Adaptive T-spline Refinement for Isogeometric Analysis in Planar Geometries

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The Meccano Method (based on tetrahedral mesh optimization)
- The initial algorithm for tetrahedral mesh generation
- Volumetric parameterization of tetrahedral meshes
- Application to isogeometric solid modeling and analysis

The Meccano Method (based on a new 2D T-mesh optimization)
- The new algorithm for two-dimensional T-mesh generation
- T-spline parameterization of 2D geometries
- Geometries with several materials or holes
- Application to isogeometric modeling and analysis

Comments and Future Research
Meccano Method for Complex Solids
Volume parameterization based on SUS of tetrahedral meshes

- Parameterization
- Refinement
- Untangling/Smoothing
Meccano Method for Complex Solids
Volume parameterization based on SUS of tetrahedral meshes

Target Element $T_t$

Physical Element $T$

Parameter space (meccano mesh)

Physical space (tangled mesh)

Physical space (optimized mesh)

(to get less distortion in the parameterization)
Meccano Method for Complex Solids

Algorithm steps

1. Meccano construction
2. Parameterization of the solid surface
   - Solid boundary partition
   - Floater’s parameterization
3. Coarse tetrahedral mesh of the meccano
   - Partition into hexahedra
   - Hexahedron subdivision into six tetrahedra
4. Solid boundary approximation
   - Kossaczky’s refinement
   - Distance evaluation
5. Inner node relocation
   - Coons patches
6. Simultaneous untangling and smoothing
   - SUS
Meccano Method for Complex Solids
Volume parameterization based on SUS of tetrahedral meshes

- Octree subdivision
INPUT DATA: Surface Triangulation
http://www.cyberware.com/
Adaptive Isogeometric Refinement (EWC 2012)
Application in Igea: Poisson problem with a central source

\[ \Delta u = \frac{1}{25} e^{\frac{(x^2+y^2+z^2)^{10}}{10}} \left( -15 + x^2 + y^2 + z^2 \right) \text{ in } \Omega \]

\[ u|_{\partial \Omega} = 0 \]

Exact solution:

\[ u \approx e^{\frac{(x^2+y^2+z^2)^{10}}{10}} \]
Adaptive Isogeometric Refinement (EWC 2012)
Igea: T-spline of Numerical Solution

Initial T-mesh
5692 cells, 9304 DOF
Adaptive Isogeometric Refinement (EWC 2012)
Igea: T-spline of Numerical Solution

\[ \eta(\Omega_e)^2 = \int_{\Omega_e} h^2 (f + \Lambda u_h)^2 \, d\Omega \]

2nd local refinement
6021 cells, 9807 DOF
Adaptive Isogeometric Refinement (EWC 2012)

Igea: T-spline of Numerical Solution

Error indicator: \[ \eta(\Omega_e)^2 = \int_{\Omega_e} h^2 (f + \Lambda u_h)^2 \, d\Omega \]

5th local refinement
6756 cells, 10838 DOF
1. The key of the meccano method is the simultaneous untangling and smoothing (SUS) procedure.

2. The quality of the T-spline mapping (i.e., positive Jacobian, good uniformity and orthogonality of the isoparametric curves) depends on the quality of the T-mesh in the physical space. We have to fix a quality metric for this mapping.

3. In order to simplify the procedure and to get less distortion in the volume parameterization, it should be interesting to **directly apply the meccano method on T-meshes instead of tetrahedral meshes**.

4. We have started analysing the problem in 2-D.
The Meccano Method on T-meshes in 2-D

Input data: Boundary representation of the object
Objective: Construction of a high quality T-spline parameterization
Select four points (A, B, C, D) of the input boundary
Boundary parameterization via chord-length
Step 2: Coarse quadrilateral mesh of the meccano (parameter space)
The Meccano Method on T-meshes in 2-D

Step 3: Refine mesh with quadtree subdivisions to approach the boundary
The Meccano Method on T-meshes in 2-D

Step 4: Move the meccano boundary nodes to the object boundary
The Meccano Method on T-meshes in 2-D
Step 5: Inner node relocation with Coons patch to facilitate the optimization

\[ x(\xi, \eta) = (1 - \xi)x(0, \eta) + \xi x(1, \eta) + (1 - \eta)x(\xi, 0) + \eta x(\xi, 1) \]

\[ - \begin{bmatrix} 1 - \xi & \xi \\ 1 - \eta & \eta \end{bmatrix} \begin{bmatrix} x(0, 0) & x(0, 1) \\ x(1, 0) & x(1, 1) \end{bmatrix} \begin{bmatrix} 1 - \eta \\ \eta \end{bmatrix} \]
Step 6: Simultaneous Untangling and Smoothing (SUS) of the T-mesh
The Meccano Method on T-meshes in 2-D

Step 7: T-spline representation of the spot
Boundary Approach in 2-D

Input data: Boundary polyline approximation (red color line)

Boundary edge refinement criterion:

\[ \exists j : A_j > \varepsilon \Rightarrow \text{refine} \]

\[ A_j = \text{Area}(P_k, P_j, P_{k+l}) \]

\[ k + 1 \leq j \leq k + l - 1 \]
Local optimization

Objective: Improve the quality of the local mesh by minimizing an objective function

Local mesh

Optimized local mesh
Local optimization

**Objective:** Improve the quality of the local mesh by minimizing an objective function

- Free node
- New position for the free node

Local mesh

Optimized local mesh
Simultaneous Untangling and Smoothing (CMAME 2003)
Weighted Jacobian Matrix on a Plane

Reference triangle

Physical triangle

Ideal triangle

\( A = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} \)

\( S = AW^{-1} \)

An algebraic quality metric of \( t \) (mean ratio)

\[
q = \frac{2\sigma}{||S||^2} = \frac{1}{\eta}
\]

where:

\[
||S|| = \sqrt{\text{tr}(S^T S)}
\]

\[
\sigma = \text{det}(S)
\]
Simultaneous Untangling and Smoothing (CMAME 2003)

Local objective function for plane triangulations


Original function: \( K(x) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2\sigma_m} \)

Modified function: \( K^*(x) = \sum_{m=1}^{M} \frac{\|S_m\|^2}{2h(\sigma_m)} \)

Modified function (blue) is regular in all \( \mathbb{R}^2 \) and it approximates the same minimum that the original function (red). Moreover, it allows a simultaneous untangling and smoothing of triangular meshes.
Simultaneous Untangling and Smoothing of T-meshes
Triangle decomposition of the T-mesh cells

Case 1: Free node is a regular node

Barriers and feasible region for a regular node

Case 2: Free node is a hanging node

Barriers and feasible region for a hanging node
Physical cell $C$ must be as similar as possible to the counterpart in the parametric space $C_p$. 

Parametric T-mesh

Optimized physical T-mesh
Simultaneous Untangling and Smoothing of T-meshes

Problems appear with the non-weighted objective function $K^*$

Satisfactory result for a conformal submesh using $K^*$

Not satisfactory result for a non-conformal submesh using $K^*$

Desirable result: Orthogonal mesh for a non-conformal case (weighted objective function $K^*_\tau$)
Simultaneous Untangling and Smoothing of T-meshes
Solution by using weighted objective functions (regular node)

\[ K^{*}(x) = \tau_1 \sum_{m=1}^{3} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_2 \sum_{m=4}^{6} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_3 \sum_{m=7}^{9} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_4 \sum_{m=10}^{12} \frac{\|S_m\|^2}{2h(\sigma_m)} \]

All possible weights for balanced quadtrees:

\[ \tau_1 = 1, \quad \tau_2 = \tau_4 = 2 \quad \text{and} \quad \tau_3 = 4 \]
Simultaneous Untangling and Smoothing of T-meshes
Solution by using weighted objective functions (hanging node)

\[ K^*_\tau(x) = \tau_1 \sum_{m=1}^{3} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_2 \sum_{m=4}^{6} \frac{\|S_m\|^2}{2h(\sigma_m)} + \tau_3 \sum_{m=7}^{11} \frac{\|S_m\|^2}{2h(\sigma_m)} \]

All possible weights for balanced quadtrees:

\[ \tau_1 = \tau_2 = 1 \quad \text{and} \quad \tau_3 = \frac{8}{5} \]

Optimized mesh without weights

Optimized mesh with weights
Objective Functions Comparison

$k^*$ (without weights) versus $k^*_{\Box}$ (with weights)

T-mesh optimized with $K^*$ (without weights)

Uniform mesh optimized with $K^*$

T-mesh optimized with $K^*_{\Box}$ (with weights)

Close to uniform mesh positions
Objective Functions Comparison

$k^*$ (without weights) versus $k^*$ (with weights)

T-mesh optimized with $K^*$
(without weights)

T-mesh optimized with $K^*$
(with weights)
Simultaneous Untangling and Smoothing of T-meshes

T-mesh transformation along the SUS process: Example

Parameter space

Physical space
Simultaneous Untangling and Smoothing of T-meshes

T-mesh transformation along the SUS process: Video
T-spline Parameterization
Determination of control points by imposing interpolation conditions

\[ S(\xi) = \sum_{\alpha \in A} P_\alpha R_\alpha(\xi) \]
Mean Ratio Jacobian $J_r(\xi)$ of Parametric Transformation

A quality metric of the T-spline mapping at any point $P_0$

$$-1 \leq J_r(\xi) = \frac{2 \det(J)}{\|J\|^2} \leq 1$$

where $J$ is the jacobian matrix of the T-spline mapping $S$
Objective Functions Comparison

$k^*$ (without weights) versus $k^*$ (with weights)

Mean ratio Jacobian

T-mesh optimized with $K^*$ (without weights)

T-mesh optimized with $K^*$ (with weights)
Applications: Isogeometric Modeling
The Spot (Mean Ratio Jacobian)
Applications: Isogeometric Modeling

The Flower (Mean Ratio Jacobian)
Applications: Isogeometric Modeling

Puzzle Piece (Mean Ratio Jacobian)
Applications: Isogeometric Modeling
Gran Canaria Island (Mean Ratio Jacobian)
Applications: Isogeometric Modeling
Gran Canaria Island (adaptive refinement to improve the mean ratio Jacobian)

No negative Jacobian after refinement!
Applications: Isogeometric Modeling
Geometries with several materials
Applications: Isogeometric Modeling

Strategy for embedded planar geometries: Three steps

1. Individual quadtree approximation for each geometry (they can be processed in parallel)
2. Insert a quadtree in a region of another quadtree
3. Balance the resulting quadtree
Applications: Isogeometric Modeling

Geometries with holes
Applications: Isogeometric Modeling
Embedded planar geometries (T-spline representation in physical space)

T-spline representation with different materials

T-spline representation with a hole
Applications: Isogeometric Modeling
Strategy for embedded planar geometries: Other example

Resulting Physical and Parametric T-mesh
Applications: Isogeometric Modeling
Strategy for embedded planar geometries: Other example
Mean ratio Jacobian
Local Nested Adaptive Refinement
Numerical solution of a Poisson problem
Concentrate source in relation to the initial mesh size

\[-\Delta u = f \quad \text{in } \Omega,\]
\[u = g \quad \text{on } \partial\Omega.\]

Exact solution:
\[u(x, y) = \exp \left[ -10^3((x - 0.6)^2 + (y - 0.35)^2) \right]\]

Residual-type error indicator:
\[\eta(\Omega_e)^2 = \|h(f + \Delta u_h)\|_{0,\Omega_e}^2 = \int_{\Omega_e} h^2 (f + \Delta u_h)^2 \, d\Omega\]
Local Nested Adaptive Refinement
Numerical solution of a Poisson problem

Error indicator:

\[ \eta(\Omega_e) = \int_{\Omega_e} h^2 (f + \Delta u_h)^2 \, d\Omega \]

Numerical solution across a section

Convergence behavior
Local Nested Adaptive Refinement
Influence of isoparametry in the convergence behavior
Poisson Problem for a Domain with Two Materials

Statement of the problem

\[-\nabla (k(x) \nabla u) = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega\]

\[k(x) = \begin{cases} 
    k_1 & \text{if } x \in \Omega_1 \\
    k_2 & \text{if } x \in \Omega_2
\end{cases}\]

\[k_1 = 1 \quad >> \quad k_2 = 0.01\]

\[f(x,y) = 10^3 \exp \left[-10^3 ((x - 0.5)^2 + (y - 0.5)^2)\right]\]
Poisson Problem for a Domain with Two Materials
Numerical solution in parametric and physical domain
Poisson Problem for a Domain with Two Materials

Dielectric cylinder in an uniform horizontal electric field $E_0$

The analytic solution in cylindrical coordinates $(\rho, \varphi)$ is given by:

$$-\nabla (k(\mathbf{x}) \nabla u) = 0 \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial \Omega.$$

$$k(\mathbf{x}) = \begin{cases} 
\varepsilon_0 \varepsilon_r & \text{if } \mathbf{x} \in \Omega_1 \quad \rho < b \\
\varepsilon_0 & \text{if } \mathbf{x} \in \Omega_2 \quad \rho \geq b
\end{cases}$$

The analytic solution is:

$$u_{\rho<b} = \frac{-2E_0\rho \cos \varphi}{(\varepsilon_r + 1)}$$

$$u_{\rho\geq b} = E_0 \cos \varphi \left(-\rho + \frac{b^2(\varepsilon_r - 1)}{\rho(\varepsilon_r + 1)}\right)$$
Comparing with the analytic solution, we have measured a maximum error in the potential of 0.88%.
Final Comments and Future Works
Automatic Construction of the Meccano
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http://www.dca.iusiani.ulpgc.es/proyect02012-2014