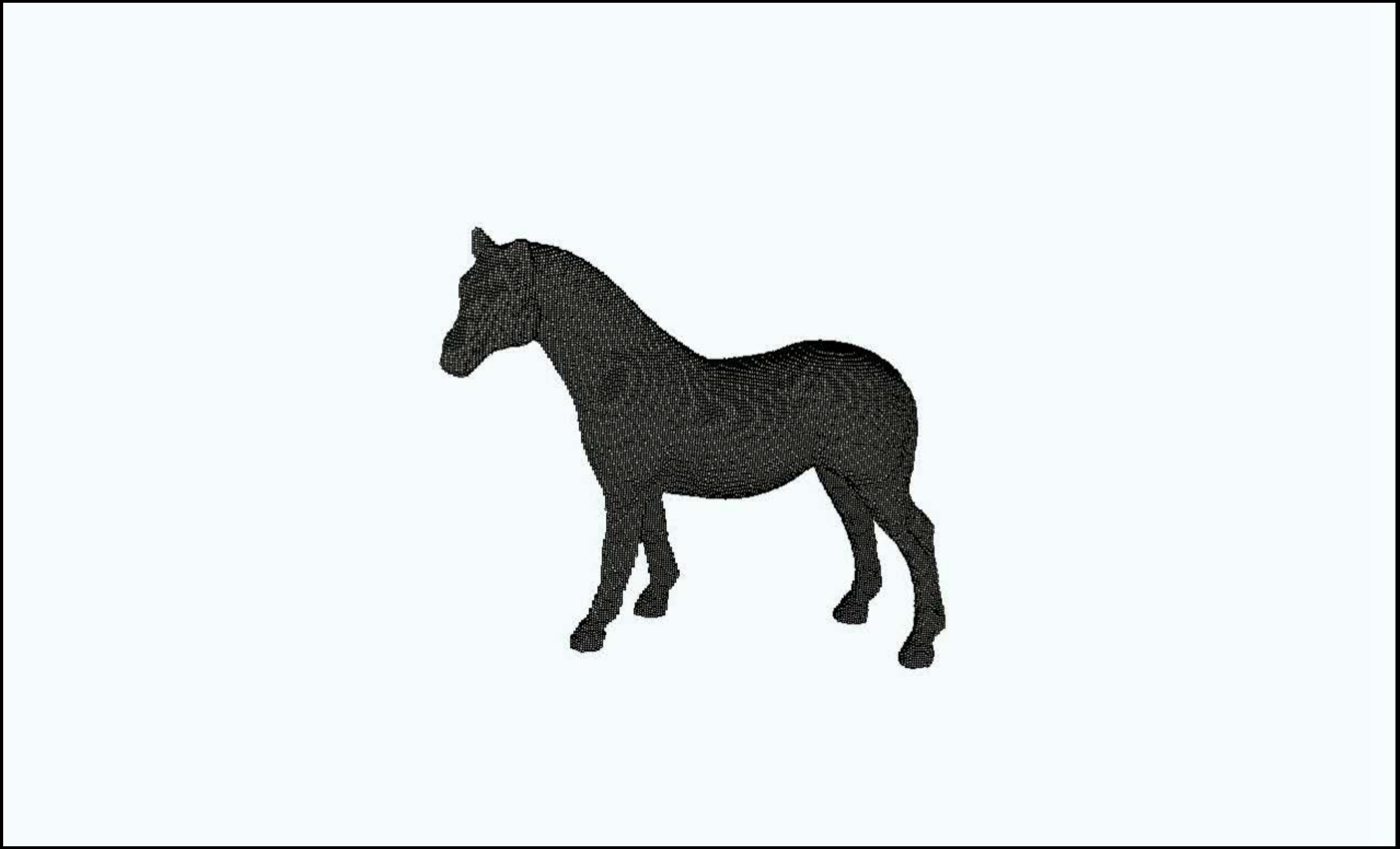
Numerical solution detail (potential and electric field) in physical domain



Comparing with the analytic solution, we have measured a **maximum error** in the potential of **0.88%**.

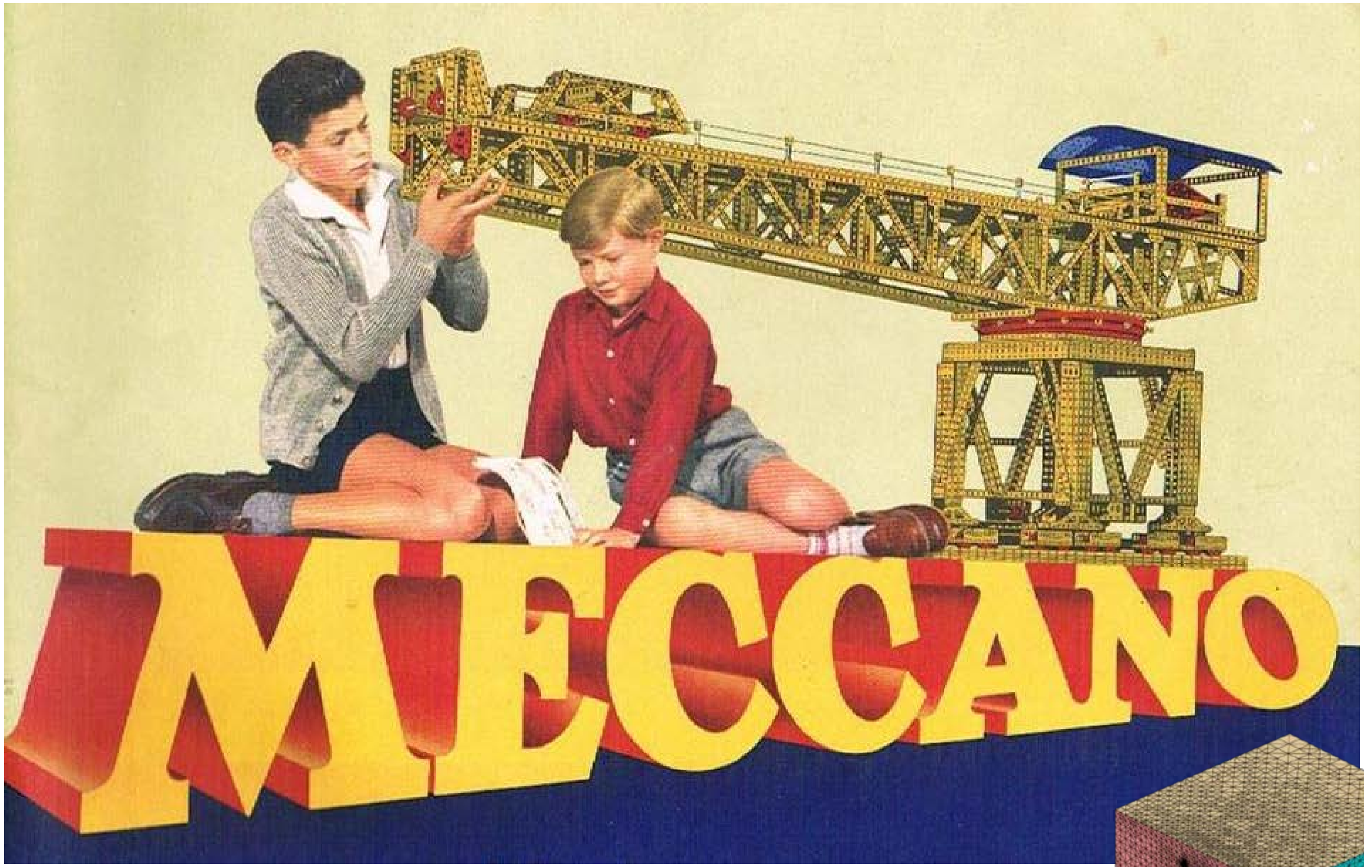
Final Comments and Future Works

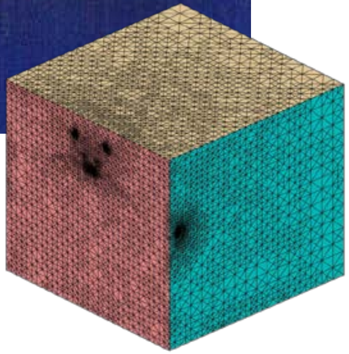
Automatic Construction of the Meccano



Final Comments and Future Works

Automatic Construction of the Meccano





Adaptive T-spline Refinement for Isogeometric Analysis in Planar Geometries

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January 8-10, 2014, Austin, Texas, USA

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CONACYT-SENER Project, Fondo Sectorial, contract: 163723

http://www.dca.iusiani.ulpgc.es/proyecto2012-2014