# A variable reactant transducer to study the oscillations of a pendulum and the value of local gravity

L. M. Curbelo, A. Hernández-Suárez, R. Pérez, A. Mujica, S. Radescu, C. González-Silgo, M. E. Torres

*Dpto. de Física, Universidad de La Laguna, Avda. Astrofísico Francisco Sánchez s/n, E-38204 La Laguna, Tenerife, Canary Islands, Spain.*

***Abstract.*—We describe a method for studying the classical experiment of the simple pendulum, consisting of a body of magnetic material oscillating through a thin conducting coil (magnetic pendulum), which according to Faraday's law of induction generates a fluctuating current in the coil that can be transferred into a periodic signal in an oscilloscope. The set up described here allows to study the motion of the pendulum beyond what is normally considered in more basic settings, including a detailed analysis of both small and large oscillations, and the determination of the value of the acceleration of gravity.**

***Index Terms.*—Simple pendulum, Faraday´s law, magnets, coil, small and large oscillations**

## I. Introduction

Every Science or Engineering student must get acquainted during their formative years with the description of oscillatory phenomena, since these are both important and ubiquitous, appearing in almost every relevant area of Science and Technology. The propagation of electromagnetic waves in vacuum, the transmission of sound in a material medium, AC currents in RLC circuits, the small, microscopic vibrations of atoms in molecules, clusters, and crystals, or the very large, macroscopic vibrations of sizeable mechanical structures, like bridges and multi-storey buildings, to name just a few, are examples where one must tackle the oscillatory behaviour of certain physical variables. All these phenomena share common general features and have some underlying formal unity in their description, with obvious variants adapted to each specific situation, and it pays to expose the student gradually into this subject, beginning with simple but illuminating examples and then proceeding towards the analysis of more elaborated and specific cases within the scope of their chosen branch of expertise. To better familiarize the student with the main general concepts used in the study of oscillations, the general theoretical formalism is also normally supplemented with a more empirical approach in the laboratory, by means of experiments which aim to provide a practical insight into various oscillatory phenomena.

Perhaps the most simple device used in the laboratory to introduce students into the field of oscillations is the simple pendulum: a small mass or bob (which is often idealized as a point mass) that hangs from an inextensible rope (ideally massless), and that oscillates frictionless in a vertical plane by the concourse of gravity, when separated from its equilibrium vertical configuration. When the dynamics of a pendulum is studied in the undergraduate Physics laboratory, one normally measures the period of its oscillations in the small-amplitude regime, changing the value of the length $L$, mass $m$, and initial displacement, to conclude that the period $T$ (in such a low-energy regime) is independent of both m and displacement, and is given theoretically by $T = 2\pi\sqrt{L/g}$, where g is the acceleration of gravity. This relation between $T$, $g$, and $L$ provides then a way to obtain a value of g through the readily measured values of $T$ and $L$, which is a quite satisfactory result that can be obtained with just a meter and a chronometer - a simple watch, for that matter, if one aims to a modest precision. However, the above expression for the period remains approximate, relying on the small-amplitude approximation. It is obviously interesting to expand the scope of the experience, exploring the cases of larger oscillations, or complicating the set-up in different ways, but to gain any meaningful insight in these cases beyond the simplest set up, one must normally achieve a higher precision in the measure of the period.

Upon reviewing several previously reported laboratory experiences dealing with the mechanical oscillations of a pendulum, it seems that a natural way of achieving this is by means of a magnetic pendulum, consisting of a body of magnetic material oscillating through a relatively thin conducting coil, which through Faraday's law of induction generates a fluctuating current in the coil that can be transferred into an oscillatory signal in an oscilloscope. Consider as an example reference [1], where a method to calculate the elastic constant of a spring with an attached mass (another oscillating system) is implemented along these lines. In that case, a coil itself was connected to an oscilloscope and suspended from the spring. The induced current passes through an external coil. Looking for new arrangements, this set up can be modified to avoid the interference of plug and wires in the oscillations, replacing the internal coil by a magnet. In reference [2] the value of the local gravity is

ISBN: 978-84-608-3145-7

similarly calculated using a sound card and adequate signal-processing software.

As will be described below, the set up and device presented here allows a simple and economical way to approach properties of the dynamics of the pendulum that are not normally considered in elemental descriptions. The experiment described in this paper can be easily implemented in a Physics undergraduate laboratory. This novel and economical method can be used as a complement in the formation of students of Sciences or Engineering, yielding results with improved precision over the most traditional set ups, that can be also easily analysed and profitably discussed by the students.

## II. Theoretical Background

In this section we summarily review the main theoretical physical concepts at the core of the experimental set up, both regarding the mechanical oscillations of a pendulum and Faraday's law of magnetism. Furthermore, it is necessary to distinguish between harmonic and anharmonic oscillations, since both can be observed in the movement of the pendulum. Concluding this section there is a description of the local gravity formula which has been used in order to determine the value of $\boldsymbol{g}$ which will be considered as a statistically *true value*.

### A. Oscillations of a pendulum, Simple Harmonic Movement (SHM) and anharmonicity

A scalar physical variable $x$ (for example, the elongation angle in a simple pendulum, see **Fig.1**) is said to undergo harmonic oscillations if its time-evolution is described by means of the well-known differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{1}$$

whose general solution is

$$x = A\, sin\,(\omega t + \alpha) \tag{2}$$

where $\boldsymbol{A}$ (amplitude of the motion) and $\boldsymbol{\alpha}$ (initial phase) are constants determined from the initial conditions in $x$ and its first derivative. In the above expressions, $\omega$ is the angular frequency of the movement, which through (2) is shown to be periodic with period $T = 2\pi/\omega$. On the other hand, the frequency f of the movement is given by $f = 1/T = \omega/2\pi$.

For a SHM, in the laboratory one normally measures the time elapsed in a certain number of oscillations and from here one obtains the period and frequency of the movement, which can be compared with the theoretical values given by the above expressions in terms of the different parameters of the system.

A particular case of SHM is the motion of a pendulum when it is released from rest with a certain initial elongation that corresponds to a very small angle $\theta$. In a simple pendulum, consisting of a point mass suspended from one end of a massless, frictionless thread with constant length $L$ that is fixed at the other end (see Fig. 1), the forces that act on the mass are the tension transmitted by the thread $\overrightarrow{T}$ and the gravitational force of the mass $\overrightarrow{F_g}$ (its weight), and the resulting trajectory is an arc of circumference of radius $L$.

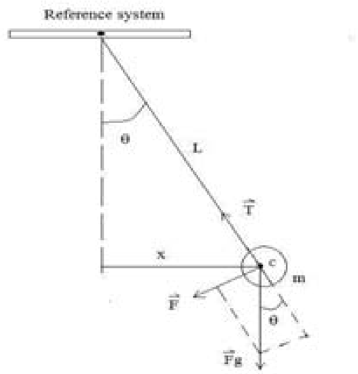


**Fig. 1.** Scheme of a simple pendulum

The tangential component of the weight is a restoring force that opposes the displacement of the mass from its equilibrium position at $\theta = 0$. Applying Newton's laws of Mechanics, one gets:

$$mL\frac{d^2\theta}{dt^2} = -mg\, sin\theta \tag{3}$$

that is,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0 \tag{4}$$

The motion described by this equation is in general oscillatory (periodic), but will be of the simple harmonic type only in the case of small oscillations for which one can approximate $sin\,\theta \cong \theta$ : In this case, $\omega = \sqrt{g/L}$ and $T = 2\pi\sqrt{L/g}$, or:

$$L = \frac{g}{4\pi^2}\, T^2 \tag{5}$$

Thus, in an experiment with pendulums of different lengths, each moving within the small oscillation regime, one must obtain that, within the experimental precision, their periods do not depend on mass or on the (small) starting displacement, and if $L$ is represented against $T^2$, one gets a linear dependence from which the value of $\boldsymbol{g}$ can be directly obtained from the slope of a linear fitting.

However, in the case of large oscillations, this harmonic approximation is insufficient and one enters the so-called anharmonic regime. This begins to be noticeable (using traditional set ups) when the initial angle exceeds around approximately 20-25º [3, 4]. In this case the period of the oscillations can be expressed as the period in the small oscillations regime corrected by a power expansion that depends on the initial angle $\theta_0$:


ISBN: 978-84-608-3145-7

$$\left(1+\frac{1}{4}sin^2\frac{1}{2}\theta_0\frac{9}{64}sin^4\frac{1}{2}\theta_0+\cdots\right)=1+\alpha\,(\theta_0) \quad (6)$$

and from this:

$$T=2\pi\sqrt{\frac{L}{g}}\,(1+\alpha\,(\theta_0))\ \rightarrow \quad (7)$$

$$\rightarrow\quad L=\frac{g}{4\pi^2}\frac{1}{(1+\alpha(\theta_0))^2}T^2 \quad (8)$$

It is possible to check experimentally this expression for the dependence of the period on the elongation.

### B. Local gravity

In all the above, one wants to consider the local value of the acceleration of gravity g as exactly as possible, to be able to compare with the experimental results. We remind here that the formula for g recommended by the International Organization of Legal Metrology (IOLM) (see [5, 6]) depends on the latitude and height over sea level, and is given, to an alleged 0.01 % precision, by:

$$g_l=g_{ec}\,(1+\beta sin^2\boldsymbol{\varphi}+\beta_1 sin^2 2\boldsymbol{\varphi})-3{,}086x10^{-6}\boldsymbol{h} \quad (9)$$

where $g_{ec}=9{,}780327$ m/s$^2$ is the value of local gravity in the equator; $\beta=0{,}0053024$ is the flat gravimetric effect; $\beta_1=0{,}0000058$ is related with the centrifugal effect, the geometrical flat effect and the value of g in the equator; $\varphi$ is the latitude in degrees; $h$ is the height above sea level (in m). Our experiments have been performed in San Cristobal de La Laguna (Tenerife, Spain), and with $\varphi$ and $h$ for this location taken from Ref [7], one gets $\boldsymbol{g_l}=$ **9,7903 m/s$^2$.**

### C. Faraday's Law of induction

The set up that will be described in the next sections makes use of Faraday's law of induction, which establishes that an electromotive force $\xi$ (and a current) on a closed circuit is generated when the magnetic flux enclosed by the circuit changes with time, as is the case when a magnetic body swings through a fixed coil [8]:

$$\oint_c \vec{\boldsymbol{E}}\cdot d\vec{\boldsymbol{l}}=-\frac{d}{dt}\int_s \vec{\boldsymbol{B}}\cdot d\vec{\boldsymbol{s}} \quad (10)$$

where $\Phi=\int_s \vec{\boldsymbol{B}}\cdot d\vec{\boldsymbol{s}}$ is the flux that crosses the surface *S* enclosed by the circuit *C* (with ***B***, the magnetic field) and $\oint_c \vec{\boldsymbol{E}}\cdot d\vec{\boldsymbol{l}}$ is the electromotive force $\boldsymbol{\xi}$ induced in the contour circuit *C*. Therefore it is possible to write (10) as:

$$\xi=-\frac{d}{dt}\Phi \quad (11)$$

The minus sign indicates that the induced current circulates in such a sense as to oppose the variation in the magnetic flux. In the case of a coil of *N* loops, the total $\boldsymbol{\xi}$ induced in the coil is the sum of the individual contributions, that is:

$$\xi=-N\frac{d\Phi}{dt} \quad (12)$$

## III. Experiment

### A. Design of the experimental set up

Our experimental device (Fig. 2) is a "simple" pendulum that consists of the following parts: a holed and telescopic methacrylate rod, in whose bottom end there is an attached magnetic mass. The pendulum oscillates in a perpendicular plane containing a coil. This movement implies a variation of the magnetic flux through the area defined by the coil, that according to Faraday's law generates an electromotive force, which is amplified and then sent to an oscilloscope.

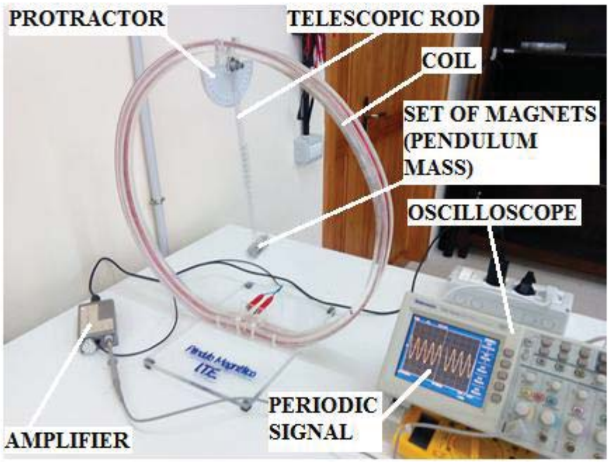


**Fig. 2.** Experimental set up.

The characteristics of the main elements employed in this experimental set up, are the following:

- A coil of 220 copper whorls with an approximate diameter of 50 cm, which was constructed over a methacrylate base.
- Neodymium magnets [9], acting as main oscillating body. The total mass of the whole set of magnets is approximately 260 g.
- Telescopic methacrylate rods in which several holes have been made, with a distance of 0.5 cm between them. This allows changing the length of the rods. The total mass of the rod is 20 g.
- A device to hold the system rod-magnets, that allows its movement contained in a plane perpendicular to the coil. The mentioned system is located in the highest point of the coil, where a protractor has also been disposed, in order to measure the angle from which the pendulum is separated from its vertical equilibrium position.
- An amplified filter formed by a High Speed FET-Input Instrumentation Amplifier, INA111, with a fixed gain of 1000, followed by a buffer amplifier

and a 5th Order Lowpass Butterworth filter, with Sallen Key topology, overall filter gain of 1, and a cut-off frequency (approximately the -3dB frequency) of 30 Hz.

- An oscilloscope to visualize the amplified signal.

*B. Procedure*

The set up allows the performance of two different procedures:

a) Fix an angle from which the pendulum is initially separated from its vertical position, while varying the length of the pendulum, by means of pins through holes in the pendulum's rod. The length can be modified from a minimum value of 20 cm to a maximum of 45 cm, with a step of 0.5 cm.

b) Fix the length of the pendulum and vary the angle from 5 to 90º, with a step of 1º.

In both cases, the signal in the oscilloscope reaches a stable value after a transient period of two or three oscillations of the pendulum.

Previously to taking measurements properly, the student can observe how an electromotive force is generated in the coil, as well as the period of the signal shown in the oscilloscope, comparing it with a rough estimate of the period of the pendulum obtained using a chronometer. For that, a student has to release the pendulum through the coil from a certain angle, whereas another student stops it at the other side, so that a half-period movement is completed.

As regards procedure *a)*, the student can see that for small angles (below approximately 30º) and using different lengths, the signal observed in the oscilloscope is periodic, and its period increases with the length of the pendulum. For large initial displacements the signal is still periodic, but has a peculiar shape rather different from the small-displacement case, which latter resembles more an harmonic signal.

Following procedure *b)*, the student can observe that, for a certain length, and if the angle from which the pendulum is released is small, the period of the signal does not change significantly, that is, it does not noticeably dependent with the initial displacement. However, for angles higher than 30º, the period of the signal increases significantly if the angle does.

## IV. RESULTS AND DISCUSSION

*A. Representation of the periodic movement*

In Figure 3 we show the different behaviour of the signal shown in the oscilloscope when the pendulum oscillates under different conditions. This case corresponds to a pendulum with a fixed length of 30 cm. The image on the top corresponds to an initial elongation of 50º, that in the middle to 30º and the bottom one to 20º. All the images were taken from the same experiment. Note that the scale is the same in all three pictures.

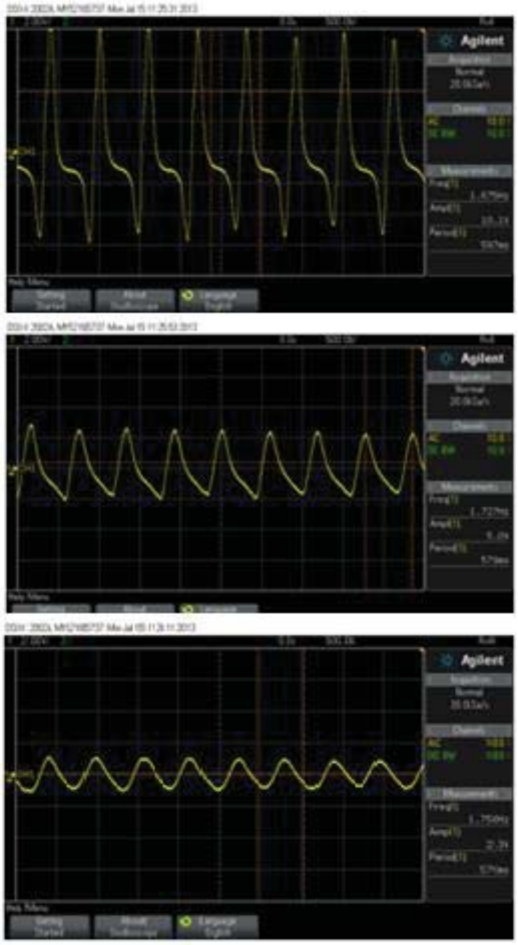


**Fig. 3.** Signal obtained in the oscilloscope for harmonic and anharmonic oscillations of a pendulum of length 30 cm. These signals are slowly changing with time as the amplitude of the movement is damped.

A periodic evolution is clearly shown and for small amplitudes the signal looks quasi-harmonic, whereas for large amplitudes it has a rather peculiar shape. The induced voltage increases with the amplitude of the oscillations, which is an effect of Faraday´s Law: a larger elongation implies a larger pendulum maximum velocity when crossing the coils, thus a larger variation of the magnetic flux and therefore the value of the induced voltage will be larger. For large oscillations, the flat part of the signal corresponds to the greatest perpendicular distances between magnet and coil. These imply low velocity and, consequently, low flux variation. When the magnet starts its accelerated movement in the direction of the center of the coil the induced voltage increases to reach a maximum. When the magnet crosses the coil, the voltage decreases to reach the opposite maximal amplitude: now, the induced current goes in the opposite direction.

Some qualitative features of the movement of the pendulum are best seen in Figures 4 and 5, where the length L of the pendulum is represented against the square of its period T for different initial elongation angles. This is represented in Fig. 4 for small displacements and in Fig. 5 for large ones. The


ISBN: 978-84-608-3145-7

period increases with the length of the pendulum, following the expected dependence for small displacements. For the sake of comparison the elongation angle that corresponds to 30º is included in both figures. Figure 5 shows an increase of the period with large elongation angles. On the other hand, one can readily notice that the period increases with the elongation angle, as plotted in Fig. 6, where it can be observed that for each chosen length the period approaches the small-displacement constant value at low angles while it increases significantly at large angles, with differences being particularly noticeable above 20-30º. A value of 23º is considered in [4] as an approximate limit at which the small-displacements approximations must be abandoned.

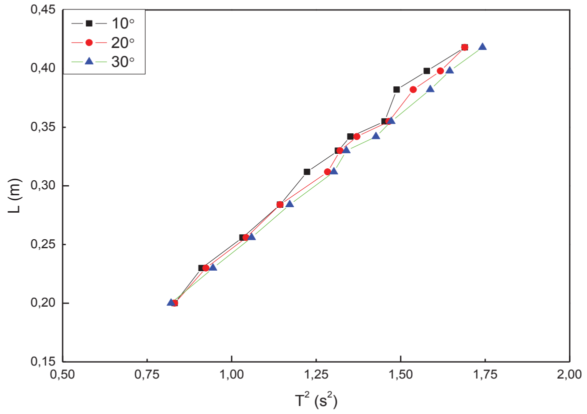


**Fig. 4.** L-T² relation for small oscillations.

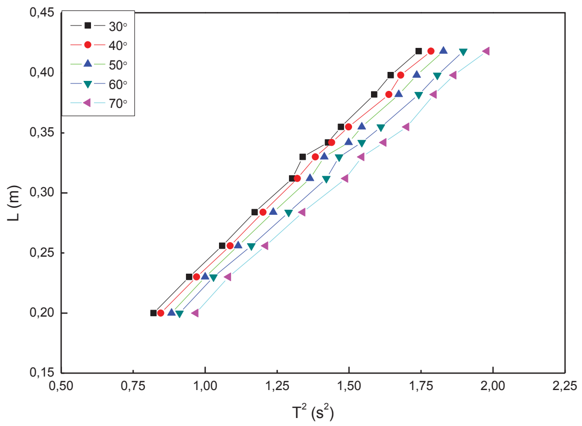


**Fig. 5.** L-T² relation for large oscillations.

## B. *Estimation of* $g$

The period $T$ was measured at seven different angles (10-70º) and eleven different pendulum lengths (20-41.8 cm). Every value of $T$ was the average of three different measurements. From a $L$-$T^2$ representation, $g$ can be obtained from linear fitting using (8) for each angle, once the set of lengths has been given. This procedure yields the value for $g$ from the slope of the fitting. These results are shown in **Table I** of the **Appendix**.

As previously explained, the coefficient $\alpha(\theta_0)$ can be calculated for each angle. Similar results were observed within the standard deviation range and the propagated error range, for $g$ calculated respectively by linear fitting and individually for each pendulum length. There is no dependence with the length of the pendulum, considering that the magnet passes through the vertical axis of the coil at a different height (out of its center). However, the accuracy in the value of $g$ depends on the initial elongation angle, $\theta_0$. A larger deviation from the theoretical value of $g$ has been obtained for 10º and 70º, whereas better values are obtained for the rest of the angles. This is probably due to the fact that measurements on the movement of the pendulum are more prone to errors for small elongation angles, due to a low flux variation, whereas for the largest angles a pitch effect in the movement appears, and movement is also probably more sensible to friction.

An equation has been searched in order to extract a value for $g$ value by means of the compilation of all the obtained data. For this the student has to perform the following steps:

**Step 1**: for each length $L$ there is a set of different measured periods $T$, which correspond to different initial angles $\theta_0$. Three selected $T$- $\theta_0$ curves (corresponding to three different lengths of the pendulum) are represented in Fig. 6. One compares then the length with the square of the period defined by (7), where $T_0$ (period for small oscillations) is corrected by means of an expansion coefficient $\alpha(\theta_0)$. It is possible to adjust the following equation, where $\alpha(\theta_0)$ is the independent variable and T the dependent one.

$$T = T_0(1+\alpha(\theta_0)) \tag{13}$$

The period for small oscillations $T_0$, can be estimated by linear fitting for every length. Note that the value of $T_0$ can be obtained from the slope $b = T_{0\alpha}$ and the intercept $a = T_{0c}$ of the lineal fitting. The value obtained for the intercept is rather good as can be seen in Fig. 7. In the same figure, is also shown the $T$ calculated by (13), using the $T_{0c}$ fitted value. As it is shown, for large elongation angles the value of $T$ is lower than the fitted value which may be partially due to pitch and friction effects previously commented.

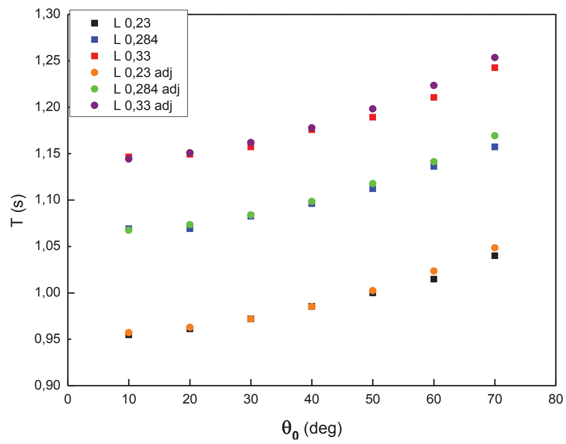


**Fig. 6.** Variation of period $T$ versus elongation $\theta_0$ for three different pendulum lengths $L$. Squares represent experimental measurements taken in the laboratory and circles correspond to calculated periods $T$ for each length by linear fitting.

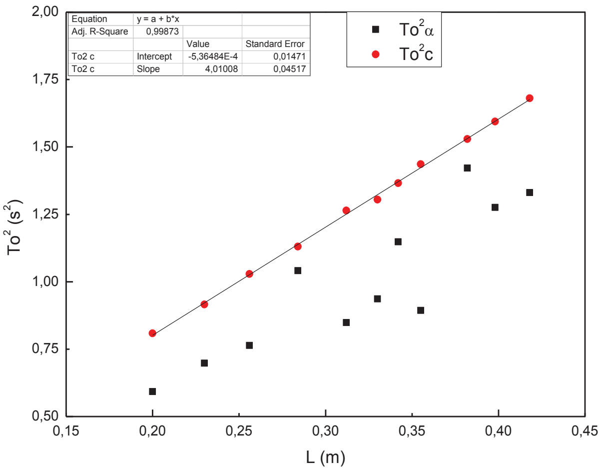


**Fig. 7 .** Difference between periods and linear fitting of $T_{0c}^2$.

**Step 2**: the two different values of $T_{0c}^2$ ($T_{0\alpha}$ and $T_{0c}$) versus $L$ have been represented in Fig. 7. In all cases, $T_{0c}$ is larger than $T_{0\alpha}$ and this reports a more regular behavior. It is another test of the effect produced by the friction between the rotation axis and the rod, because $T_{0\alpha}$ is affected by the oscillation quality ($\alpha$ value decreases because experimental amplitudes are smaller than theoretical ones). More dispersed values are due to the damping effect and additionally, it is not easy to take very good measures of initial elongation angles (an optimistic error in the measure of 1º was initially considered). So, if $T_{0c}$ is calculated for each length $L$, this dataset can be used to fit with (5), what allows calculating a value for $\boldsymbol{g}$. On the other hand, it can be calculated an individual value for each $L$, directly from (5). It was found that $\boldsymbol{g}$ does not depend on length, presenting a magnitude range that goes from 9.71 ± 0.07 m/s$^2$ to 9.98 ± 0.04 m/s$^2$. Finally, the value of $\boldsymbol{g}$ obtained by linear fitting and its deviation (given by the standard deviation obtained for the slope $b$, which is propagated because $\boldsymbol{g} = 4\pi^2/b$ ) are: $\boldsymbol{g}$ = 9.84 ± 0.11 m/s$^2$. As one could expect by arguments already mentioned, this value is relatively precise and quite accurate. Considering the exact value of $\boldsymbol{g}$ = 9,790 m/s$^2$ [5-7], this result means a difference of 5,1%.

## V. CLOSING REMARKS

The experimental set up for the magnetic pendulum described in the above sections was done in-house, completely handmade and starting from scratch. Although illustrated here with results for the most basic and ideal case of a simple pendulum, nothing precludes the use of a similar set up with a more realistic physical pendulum, with the obvious changes in the theoretical formulae that have to be taken into account in that case. Essentially the same set up has been also employed for studying the oscillations of other systems, like a spring with a mass. Improvements over the existing set up can still be made, particularly aiming at the reduction of friction. It would also be interesting to allow even larger amplitudes of operation. It is in particular of interest the extension of the study to include anharmonic oscillations, exploring the limit between harmonic and anharmonic behavior. The difficulty to keep movement in a vertical plane and the damping of oscillations affect the accuracy of the experimental data and the values obtained for $\boldsymbol{g}$ from them. A procedure of analysis which relates all data together was followed, and an approximation for the period dependence on the elongation´s angles was used. This last one allows detecting the effect in the period of oscillations by means of the angle´s measurements dispersion. Although the values obtained for the periods are affected by error, the value obtained for $\boldsymbol{g}$ by working with the whole dataset has a deviation of 5% compared to the exact value, with a relative standard deviation of 1%, which can be considered acceptable.

The utility of this device as a teaching laboratory experiment has been tested with first-year undergraduate students in degrees of Physics, Chemistry, and Engineering, allowing access to some features which are not usually considered at an elementary level. The students have had the general impression that the use of this device contributes positively to their understanding of different types of phenomena, and their analysis in a simultaneous way. This represents a positive contribution to methodological resources and the development of practical contents in Science degrees. As a result, the teaching-learning process has been also improved.

## REFERENCES

[1] J. Alba Fernández, Jaime Ramis Soriano, Romina del Rey Tormos, Eva Escuder Silla, "Una variación del método dinámico para la obtención de la constante elástica de un muelle". Revista Española de Física, vol. 22, no.4, 2008, pp. 47-50.

[2] M. J. Santos, J.A. White, A. González, S. Velasco, "Escuchando la gravedad: muelle oscilante", Departamento de Física Aplicada, Facultad de Ciencias de la Universidad de Salamanca, 2010. (http://diarium.usal.es/smjesus/files/2013/05/PosterMuelleOscilante.pdf)

[3] P.A.Tipler and G. Mosca, "Física para la Ciencia y la Tecnología", Reverté, vol. 1, 5ª ed., p.410, 2008.

[4] M. Alonso and E.J. Finn, "Física", Addison Wesley, vol. 1, pp. 366-369,1989.

[5] Organización Internacional de Metrología Legal. "OIML boletín 127", Junio 1992. Available: www.oiml.org.

[6] www.metas.com.mx/La-Guia-MetAs-02-05-gl.pdf

[7] www.aemet.es./es/eltiempo/predicción/municipios/horas/san-cristobal-de-la-laguna-id38023.

[8] D. K. Cheng, "Fundamentos de Electromagnetismo para Ingeniería", Addison Wesley, pp. 230-231, 1998.


ISBN: 978-84-608-3145-7

[9] http://www.amazon.es/first4magnets-f999n-di%C3%A1metro-resistente-tirador/dp/B0022T7XDE

## APPENDIX

TABLE I

Values of ***g*** obtained by linear fitting using (8)

| Angle α (deg) | a | b | $\chi^2$ ($\cdot 10^{-5}$) | $R^2$ | g ($m/s^2$) |
|---|---|---|---|---|---|
| 10 | -0.005 ±0.008 | 0.254 ±0.007 | 31.33 | 0.99294 | **10.08 ±0.26** |
| 20 | -0.003 ±0.005 | 0.248 ±0.004 | 14.10 | 0.99628 | **9.97 ±0.17** |
| 30 | 0.005 ±0.003 | 0.237 ±0.003 | 6.212 | 0.99860 | **9.71 ±0.11** |
| 40 | 0.003 ±0.003 | 0.233 ±0.002 | 5.067 | 0.99886 | **9.80 ±0.10** |
| 50 | 0.000 ±0.003 | 0.229 ±0.002 | 4.769 | 0.99892 | **9.97 ±0.10** |
| 60 | 0.002 ±0.003 | 0.219 ±0.002 | 6.153 | 0.99861 | **9.93 ±0.11** |
| 70 | -0.004 ±0.004 | 0.214 ±0.003 | 7.859 | 0.99823 | **10.18 ±0.13** |

ISBN: 978-84-608-3145-7 

ISBN: 978-84-608-3145-7