

Which spheres admit a topological group structure?

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Abstract

It is well known since the 1940's that $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$ are the only spheres admitting a topological group structure. In this short note we provide an easy and direct proof (without using Lie group theory nor dimension theory) of the fact that $\mathbb{S}^{2n}$ does not admit such a structure for any $n > 0$. The proof is based upon the notion of group actions on a topological space; loosely speaking what makes possible this argument is that there are more self-homeomorphisms of a topological group than of an even sphere.

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1 Introduction

The proof of the fact that the only Euclidean spheres that can be made into topological groups are $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$ is given in [7, p. 144], it refers to a result from Cartan [2, p. 179] and uses Lie group theory and dimension theory for the proof.

Other proofs found in the literature use the fact that the only H-spaces among $\mathbb{S}^n$, $n \geq 0$, are those given by $n = 0, 1, 3, 7$ (see [1, Theorem 1.1.1]). Since a topological group must be an H-space, the only ones to test are for those values of n . Moreover, in [4, Theorem 1] it can be found a direct proof of the fact that $\mathbb{S}^7$ is not a Lie group and therefore not a topological group, because “*all locally euclidean groups are isomorphic to Lie groups*” [5, p. 213].

None of these proofs are elementary. The goal of this note is to present a more simple proof of the fact that no even dimensional sphere admits a topological group structure.

In order to make the paper self-contained we define the tools required for its development.

2 Preliminaries

In this section we provide the definition, some examples and some properties of topological groups. For a deeper comprehension of the subject read [6].

Definition 1. *Let G be a group (with multiplicative notation) and a topological space at the same time. Then G is said to be a topological group if the group operation and the inverse operation are continuous mappings:*

$$\begin{array}{ll} G \times G & \rightarrow G \\ (f, g) & \mapsto f \cdot g \end{array} \qquad \begin{array}{ll} \text{inv} : G & \rightarrow G \\ g & \mapsto g^{-1} \end{array}.$$

Example 1. *We now give the topological group structure for $\mathbb{S}^n$, $n \in \{0, 1, 3\}$.*

($n = 0$) As a topological space it is the discrete space $\{-1, 1\}$ and if it is given a group structure we get $\mathbb{Z}_2$, so $\mathbb{S}^0 \approx \mathbb{Z}_2$.

($n = 1$) Let us consider $\mathbb{S}^1$ as the unitary complex numbers, endowed with the complex product and with the topology induced from the usual one of $\mathbb{C}$. Clearly, $\mathbb{S}^1$ is a topological group and it is usually denoted by $\mathbb{T}$.

($n = 3$) We manage to see $\mathbb{S}^3$ in $\mathbb{C} \times \mathbb{C}$ by means of the following product:

Let us represent each element (z, w) of $\mathbb{C} \times \mathbb{C}$ by a matrix of the form

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

and define the product of two elements as the matrix multiplication.

This representation permits us to write the Euclidean norm of such a (z, w) as

$$||(z, w)|| = \sqrt[2]{\det(z, w)}.$$

When $\mathbb{C} \times \mathbb{C}$ is seen in this fashion it is called the quaternionic space and denoted by $\mathbb{H}$. Now it is clear that $\mathbb{S}^3$ is the set of unitary quaternions and that this product operation makes it a topological group since, as a complex function, it is continuous and the inversion map (which is the matrix inversion): $(z, w)^{-1} = (\overline{z}, -w)$ is also a continuous complex function in $\mathbb{S}^3$.

Proposition 1. *Let G be a topological group. Then for each $a \in G$ the left and right translations, given by a , are homeomorphisms. Furthermore, the inversion map is also an homeomorphism.*

Proof. Let $a \in G$ and $L_a : G \rightarrow G$ (with $L_a(g) = a \cdot g$) be the left translation given by a . Then L_a is a composition of continuous functions and its inverse function is $L_{a^{-1}}$. Hence L_a is a homeomorphism (it follows analogously for the right translation $R_a : G \rightarrow G$, $R_a(g) = g \cdot a$).

To show that the inversion map is a homeomorphism, just notice that it is its own inverse (it is continuous by hypothesis). $\square$

Next, we define homogeneity in the context of topological spaces and we show that every topological group is homogeneous.

Definition 2. A topological space X is said to be homogeneous if for any $x, y \in X$, there exists a homeomorphism $f : X \rightarrow X$ such that $f(x) = y$.

Observation 1. Every topological group is homogeneous. This follows by proposition 1, since given $a, b \in G$ we can define the homeomorphism $f = L_{a^{-1}} \circ R_b$, which verifies $f(a) = b$.

But not every homogeneous topological space can be made into a topological group, the Sorgenfrey line is a counterexample.

Example 2. The Sorgenfrey line is a homogeneous topological space but not a topological group.

It is homogeneous: given a, b in the Sorgenfrey line, the map $f(x) = x + b - a$ sends a to b and is an homeomorphism since it is continuous, and its inverse, $f^{-1}(y) = y - b + a$, is also continuous.

It is not a topological group with the addition of real numbers because the inverse operation fails to be continuous.

Furthermore, it does not admit a topological group structure at all: every first countable topological group is metrizable but the Sorgenfrey line is first countable and not metrizable. Therefore, there is no group operation in $\mathbb{R}$ that could turn the Sorgenfrey line into a topological group.

The two dimensional sphere is another example of an homogeneous topological space (think of rotations) but not a topological group, as we shall see.

3 $\mathbb{S}^{2n}$ is not a topological group.

First of all, we recall some notions about group actions on topological spaces and the concept of degree of a continuous map from a topological space to itself. Let us also fix some notation: X is a topological space, G a group and $\text{Homeo}(X)$ the group of self-homeomorphisms of X .

Definition 3. We say that G acts on X if there exists an homomorphism $\Theta : G \rightarrow \text{Homeo}(X)$. For each $g \in G$ we write Θ_g instead of $\Theta(g)$.

If, in addition, $\Theta_g : X \rightarrow X$ has no fixed points for any $g \in G$ but for the identity (i.e., $\Theta_g(x) \neq x$, $\forall x \in X$ and $\forall g \in G \setminus \{e\}$), then the action is said to be free (or that G acts freely on X).

Observation 2. Every topological group acts freely on itself. Indeed, since translations are homeomorphisms we can define the map $L : G \rightarrow \text{Homeo}(G)$ such that $g \mapsto L_g$, which is an homomorphism. Moreover, L_g is a translation, so if $g \neq e$ then L_g has no fixed points.

The following definition, properties and proposition can be found in [3, pp. 134-135]. They are the tools we use to reach our goal. First, let us fix $n \in \mathbb{N} \setminus \{0\}$.

Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a continuous map. Now consider the induced endomorphism of the n^{th} -homology group of the n -sphere, $f_* : H_n(\mathbb{S}^n) \rightarrow H_n(\mathbb{S}^n)$. Since $H_n(\mathbb{S}^n) \simeq \mathbb{Z}$, there exists a unique integer d such that $f_*(\alpha) = d\alpha$ for every $\alpha \in H_n(\mathbb{S}^n)$.

Definition 4. We define the degree of f as $\deg(f) = d$.

Some properties of the degree that we shall use further on are:

- (i) $\deg(\text{id}) = 1$ where $\text{id} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is the identity map;
- (ii) if $f, g : \mathbb{S}^n \rightarrow \mathbb{S}^n$ are continuous maps, then $\deg(f \circ g) = \deg(f) \cdot \deg(g)$;
- (iii) let f and g be as above. Then, they are homotopic if and only if $\deg(f) = \deg(g)$;
- (iv) if f is the reflection of $\mathbb{S}^n$ that interchanges the hemispheres determined by a fixed equator, then $\deg(f) = -1$;
- (v) let id^- be the antipodal map, then $\deg(\text{id}^-) = (-1)^{n+1}$;
- (vi) if $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is a continuous map without fixed points, then $\deg(f) = (-1)^{n+1}$.

Hence from (iii), (v) and (vi) we get that every continuous map from the n -sphere in itself without fixed points is homotopic to the antipodal map. The converse statement is false, for if n is odd then (i), (iii) and (v) imply that the antipodal map is homotopic to the identity; and if n is even then (iv), (v) and (vi) imply that a reflection as in (iv) is homotopic to the antipodal map.

As a direct consequence of the definition of degree and its properties we get the next result [3, Proposition 2.29].

Proposition 2. $\mathbb{Z}_2$ is the only nontrivial group that can act freely on $\mathbb{S}^n$ if n is even.

Proof. Let n be even and let G act freely on $\mathbb{S}^n$. Then there exists a homomorphism $\Theta : G \rightarrow \text{Homeo}(\mathbb{S}^n)$. Hence for each $g \in G$ we can compute the degree $\deg(\Theta_g)$. Furthermore, since Θ is an homomorphism, from (i) and (ii) we have the following equalities:

$$\deg(\Theta_g) \cdot \deg(\Theta_{g^{-1}}) = \deg(\Theta_g \circ \Theta_{g^{-1}}) = \deg(\Theta_e) = \deg(\text{id}) = 1.$$

So $\deg(\Theta_g) = \deg(\Theta_{g^{-1}}) = \pm 1$, because the degree must be an integer.

We can now construct a degree function $d : G \rightarrow \mathbb{Z}_2 = \{\pm 1\}$ as $d(g) = \deg(\Theta_g)$, which is an isomorphism. Indeed, d is an homomorphism by (ii) and because G acts freely on S^n we know that Θ_g has no fixed points for any $g \neq e$. So (vi) implies $\deg(\Theta_g) = -1$ and from (i) we have $\deg(\Theta_e) = 1$. Hence d is onto and has trivial kernel. $\square$

Now it is clear that $\mathbb{S}^{2n}$ is not a topological group for any $n > 0$. Otherwise it would act freely on itself and by proposition 2 it would be isomorphic to $\mathbb{Z}_2$, which is clearly a contradiction.

This statement can be set in a naive way: “If an even-dimensional sphere were a topological group, it would be isomorphic to the zero-dimensional one”.

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