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Positive Mild Solutions of A Fractional Boundary Value Problem on the Half-Line

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Abstract. In this paper, we investigate the existence and uniqueness of a mild solution to a fractional boundary value problem involving a Riemann-Liouville type fractional derivative. Our approach is based on the application of a relatively recent fixed point theorem for $\mathcal{F}$-contractions in complete metric spaces, which allows us to work under more general conditions than classical contraction principles. This framework provides a powerful and flexible tool for dealing with nonlocal problems arising in various applied fields. In addition to establishing existence and uniqueness, we demonstrate that, under certain additional assumptions, the mild solution is positive. This qualitative property is of particular interest in real-world applications where negative solutions may lack physical meaning. Finally, to illustrate the theoretical results, we present a concrete example that satisfies all the hypotheses and confirms the main conclusions.

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Keywords. Riemann–Liouville fractional derivative, Fractional boundary value problem, Fixed point theorem, Mild solution, Positive solution.

1. Introduction

Fractional differential equations appear in a great number of engineering and scientific disciplines as mathematical models of processes in the fields of physics, chemistry, polymer rheology, electrical circuits, biology and control theory, among others (see [1, 7–9, 12] and the references therein, for example).

Fractional derivatives are a very good tool for the description of memory and hereditary properties of a great number of materials and processes and this constitutes the main advantage of fractional differential equations in comparison with classical integer-order models.

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A lot of papers have appeared in the literature studying the existence of solutions to fractional boundary value problems in finite interval [4,6,10, 14,19].

In the last decade, fractional boundary value problems on infinite intervals have been studied by many authors. These types of problems arise in the study of radially symmetric solutions of nonlinear elliptic equations and a great number of physical phenomena [2,3].

Among the main tools used in the theory of solutions to these types of problems, we can mention the monotone iterative technique, fixed point index theory, and Guo-Krasnoselskii's, Schauder's and Avery–Peterson fixed point theorems. [13,15–18].

In this present paper, we study the existence and uniqueness of solutions to the following fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}\, x(t) + f\left(t, x(t)\right) = 0, \quad t \in (0,\, \infty), \\ x(0) = 0, \quad \lim\limits_{t\to\infty} D^{\alpha-1}_{0+}\, x(t) = \lambda \geq 0, \end{cases} \tag{1}$$

with $1 < \alpha < 2$. Here D^{α}_{0+} denotes the standard Riemann-Liouville fractional derivative.

Our main tool in our study is a fixed point theorem for F-contractions [11].

2. Preliminaries

Our starting point in this section is the fixed point theorem that we will use in our study. This results appears in [11].

First, we need to introduce the following class of functions $\mathcal{F}$. By $\mathcal{F}$, we denote the class of functions $\varphi : (0,\, \infty) \to \mathbb{R}$ that satisfy the following three conditions:

1. φ is a strictly increasing function.
2. $inf(\varphi) = -\infty$.
3. φ is a continuous function.

Functions φ belonging to the class $\mathcal{F}$ are $\varphi(t) = -\dfrac{1}{\sqrt{t}}$, $\varphi(t) = \ln(t)$, $\varphi(t) = \ln(t) + t$, among others.

The above mentioned fixed point theorem is the following.

Theorem 1. *Let T be a self-mapping of a complete metric space $(X,\, d)$ such that there exist $\tau > 0$ and $\varphi \in \mathcal{F}$ satisfying that, for any $x,\, y \in X$ with $d\left(Tx, Ty\right) > 0$,*

$$\tau + \varphi\left(d(Tx, Ty)\right) \leq \varphi\left(d(x, y)\right).$$

Then, T has a unique fixed point x^* in X. Moreover, for any $x_0 \in X$, the sequence $(T^n x_0)$ converges to x^*.

For our study, we need the following lemma which corresponds to Lemma 2.3 of [17].

Lemma 1. *Suppose that* $g \in C[0,\infty)$*. Then, the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha}\, x(t) + g(t) = 0, & 0 < t < \infty, \\ x(0) = 0, & \lim\limits_{t\to\infty} D_{0^+}^{\alpha-1}\, x(t) = \lambda \geq 0, \end{cases} \tag{2}$$

where $1 < \alpha < 2$ *has a unique solution*

$$x(t) = \int_0^{\infty} G(t,\, s)\, g(s)\, ds + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1},$$

for $t \in [0,\, \infty)$*, where*

$$G(t,\, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t < \infty, \\ t^{\alpha-1}, & 0 \leq t \leq s < \infty. \end{cases}$$

Remark 1. In Lemma 2.4 of [17], the authors proved the following facts:

1. $G(t,\, s)$ is a continuous function on $[0,\infty) \times [0,\infty)$ and $G(t,s) \geq 0$ for any $t,\, s \in [0,\infty)$.
2. For any $t,\, s \in [0,\infty)$, $G(t,s) \leq \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ and, consequently,

$$\frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}.$$

Next, we present the following definition.

Definition 1. A function $x(t)$ is said to be a mild solution to Problem (1) if it satisfies the integral equation

$$x(t) = \int_0^{\infty} G(t,s)\, f\left(s, x(s)\right)\, ds + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1},$$

for any $t \in [0,\infty)$.

3. Main Results

In this section, we start with a technical result which is interesting for our aim.

The following lemma appears in Lemma 4 of [5] for $p > 1$, but the result also works when $p > 0$. To make the paper self-contained, we give a proof of this fact.

Lemma 2. *Let* $p > 0$ *and* $\tau > 0$*, and consider the function* $\phi_p^{\tau} : [0,\infty) \to [0,\infty)$ *given by*

$$\phi_p^{\tau}(t) = \frac{t}{\left(1 + \tau\, t^{1/p}\right)^{p}},$$

for $t \in [0,\infty)$*.*

Then

1. $\phi_p^{\tau}(t)$ *is increasing.*
2. $\phi_p^{\tau}(t)$ *is subadditive.*

3. *For any* $t, s \in [0, \infty)$*, the following inequality*
$$\left|\phi_p^\tau(t) - \phi_p^\tau(s)\right| \le \phi_p^\tau\left(|t-s|\right),$$
holds.

Proof.

(i) ϕ_p^τ is increasing since
$$\phi_p^\tau(t)' = \frac{1}{(1+\tau t^{\frac{1}{p}})^{p+1}} \ge 0 \quad \text{for } t \in [0,\infty).$$

(ii) As $\phi_p^\tau(0) = 0$ and
$$\left(\phi_p^\tau\right)'' = -\frac{(p+1)\frac{\tau}{p}}{t^{1-\frac{1}{p}}(1+\tau t^{\frac{1}{p}})^{p+2}} \le 0 \quad \text{for } t \in (0,\infty),$$
ϕ_p^τ is a concave function, and, by a well-known result, ϕ_p^τ is subadditive.

(iii) Without loss of generality, suppose that $s < t$ and $s, t \in [0, \infty)$. By using (ii), it follows
$$\phi_p^\tau(t) = \phi_p^\tau(s + t - s) \le \phi_p^\tau(s) + \phi_p^\tau(t-s)$$
and this gives us
$$\phi_p^\tau(t) - \phi_p^\tau(s) \le \phi_p^\tau(t-s).$$
Since ϕ_p^τ is increasing by (i),
$$|\phi_p^\tau(t) - \phi_p^\tau(s)| = \phi_p^\tau(t) - \phi_p^\tau(s) \le \phi_p^\tau(t-s) = \phi_p^\tau(|t-s|).$$
This is the desired result.

□

Next, we present the space where the solutions of our problem live. By E, we denote the space defined by
$$E = \left\{x \in C[0,\,\infty) : \sup\left\{\frac{|x(t)|}{1+t^{\alpha-1}} : t \in [0,\infty)\right\} < \infty\right\}.$$
The space E equipped with the norm
$$||x|| = \sup\left\{\frac{|x(t)|}{1+t^{\alpha-1}} : t \in [0,\infty)\right\},$$
is a Banach space.

Notice that the distance in E is given by
$$d(x,y) = \sup\left\{\frac{|x(t)-y(t)|}{1+t^{\alpha-1}} : t \in [0,\,\infty)\right\}.$$

Next, we present the main result of the paper.

Theorem 2. *Under the following assumptions:*

1. $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function. Moreover,* $\int_0^\infty f(s,0)\,ds < \infty$.

2. *There exist* $\tau > 0$ *and* $p > 0$ *such that, for any* $t \in [0, \infty)$ *and* $x, y \in [0, \infty)$,

$$|f(t,x) - f(t,y)| \leq a(t)\, \frac{|x-y|}{\left(1+\tau|x-y|^{1/p}\right)^p} = a(t)\, \phi_p^\tau\left(|x-y|\right),$$

where $a : [0,\infty) \to [0,\infty)$ *and it satisfies* $\int_0^\infty \left(1+s^{\alpha-1}\right) a(s)\, ds \leq \Gamma(\alpha)$,

then Problem (1) *has a unique non-negative mild solution in* E*.*

Proof. Let P be the cone given by $P = \{x \in E : x \geq 0\}$.

It is clear that P is a closed subset and, therefore, (P, d) is a complete metric space being d the distance defined by

$$d(x,y) = \sup\left\{ \frac{|x(t)-y(t)|}{1+t^{\alpha-1}} : t \in [0,\infty) \right\}.$$

for $x, y \in P$.

Consider the operator T defined on P by

$$(Tx)\,(t) = \int_0^\infty G(t,s)\, f(s,x(s))\, ds + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1},$$

for any $x \in P$.

Notice that a fixed point of T is a mild solution to our Problem (1).

By Remark 1, $G(t,s) \geq 0$ for $t, s \in [0,\infty)$, by *2*, $f(t,s) \geq 0$ for $t, s \in [0,\infty)$ and as $\lambda \geq 0$, we have that, for $x \in P$, $Tx \geq 0$.

To prove that T applies P into itself, we have to see that if $x \in P$ then $Tx \in C[0,\infty)$.

To do this, we take $t_0 \in [0,\infty)$ and $(t_n) \subset [0,\infty)$ such that $t_n \to t_0$ and we have to prove that $(Tx)\,(t_n) \to (Tx)\,(t_0)$.

In fact,

$$\begin{aligned} |(Tx)\,(t_n) - (Tx)\,(t_0)| \leq & \left| \int_0^\infty \left(G(t_n,s) - G(t_0,s)\right)\, f\left(s, x(s)\right)\, ds \right| + \\ & + \frac{\lambda}{\Gamma(\alpha)} \left| t_n^{\alpha-1} - t_0^{\alpha-1} \right| = I_1^n + I_2^n. \end{aligned}$$

It is clear that $I_2^n \to 0$ when $n \to \infty$. On the other hand, and taking into account our assumptions, we have

$$\begin{aligned}
I_1^n \quad & \leq \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; f\left(s,x(s)\right)\, ds \\
& \leq \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; \Big[|f\left(s,x(s)\right)-f(s,0)|+f(s,0)\Big]\, ds \\
& \leq \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; \left[a(s)\, \dfrac{|x(s)|}{\left(1+\tau|x(s)|^{1/p}\right)^p}+f(s,0)\right] ds \\
& = \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; \left[a(s)\, \dfrac{x(s)}{\left(1+\tau x(s)^{1/p}\right)^p}+f(s,0)\right] ds \\
& = \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; \left[\left(1+s^{\alpha-1}\right)\, \dfrac{a(s)\,\frac{x(s)}{\left(1+s^{\alpha-1}\right)}}{\left(1+\tau x(s)^{1/p}\right)^p}+f(s,0)\right] ds \\
& \leq \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; \left[\left(1+s^{\alpha-1}\right)\, \dfrac{a(s)\,\frac{x(s)}{1+s^{\alpha-1}}}{\left(1+\tau\left(\frac{x(s)}{1+s^{\alpha-1}}\right)^{1/p}\right)^p}+f(s,0)\right] ds \\
& \leq \textstyle\int_0^\infty |G(t_n,s)-G(t_0,s)|\; \left[\left(1+s^{\alpha-1}\right) a(s) \dfrac{||x||}{\left(1+\tau||x||^{1/p}\right)^p}+f(s,0)\right] ds, \quad (3)
\end{aligned}$$

where we have used Lemma 2.

Taking into account the continuity of $G(t,s)$ and our assumptions, for $\varepsilon>0$ we find $n_0\in\mathbb{N}$ such that, for $n\geq n_0$, we have

$$|G(t_n,s)-G(t_0,s)|\leq \frac{\varepsilon}{\displaystyle\int_0^\infty\left(\left(1+s^{\alpha-1}\right)\, a(s)\, \frac{||x||}{\left(1+\tau||x||^{1/p}\right)^p}+f(s,0)\right) ds}.$$

From the last inequality and (3), it follows that $I_1^n\leq\varepsilon$ for $n\geq n_0$. This says that $\left(Tx\right)\left(t_n\right)\to\left(Tx\right)\left(t_0\right)$, this is $Tx\in C[0,\infty)$.

Now, we will prove that for $x\in P$, then

$$\sup\left\{\frac{\left(Tx\right)(t)}{1+t^{\alpha-1}}: t\in[0,\infty)\right\}<\infty.$$

In fact, we take $x\in P$ and $t\in[0,\infty)$ and, by Remark 1 and our assumptions, we infer

$$\begin{aligned}
\frac{\left(Tx\right)(t)}{1+t^{\alpha-1}} &= \frac{\displaystyle\int_0^\infty G(t,s)\, f\left(s,x(s)\right)\, ds+\frac{\lambda}{\Gamma(\alpha)}t^{\alpha-1}}{1+t^{\alpha-1}} \\
&= \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}\, f\left(s,x(s)\right)\, ds+\frac{\lambda}{\Gamma(\alpha)}\,\frac{t^{\alpha-1}}{1+t^{\alpha-1}} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\infty f(s,x(s))\, ds+\frac{\lambda}{\Gamma(\alpha)} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(|f(s,x(s))-f(s,\,0)|+f(s,0)\right)\, ds+\frac{\lambda}{\Gamma(\alpha)}.
\end{aligned}$$

Now, using a similar argument to the previous proof of $I_1^n\to 0$ when $n\to\infty$, we get

$$\frac{(Tx)\,(t)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}\left[\frac{||x||}{\left(1+\tau||x||^{1/p}\right)^{p}} + \int_0^\infty \left(1+s^{\alpha-1}\right) a(s)\,ds \right.$$
$$\left. + \int_0^\infty f(s,\,0)\,ds\right] + \frac{\lambda}{\Gamma(\alpha)} < \infty.$$

This proves our claim. Therefore, T maps P into itself.

In the sequel, we check that T satisfies the condition appearing in Theorem 1. In fact, we take $x,\, y \in P$ with $d(Tx,\, Ty) > 0$, then we deduce

$$\begin{aligned}
d(Tx, Ty) &= \sup\left\{\frac{|(Tx)(t)-(Ty)(t)|}{1+t^{\alpha-1}} : t\in[0,\infty)\right\}\\
&= \sup\left\{\frac{\displaystyle\int_0^\infty G(t,s)\,f(s,x(s))\,ds - \int_0^\infty G(t,\,s)\,f(s,y(s))\,ds}{1+t^{\alpha-1}} : t\in[0,\infty)\right\}\\
&\leq \sup\left\{\frac{\displaystyle\int_0^\infty G(t,s)\;|f(s,x(s))-f(s,y(s))|\;ds}{1+t^{\alpha-1}} : t\in[0,\infty)\right\}\\
&\leq \sup\left\{\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}}\;|f(s,x(s))-f(s,y(s))|\;ds : t\in[0,\infty)\right\}.
\end{aligned}$$

Now, using item 2 of Remark 1 and hypothesis (2) from the theorem statement, we obtain

$$\begin{aligned}
d(Tx,Ty) &\leq \frac{1}{\Gamma(\alpha)}\int_0^\infty a(s)\,\frac{|x(s)-y(s)|}{\left(1+\tau\,|x(s)-y(s)|^{1/p}\right)^{p}}\,ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(1+s^{\alpha-1}\right)\,a(s)\,\frac{\left|\frac{x(s)}{1+s^{\alpha-1}}-\frac{y(s)}{1+s^{\alpha-1}}\right|}{\left(1+\tau\left|\frac{x(s)}{1+s^{\alpha-1}}-\frac{y(s)}{1+s^{\alpha-1}}\right|^{1/p}\right)^{p}}\,ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(1+s^{\alpha-1}\right)\,a(s)\,\frac{d(x,y)}{\left(1+\tau\,d(x,y)^{1/p}\right)^{p}}\,ds\\
&\leq \frac{d(x,y)}{\left(1+\tau\,d(x,y)^{1/p}\right)^{p}}\cdot\frac{1}{\Gamma(\alpha)}\int_0^\infty \left(1+s^{\alpha-1}\right)\,a(s)\,ds\\
&\leq \frac{d(x,y)}{\left(1+\tau\,d(x,y)^{1/p}\right)^{p}},
\end{aligned}$$

where we have used that ϕ_p^τ is an increasing function and, consequently, $\phi_p^\tau\left(\left|\dfrac{x(s)}{1+s^{\alpha-1}}-\dfrac{y(s)}{1+s^{\alpha-1}}\right|\right) \leq \phi_p^\tau\left(d(x,\,y)\right)$ and 2.

Summarizing, we have proved that, for any $x, y \in P$ with $d(Tx, Ty) > 0$, we have

$$d(Tx,\, Ty) \leq \frac{d(x,y)}{\left(1+\tau\, d(x,y)^{1/p}\right)^{p}},$$

or, equivalently.

$$d(Tx,\, Ty)^{1/p} \leq \frac{d(x,y)^{1/p}}{1+\tau\, d(x,y)^{1/p}}.$$

From this, we have

$$\frac{1}{d(x,y)^{1/p}} + \tau \leq \frac{1}{d(Tx,Ty)^{1/p}},$$

this is

$$\tau - \frac{1}{d(Tx,Ty)^{1/p}} \leq -\frac{1}{d(x,y)^{1/p}}.$$

This proves that the contractive condition in Theorem 1 is satisfied with $\varphi(t) = -\dfrac{1}{t^{1/p}}$ and it is easily seen that $\varphi \in \mathcal{F}$.

Finally, by Theorem 1, the operator T has a unique fixed point x^* in P. This finishes the proof. $\square$

From a practical point of view an interesting question is if the mild solution to Problem (1) obtained by Theorem 2 is positive, this is, $x^*(t) > 0$ for $t \in [0, \infty)$.

In what follows, we will present a sufficient condition for this to occur.

Theorem 3. *Under the conditions of Theorem 2,*

1. *if $\lambda > 0$, then the mild solution to Problem (1) x^* is positive.*
2. *suppose that $\lambda = 0$. If $f(t,x)$ is increasing respect to the second variable x and there exists $t_0 \in [0, \infty)$ such that $f(t_0, 0) > 0$, then the mild solution x^* is positive.*

Proof.

1. Let $\lambda > 0$. Since x^* is a fixed point of the operator T given by

$$(Tx)(t) = \int_0^\infty G(t,s)\, f(s, x(s))\, ds + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1},$$

for $t \in [0, \infty)$, this is

$$x^*(t) = \int_0^\infty G(t,s)\, f(s, x(s))\, ds + \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1},$$

for $t \in [0, \infty)$, it follows $x^*(t) \geq \dfrac{\lambda}{\Gamma(\alpha)\, t^{\alpha-1}}$ for $t \in [0, \infty)$.

Therefore, x^* is positive.

2. Let $\lambda = 0$. Suppose in contrary case that x^* is not positive.

Therefore, we can find $t^* \in (0, \infty)$ such that $x^*(t) = 0$ and, consequently,

$$0 = x^*(t^*) = \int_0^\infty G(t^*, s)\, f(s, x(s))\, ds.$$

Taking into account the non-negative character of the functions $G(t,s)$ and $f(s,x(s))$ and our assumption, we infer

$$\begin{aligned} 0 = x^*(t^*) &= \int_0^\infty G(t^*,s)\, f(s,x(s))\, ds \\ &\geq \int_0^\infty G(t^*,s)\, f(s,0)\, ds \geq 0, \end{aligned}$$

and, therefore

$$\int_0^\infty G(t^*,s)\, f(s,0)\, ds = 0.$$

As the integrand is non-negative, we have that

$$G(t^*,\, s)\, f(s,\, 0) = 0 \quad a.e\,(s).$$

Since $G(t^*,\, s)$ is of polynomial type, $G(t^*,s) \neq 0\, a.e\,(s)$ and from the last expression, we deduce

$$f(s,\, 0) = 0 \quad a.e\,(s).$$

Now, using our assumption, this is, the existence of $t_0 \in [0,\infty)$ such that $f(t_0,0) > 0$, this fact and the continuity of f tell us that there exists a set A such that $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure, and $f(s,0) > 0$ for $s \in A$. This contradicts the above obtained fact. Therefore, x^* is a positive mild solution to Problem (1).

This finishes the proof. □

To end this section, we present a numerical example illustrating our results.

Consider the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{3/2}\, x(t) + \dfrac{\mu}{(3t+2)^3\,(1+\sqrt[4]{t})} \left(\dfrac{x(t)}{(1+5\sqrt[3]{x(t)})^3} + \dfrac{1}{(t^2+1)^4} \right) = 0, \\ x(0) = 0, \quad \lim\limits_{t\to\infty} D_{0^+}^{1/2}\, x(t) = 1, \end{cases} \tag{4}$$

where $\mu > 0$ and $t \in [0,\, \infty)$.

Problem (4) is a particular case of Problem (1), with $\alpha = \dfrac{3}{2}$, $\lambda = 1$ and

$$f(t,\, x) = \frac{\mu}{(3t+2)^3\,(1+\sqrt[4]{t})} \left(\frac{x}{(1+5\sqrt[3]{x})^3} + \frac{1}{(t^2+1)^4} \right).$$

Since $\mu > 0$, it is clear that $f : [0,\, \infty) \times [0,\, \infty) \to [0,\, \infty)$ is continuous and, moreover, we have

$$\begin{aligned} \int_0^\infty f(s,\, 0)\, ds &= \int_0^\infty \frac{\mu}{(3s+2)^3\,(1+\sqrt[4]{s})\,(s^2+1)^4}\, ds \\ &\leq \int_0^\infty \frac{\mu}{(3s+2)^3}\, ds = \frac{\mu}{24} < \infty. \end{aligned}$$

This proves that assumption *2* of Theorem 2 is satisfied.

On the other hand, for any $t \in [0, \infty)$ and $x,\, y \in [0, \infty)$, we deduce

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{\mu}{(3t+2)^3\,(1+\sqrt[4]{t})} \left| \frac{x}{(1+5\sqrt[3]{x})^3} - \frac{y}{(1+5\sqrt[3]{y})^3} \right| = \\ &= \frac{\mu}{(3t+2)^3\,(1+\sqrt[4]{t})} \left| \phi_3^5(x) - \phi_3^5(y) \right| \leq \\ &\leq \frac{\mu}{(3t+2)^3\,(1+\sqrt[4]{t})}\, \phi_3^5\left(|x-y|\right), \end{aligned}$$

where in the last inequality, we have used Lemma 2. Therefore, in this case, $a(t) = \dfrac{\mu}{(3t+2)^3\,(1+\sqrt[4]{t})}$ and it is clear that $a:\, [0,\, \infty) \to [0, \infty)$, and, moreover,

$$\int_0^\infty (1+s^{1/2})\, \frac{\mu}{(3\,s+2)^3\,(1+\sqrt[4]{s})}\, ds \leq \mu \int_0^\infty \frac{ds}{(3\,s+2)^3} = \frac{\mu}{24}.$$

As $\Gamma\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}$, if $\mu \leq 12\sqrt{\pi} \approx 21.26$, then assumption *2* is satisfied with $\tau = 5$ and $p = 3$.

From this and Theorem 2, it follows that Problem (4) has a unique nonnegative mild solution in the space E when $\mu \leq 21.26$. Additionally, as $\lambda = 1$, assumption *3* in Theorem 3 is satisfied and, consequently, Problem (4), has a unique positive mild solution in E when $\mu \leq 21.26$.

Notice that for the existence of mild solution to Problem (4) we cannot use Banach's contraction principle with distance d. To see this, taking into account that the operator T defined on the cone $P = \{x \in E:\; x \geq 0\}$, where

$$E = \left\{ x \in C[0,\infty):\, \sup\left\{ \frac{|x(t)|}{1+t^{1/2}}\, :\, t \in\; [0,\infty) \right\} < \infty \right\}$$

equipped with the distance

$$d(x,y) = \sup\left\{ \frac{|x(t)-y(t)|}{1+t^{1/2}}\, :\, t \in\; [0,\, \infty) \right\},$$

is given by

$$(Tx)(t) = \int_0^\infty G(t,s)\, \frac{\mu}{(3\,s+2)^3\,(1+\sqrt[4]{s})} \left(\frac{x(s)}{\left(1+s\sqrt[3]{x(s)}\right)^3} + \frac{1}{(s^2+1)^4} \right) ds,$$

for $x \in P$ and $G(t,s)$ is the Green's function appearing in Lemma 1. Moreover, for $\mu \leq 21.26$, we have that, for any $x, y \in P$,

$$d(Tx,\, Ty) \leq \frac{d(x,y)}{\left(1+5\,d(x,y)^{1/3}\right)^3}.$$

From the last inequality, we get

$$\frac{d(Tx,\, Ty)}{d(x,y)} \leq \frac{1}{\left(1+5\,d(x,y)^{1/3}\right)^3},$$

for any $x, y \in P$ with $x \neq y$.

This tells us that, when $d(x,y) \to 0$, $\dfrac{d(Tx, Ty)}{d(x,y)} \to 1$, and this means that in our example the Banach's contraction principle does not work.

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Author contributions All authors whose names appear on the submission contributed equally to this work.

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Declarations

Conflict of Interest The authors declare no conflict of interest.

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