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Attractors associated to iterated function systems of enriched φ-contractions in Banach spaces

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ABSTRACT. In this paper, we consider iterated function systems built using a finite family of condensing enriched φ-contractions w.r.t. a measure of noncompactness in Banach spaces. We prove the existence of a fractal associated to the above mentioned iterated function system.

1. INTRODUCTION

Nowadays, fractal sets and fractal interpolation functions play a very important role in mathematics and in other areas of science, namely biology, quantum physics, wavelet analysis, computer graphics, metallurgy, earth sciences, surface physics, chemistry, economics and medical sciences, see [18] and references therein.

In this paper, we discuss the existence of fractal sets which are obtained as attractors of an iterated function system (I.F.S., in short). More precisely, suppose that X is a nonempty set, then every nonempty family of self-maps $\mathcal{F}$ on X is called an iterated function system. Associated to I.F.S. given by $\mathcal{F}$, we define the invariance operator $T\colon \mathcal{P}(X) \to \mathcal{P}(X)$ given by

$$T(H) = \bigcup_{f \in \mathcal{F}} f(H),$$

for any $H \in \mathcal{P}(X)$.

When $T(H) = H$, that is, H is a fixed point by T, we say that H is $\mathcal{F}$-invariant. In the case that $H \subset T(H)$, H is said to be a subinvariant set.

Now, we suppose that (X, d) is a metric space, by a $\mathcal{F}$-fractal we mean a nonempty compact $\mathcal{F}$-invariant subset H of X.

The aim of this paper is to study iterated function systems built using a finite family of condensing enriched φ-contractions for a measure of noncompactness in Banach spaces and to prove the existence of a fractal associated to this iterated function system.

2. PRELIMINARIES

In the sequel, we present some definitions which we will use later.

Suppose that $(E, \|\cdot\|)$ is a real Banach space. By $\mathfrak{M}_E$ we denote the class of nonempty bounded subsets of E and by $\mathfrak{N}_E$ the subfamily of the relatively compact subsets of E. For $X, Y \in \mathfrak{M}_E$, by $X + Y$ and λX with $\lambda \in \mathbb{R}$, we denote the usual algebraic operations on sets; by $\overline{X}$, $conv\, X$ and $Conv\, X$ are denoted the closure, the convex ball and the convex closed ball of X, respectively.

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Definition 2.1. *A mapping $\mu\colon \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness (m.n.c., for short) in E if it satisfies the following conditions:*

(1) The family $Ker\,\mu = \{X \in \mathfrak{M}_E \colon \mu(X) = 0\}$ is nonempty and $Ker\,\mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \implies \mu(X) \leq \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(X)$.
(4) $\mu(Conv\,X) = \mu(X)$.
(5) $\mu(\lambda X + (1-\lambda) Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0,1]$.
(6) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ satisfying that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$ then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Moreover, if $\mu(X+Y) \leq \mu(X) + \mu(Y)$ and $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$, then we say that μ is subadditive and homogeneous, respectively.

When $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$, then we say that the m.n.c. μ satisfies the maximum property and when $Ker\,\mu = \mathfrak{N}_E$ μ is said to be full.

A m.n.c. μ is said to be regular when it is sublinear, full and satisfies the maximum property.

In the theory of measures of noncompactness there are two regular measures that have a very important relevance, they are the Kuratowski $\alpha(X)$ and the Hausdorff $\chi(X)$, which are defined by

$$\alpha(X) = \inf\Big\{\epsilon > 0\colon \text{there exists } A_i \subset E\ i = 1, 2, \ldots, n \\ \text{such that } X \subset \bigcup A_i \text{ with diam } A_i < \epsilon\Big\}$$

and

$$\chi(X) = \inf\left\{\epsilon > 0\colon \text{there exists } F \subset E \text{ finite satisfying } X \subset \bigcup_{x\in F} B(x,\epsilon)\right\}.$$

In what follows, we recall some concepts and results about enriched φ-contractions taken from [4], see also [8].

Definition 2.2. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if it satisfies the following conditions:*

(i) φ is nondecreasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$.
(ii) $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$, where φ^n denotes the n-iteration of φ.

Examples of comparison functions are:

$$\varphi(t) = \frac{t}{t+1}, \qquad \varphi(t) = \arctan t, \qquad \varphi(t) = \ln(1+t),$$

$$\varphi(t) = \begin{cases} \dfrac{t}{2}, & t \in [0,1], \\ t - \dfrac{1}{3}, & t \in (1,\infty), \end{cases}$$

It is easily seen that any comparison function satisfies the following property.

$$\varphi(t) < t \text{ for any } t > 0.$$

For a systematic account on iterated function systems of various φ-contractions (Browder, Rakotch, Edelstein, Matkowski etc.), called there weak contractions, we refer to [15]. We note that (b,φ)-μ-enriched contractions considered in the present paper are independent of the classes of mappings treated in [15].

Definition 2.3. *Let $(E, \|\cdot\|)$ be a Banach space ante $T\colon E \to E$ be a self mapping. We say that T is an enriched φ-contraction if there exist a constant $b \in [0, \infty)$ and a comparison function φ such that, for any $x, y \in E$,*

$$\|b(x-y) + Tx - Ty\| \le (b+1)\varphi(\|x-y\|).$$

In short, we say that T is a (b, φ)-enriched contraction.

One of the results proved in [4] is the next theorem.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a Banach space and $T : E \to E$ an enriched (b, φ)-contraction. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \text{ for } n \ge 0,$$

and $x_0 \in E$ arbitrary, converges strongly to p.

For other related results on other types of enriched contractive mappings we refer to [3]-[9].

3. Main result

Before presenting our main result, we need the following definition.

Definition 3.4. *Let Ω be a nonempty, bounded, closed convex subset of a real Banach space $(E, \|\cdot\|)$ and $T\colon \Omega \to \Omega$.*

We say that T is a (b, φ)-μ-enriched contraction, where μ is a m.n.c., if there exist $b \in [0, \infty)$ and φ a comparison function such that

$$\mu\left(\{bx + Tx\colon x \in B\}\right) \le \varphi(\mu(B)),$$

for any nonempty, closed and convex subset B of Ω.

Remark 3.1. *Suppose that $T\colon \Omega \to \Omega$ is a (b, φ)-μ-enriched contraction and $A \subset \Omega$. By $T^b(A)$ we denote the following set*

$$T^b(A) = \left\{ \left(1 - \frac{1}{b+1}\right) x + \frac{1}{b+1} Tx\colon x \in A \right\}.$$

In what follows, we suppose that the m.n.c. μ satisfies the maximum property. The main result of the paper is the following.

Theorem 3.2. *Let $\mathcal{F} = \{T_1, T_2, \ldots, T_n\}$ be a finite family of (b_i, φ_i)-μ-enriched contractions of a nonempty bounded, closed and convex subset Ω of a Banach space $(E, \|\cdot\|)$ into itself. Then there exists a convex compact subset P of Ω such that*

$$T^b(P) = \bigcup_{i=1}^{n} T_i^{b_i}(P) \subset P.$$

Proof. As $\Omega \ne \emptyset$, we take $x_0 \in \Omega$.

Next, we consider the following family

$$\mathcal{M} = \Bigg\{ A \subset \Omega\colon A \text{ nonempty, closed, convex, } x_0 \in A \text{ and}$$
$$T^b(A) = \bigcup_{i=1}^{n} T_i^{b_i}(A) \subset A \Bigg\}.$$

It is clear that $\mathcal{M} \neq \emptyset$ because $\Omega \in \mathcal{M}$, since $\Omega \neq \emptyset$, closed, convex and $x_0 \in \Omega$.

Moreover, the convexity of Ω gives us that, for any $i \in \{1, 2, \ldots, n\}$,

$$T_i^{b_i}(\Omega) = \left\{ \left(1 - \frac{1}{b_i + 1}\right) x + \frac{1}{b_i + 1} T_i x \colon x \in \Omega \right\} \subset \Omega,$$

and, therefore, $T^b(\Omega) \subset \Omega$.

Now, we consider the following sets,

$$P = \bigcap_{A \in \mathcal{M}} A \quad \text{and} \quad Q = Conv\left(T^b(P) \cup \{x_0\}\right).$$

We claim that $P = Q$.

In fact, firstly, we note that $P \neq \emptyset$ since for any $A \in \mathcal{M}$, $x_0 \in A$ and, consequently, $x_0 \in P$.

Moreover, for any $A \in \mathcal{M}$,

$$T^b(P) = T^b\left(\bigcap_{A \in \mathcal{M}} A\right) = \bigcup_{i=1}^{n} T_i^{b_i}\left(\bigcap_{A \in \mathcal{M}} A\right) \subset \bigcup_{i=1}^{n} T_i^{b_i}(A) = T^b(A) \subset A.$$

From this, we infer that $T^b(P) \subset P$.

On the other hand, since $x_0 \in P$, P is convex and closed (because P is intersection of convex and closed sets) and $T^b(P) \subset P$, it follows

$$Q = Conv\left(T^b(P) \cup \{x_0\}\right) \subset Conv(P \cup \{x_0\}) = Conv(P) = P,$$

this is $Q \subset P$.

For the reverse inclusion, taking into account that $Q \subset P$, we have

$$T^b(Q) \subset T^b(P) \subset Conv\left(T^b(P) \cup \{x_0\}\right) = Q,$$

and, since Q is closed, convex and $x_0 \in Q$, we infer that $Q \in \mathcal{M}$.

As $P = \bigcap_{A \in \mathcal{M}} A$ and $Q \in \mathcal{M}$, we have that $P \subset Q$.

This proves our claim.

Next, we will prove that P is compact.

To do this, we argue by contradiction. Suppose that P is not compact. Then we deduce that

$$\begin{aligned}
0 < \mu(P) &= \mu(Q) \\
&= \mu\left(Conv\left(T^b(P) \cup \{x_0\}\right)\right) \\
&= \mu\left(T^b(P) \cup \{x_0\}\right) \\
&= \mu\left(T^b(P)\right) \\
&= \mu\left(\bigcup_{i=1}^{n} T_i^{b_i}(P)\right) \\
&= \max_{1\le i\le n} \mu\left(T_i^{b_i}(P)\right) \\
&= \max_{1\le i\le n} \mu\left\{\left(1-\frac{1}{b_i+1}\right)x + \frac{1}{b_i+1}T_i x \colon x \in P\right\} \\
&= \max_{1\le i\le n} \left\{\frac{1}{b_i+1}\mu\left\{(b_i+1)\left(1-\frac{1}{b_i+1}\right)x + T_i x \colon x \in P\right\}\right\} \\
&= \max_{1\le i\le n} \left\{\frac{1}{b_i+1}\mu\left\{b_i x + T_i x \colon x \in P\right\}\right\} \\
&\le \max_{1\le i\le n} \left\{\frac{1}{b_{i+1}}\varphi_i(\mu(P))\right\},
\end{aligned}$$

where we have used that μ is a m.n.c. satisfying the maximum property, that $\mu(\{x_0\}) = 0$ and that T_i are (b_i, φ_i)-μ-enriched contractions.

Now, by using that φ_i are comparison functions and that $\mu(P) > 0$, we have that $\varphi_i(\mu(P)) < \mu(P)$ and from the last inequality, we deduce

$$0 < \mu(P) \le \max_{1\le i\le n} \left\{\frac{1}{b_{i+1}}\varphi_i(\mu(P))\right\} \le \max_{1\le i\le n} \left\{\frac{1}{b_{i+1}}\mu(P)\right\} < \mu(P).$$

This gives us a contradiction. Therefore, P is compact.

The convexity of P is clear since P is a intersection of convex subsets.

□

Before giving our result about the existence of a fractal, we need some definitions and results which appear in [14].

Suppose that $\mathcal{F}$ is a family of maps of a nonempty set X into itself and let $T \colon \mathcal{P} \longrightarrow \mathcal{P}$ be the invariance operator, that is, for any $H \in \mathcal{P}(X)$,

$$T(H) = \bigcup_{f \in \mathcal{F}} f(H).$$

Now, let $H_1 \subset X$. By the Kantorovich iteration and its limit we mean

$$H_{n+1} = T(H_n)$$

and

$$H = \bigcup_{n \in \mathbb{N}} H_n,$$

respectively.

The following two lemmas, which appear in the above mentioned paper, play a very important role in proving our next result.

Lemma 3.1. *Suppose that $\mathcal{F}$ is a nonempty family of self-maps on a nonempty set X and let $H \subset X$ be a $\mathcal{F}$-subinvariant set (that is, $H \subset T(H) = \bigcup_{f\in\mathcal{F}} f(H)$).*

Then, the limit of H under Kantorovich iteration is an $\mathcal{F}$-invariant set (which means that $T(H) = H$ for $H = \bigcup_{n\in\mathbb{N}} H_n$).

Lemma 3.2. *Let (X, d) be a metric space and $\mathcal{F}$ a finite family of continuous maps of X into itself and $H \subset X$ is a relatively compact $\mathcal{F}$-invariant set.*

Then $\overline{H}$ is also $\mathcal{F}$-invariant (here $\overline{H}$ denotes the topological closure of X).

Next, we prove the existence of a fractal.

In our case the family $\mathcal{F}$ is given by

$$\mathcal{F} = \left\{T_i^{b_i} : \Omega \to \Omega \,, i = 1, 2, \ldots, n\right\}$$

where $T_i^{b_i}(x) = \left\{\left(1 - \frac{1}{b_i+1}\right) x + \frac{1}{b_i+1} T_i x\right\}$.

Theorem 3.3. *Let $\mathcal{F} = \{T_1, T_2, \ldots, T_n\}$ be a family of (b_i, φ_i)-μ-enriched contractions continuous on Ω. Under assumptions of Theorem 3.2, there exists a relatively compact $\mathcal{F}$-invariant subset of Ω.*

Proof. Let P be the convex compact subset of Ω such that

$$T^b(P) = \bigcup_{i=1}^{n} T_i^{b_i}(P) \subset P,$$

whose existence is guaranteed by Theorem 3.2. Since $T^b(P) \subset P$, particularly, $T_1^{b_1} : P \to P$. As P is convex and compact and $T_1^{b_1}$ is continuous, by Schauder's fixed point theorem, it follows that there exists a fixed point $x_1 \in P$, that is, $T_1^{b_1}(x_1) = x_1$.

Put $H = \{x_1\} \subset P$. It is clear that

$$H \subset T^b(H) = \bigcup_{i=1}^{n} T_i^{b_i}(H) \subset \bigcup_{i=1}^{n} T_i^{b_i}(P) \subset P\,,$$

and, therefore, H is a $\mathcal{F}$-subinvariant set.

Now, we consider the Kantorovich iteration of H, that is, $H_1 = H = \{x_1\}$ and $H_{n+1} = T^b(H_n)$. Taking into account Lemma 3.1, the limit of H, that is, $G = \bigcup_{n\in\mathbb{N}} H_n$, is a $\mathcal{F}$-invariant set.

Moreover, as $H_1 = \{x_1\} \subset P$. we have

$$H_2 = T^b(H_1) \subset T^b(P) \subset P\,,$$

and, using induction, we infer that

$$H_n \subset P \quad \text{for any } n \in \mathbb{N}\,.$$

Therefore, since P is compact and $\overline{G} = \overline{\bigcup_{n\in\mathbb{N}} H_n}$ is a closed contained in a compact, we deduce that $\overline{G}$ is compact.

This gives us the desired result. □

4. Conclusions and further developments

1. In this paper we studied iterated function systems which are built by using a finite family of condensing (b, φ)-μ-enriched contractions with respect to a measure of noncompactness in Banach spaces and proved the existence of a fractal associated to such an iterated function system.

2. To our best knowledge, this is the first approach to obtaining attractors of iterated function systems consisting of enriched φ-contractive mappings in the presence of a measure of noncompactness.

3. Related results based on the use of measures of noncompactness were established in [15], Section 7.4, but in the case of multivalued case and for φ-contractions of Browder type.

4. We note that, in view of Example 3 in [4], (b, φ)-μ-enriched contractions involved in the iterated function systems considered in the present paper are independent of the classes of mappings treated in [15].

5. Existence of a fractal of iterated function systems containing condensing functions w.r.t. the degree of nondensifiability, which is different of the concept of measure of noncompactness, was established very recently in [16].

6. Further developments on the study of attractors of iterated function systems could be done by considering other classes of enriched contractions, namely enriched Banach contractions [6], enriched Kannan mappings [7], enriched Chatterjea mappings [9], , [1], [2], [10], [11], [13], [17], [19], [20], and many others that could be found in the recent survey [12]. Similar attempts were already done in [18] for Ćirić-Reich-Rus contractions, but without using the measure of noncompactness.

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