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Original articles

Convergence of the R_1^+ tetrahedra family in iterative Longest Edge Bisection

Miguel A. Padrón [a], Agustín Trujillo-Pino [b], Jose Pablo Suárez [a],*

[a] IUMA Information and Communications System, University of Las Palmas de Gran Canaria, Campus Universitario de Tafira, Las Palmas de Gran Canaria, 35017, Las Palmas, Spain

[b] Instituto Universitario de Cibernética, Empresa y Sociedad, University of Las Palmas de Gran Canaria, Campus Universitario de Tafira, Las Palmas de Gran Canaria, 35017, Las Palmas, Spain

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ABSTRACT

We study the similarity classes appearing in the iterative Longest Edge Bisection (LEB), of an improved family of nearly equilateral tetrahedra. We focus here on the R_1^+ family as a generalization of the family mentioned by Adler in Adler (1983). We characterize the finite convergence of similarity classes using the Similarity Classes Longest Edge Bisection (SCLEB) algorithm. We prove that below the bound of 37 similarity classes, a number $n \leq 37$ classes are generated where $n \in \{4, 8, 9, 13, 21, 37\}$. Using a tetrahedra sextuple representation and SCLEB, all the generated classes are clearly delimited, thereby improving the results by Adler and others.

1. Introduction

The Longest Edge Bisection (LEB) of tetrahedra is the simplest way of subdivision scheme that have been used extensively in decades for many applications in Numerical Methods. It first selects the longest edge of the tetrahedron, and then its midpoint is connected to the opposite edge, generating 2 new tetrahedra.

Some attractive properties of the LEB include: easy to implement in a programming language, cheap computational cost, and the fact that interior angles do not degenerate, [1,2]. Empirical studies suggest that in 3D we obtain quality meshes when LEB is applied iteratively to any initial mesh, [3,4]. This is a desirable property in finite element applications, [5–7].

However, an important property regarding the LEB remains open. This is the question of whether the iterative application of the LEB produces a finite or infinite number of similarity classes. Briefly, any pair of tetrahedra are in the same similarity class if their shapes are equal to each other after any affine transformation. A finite and low number of similarity classes is necessary for numerical stability in the Finite Element Method, [8]. The main benefit is that the element matrix can be computed only once for each similarity class of tetrahedra, and then the construction of the stiffness matrix can be performed much faster. It should be noted, that many subdivision algorithms for tetrahedra have been designed to shorten the number of new generated similarity classes, and to preserve tetrahedra quality [5,9–11]. However, those algorithms are not purely based on the LEB and, in some cases, the computer implementation is a bit tricky.

A previous study on similarity classes dates back to Adler in 1983 [12]. Adler conjectured that there exist certain classes of tetrahedra that produce up to 37 similarity classes when the LEB is iterated. Although he did not provide a proof of his conjecture, he noted -without proof- that the tetrahedra should have the following properties: (1) all edge lengths are within 5% of each other,

* Corresponding author.
E-mail addresses: miguel.padron@ulpgc.es (M.A. Padrón), agustin.trujillo@ulpgc.es (A. Trujillo-Pino), josepablo.suarez@ulpgc.es (J.P. Suárez).

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and (2) the longest edge and the second longest edge are opposite each other. Experimental results and the LEB algorithm presented in [4] indicate that, in practice, Adler's statement appears to be valid. Consequently, the iterative LEB for the "nearly equilateral tetrahedra" produces a finite number n, with $n \leq 37$ similarity classes. However, no systematic treatment of the study of similarity classes can be found so far. Lastly, in [13], Adler's conjecture is proved, and the bound on the conditions between edge lengths for the convergence of LEB applied to near equilateral tetrahedra, is improved from 5% to 22.47%. This new family, called the R_1^+ family, is a generalization of the family mentioned by Adler of near equilateral tetrahedra. In [13] the Similarity Classes Longest Edge Bisection (SCLEB) algorithm is also introduced. This algorithm computes the similarity classes that arise during the iterative LEB process.

The aim of this paper is to contribute with the systematic study of similarity classes in the LEB. We focus here on the R_1^+ family as a generalization of the family mentioned by Adler, and characterize the convergence in 37 or less similarity classes in the LEB. We prove that, for any tetrahedron belonging to R_1^+, there are n classes, $n \leq 37$, where $n \in \{4, 8, 9, 13, 21, 37\}$. We delimit the subfamilies of such converging tetrahedra with $n \leq 37$ using the tetrahedra sextuple representation.

2. Preliminaries

Let us represent a *tetrahedron* $T = (A, B, C, D, E, F)$ as a sextuple with the square lengths of its 6 edges in a certain order as explained in [14]. In this way, the position and orientation of a tetrahedron are disregarded, and only its geometric shape is represented.

For brevity, we will use *edges* instead of square edges in the remainder of the paper.

Note that, the edges ABC form a face, and the opposite edges to A, B, C are F, D, E respectively. There are 24 different sextuples to represent the same tetrahedron. We call the *normalized sextuple representation* of a tetrahedron, as defined in [14], to the sextuple that places the longest edge as the first value, and its neighboring edge with the longest length as the second value. In the case of repeated edges, we choose the sextuple that places the highest possible values in the first positions of the sextuple. In this way, the sextuple of a similarity class always satisfies that $A \geq B, C, D, E, F$ and $B \geq C, D, E$.

A *similarity class* is represented by $k(A, B, C, D, E, F)$, assuming that (A, B, C, D, E, F) is a normalized sextuple representation, for $k \in \mathbb{R}^+$ being a scale factor. We can omit the factor k and use brackets to represent a similarity class, $[A, B, C, D, E, F]$, and parenthesis to represent a single tetrahedron. In this manner, $[A, B, C, D, E, F] = k(A, B, C, D, E, F)$, $\forall k \in \mathbb{R}^+$. For brevity in the reminder of the paper, we will use *classes* and similarity classes as equivalent terms. Given 2 tetrahedra, T_1 and T_2, they belong to the same class if $T_1 = kT_2$, being $k \in \mathbb{R}^+$. To denote that a class T belongs to a family, say R_1^+, we use $T \in R_1^+$.

Besides, a *sextuple expression* will be a sextuple given in the form of linear combination of the edges; e.g., the sextuple $[4F, 4B, 4B, 4B - A, 4B - A, A]$ becomes $[8, 4, 4, 2, 2, 2]$ when $A = F = 2$ and $B = 1$.

Definition 1. The Similarity Classes Longest Edge Bisection (SCLEB) of class $T = [A, B, C, D, E, F]$ is defined following these two-steps [13]:

1. Subdivision of T to get:

$$T_1 = [A, 4B, 2B + 2C - A, 2E + 2D - A, 4E, 4F] \tag{1}$$

$$T_2 = [A, 2B + 2C - A, 4C, 4D, 2E + 2D - A, 4F] \tag{2}$$

2. Normalization of T_1 and T_2

Given an input class $[A, B, C, D, E, F]$, a list of output classes are obtained through the application of the iterative SCLEB to all of the descendants of the input class. If no new classes are produced after a SCLEB iteration, we say that the input class converges in a finite number n of classes.

Definition 2. The R_1^+ family, as a generalization of the family mentioned by Adler in [12], is the set of classes [A,B,C,D,E,F], such that $A \geq F \geq B \geq C, D, E$, and the ratio between the square of the longest edge A, and the square of the shortest edge, is less than or equal to $\frac{3}{2}$.

In [13] it was proved that all the tetrahedra belonging to family R_1^+ converge in 37 or fewer classes. Focusing on the class $T = [A, B, C, D, E, F] \in R_1^+$ family, we will refer to edge A (the longest edge) and edge F (opposite to A), as *primary edges*, and we will refer to the remaining edges B, C, D and E as *secondary edges*. This distinction will play a very significant role in the study of the conditions for convergence of SCLEB in less than 37 similarity classes, which will be examined in this work.

3. The convergence of R_1^+ in less than 37 classes

We are interested in studying the cases in which SCLEB applied to an arbitrary class $T_1 = [A, B, C, D, E, F] \in R_1^+$ produces exactly 37 new classes, and the cases in which it produces a smaller number. In [13] the complete tree of the 37 similarity classes $\{T_1, T_2, \ldots, T_{37}\}$ generated in all the iterations of SCLEB is provided. Every class is represented as a linear expression of the original values A, B, C, D, E, F of the initial class.

Fig. 1 shows the first iteration of the SCLEB applied to class T_1. We find that the subdivision by the longest edge generates 2 new similarity classes. On the one hand, we obtain $T_2 = [4F, 4B, 4E, 4H, 4G, A]$, where $4H = 2D + 2E - A$ and $4G = 2B + 2C - A$

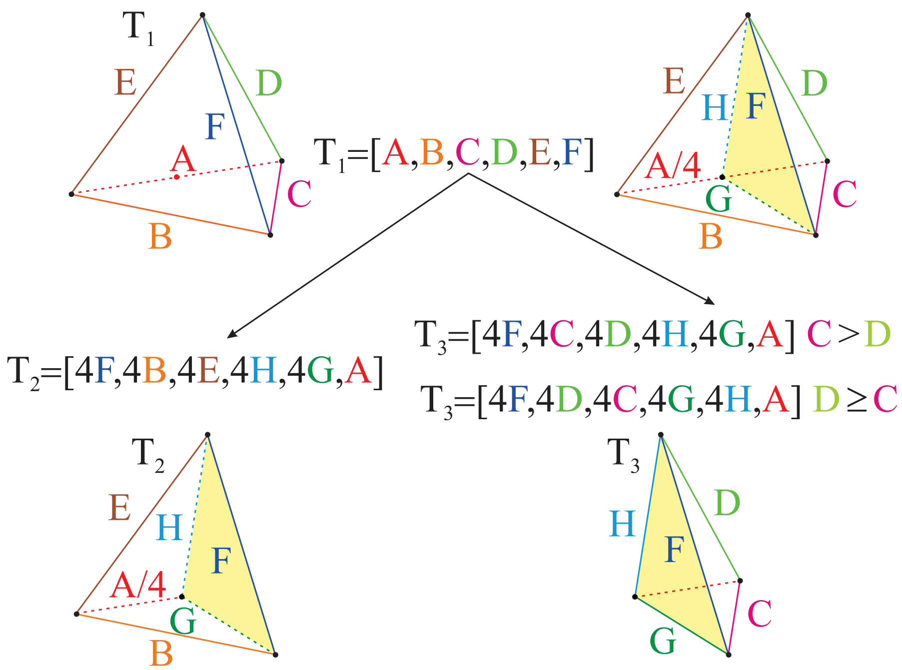

Fig. 1. First step of SCLEB of $T_1 = [A, B, C, D, E, F]$.

are the shortest medians of faces ADE and ABC, respectively. On the other hand, we obtain T_3, that has 2 different expressions depending on whether $C > D$ or not, with the expressions given by $[4F, 4C, 4D, 4H, 4G, A]$ or $[4F, 4D, 4C, 4G, 4H, A]$.

Likewise, we can continue analyzing the expressions for all the classes obtained at each step of SCLEB, up to the eighth iteration, where all generated classes have already appeared previously, and then the process stops.

In some cases, the expressions of 2 different classes, T_i and T_j, may coincide numerically, e.g., the regular tetrahedron class $R = [A, B, C, D, E, F] = [1, 1, 1, 1, 1, 1]$ produces the classes $T_2 = [4F, 4B, 4E, 4H, 4G, A]$ and $T_3 = [4F, 4D, 4C, 4G, 4H, A]$, see Fig. 1. Substituting the edge values of R into these classes yields $T_2 = T_3 = [4, 4, 4, 3, 3, 1]$. Numerically, they are exactly the same class, although the expressions hold different forms. In this case, only a single new class is generated in the first iteration, instead of 2. Therefore, in the second iteration, only 1 class will need to be subdivided instead of 2. This will reduce the number of new classes generated in the subdivision tree, producing less than 37 new classes. In fact, for the case of the regular tetrahedron, we already know that it produces only 8 classes, [3,13]. This leads us to the following Lemma.

Lemma 1. *The SCLEB applied to a class $T_1 \in R_1^+$ converges in less than 37 classes if and only if there exist indices $i, j \leq 37$, with $i \neq j$, such that $T_i = T_j$.*

Proof. Note that, the only condition for a class $T_1 \in R_1^+$ to produce less than 37 classes, is that at least 2 of the 37 sextuple expressions produced in the SCLEB coincide numerically. □

3.1. Study of class coincidences in the SCLEB

Firstly, we study the sextuple expression of classes T_2 and T_3, to see what conditions must be held for both to coincide numerically. Class T_3 is represented by 2 sextuple expressions, depending on the values of C and D, whereas T_2 has a single expression, see Fig. 1.

Let us consider $T_2 = [4F, 4B, 4E, 4H, 4G, A]$ and T_3 in the case $C > D$, so $T_3 = [4F, 4C, 4D, 4H, 4G, A]$. The first edge for both classes is $4F$, thus it will always coincide. For the second and third edges to be equal, it must be that $B = C$ and $D = E$. The edges in positions 4, 5, and 6 are the same for both sextuples.

Let us see the case $C < D$, then $T_3 = [4F, 4D, 4C, 4G, 4H, A]$. Again, comparing each edge of sextuple T_2 and T_3, we observe that for both classes to coincide numerically, it must be that $B = D$, $C = E$, and $H = G$. It can be easily deduced from the expressions H and G that this last condition follows from the first two. Finally, in the case where $C = D$, it follows that $H = G$, which in turn forces $B = E$. Therefore, we conclude that $T_2 = T_3$ if and only if at least 2 out of the 4 secondary edges are equal in pairs.

We can perform a similar analysis for the second iteration, see Fig. 2. Although the cases are considerably more numerous than in the first iteration, let us study, for example, the comparison between the sextuple expressions of T_4 and T_5, first assuming the case $J > G$. Recall that $4J = 2B + 2E - F$ and $4K = 2C + 2D - F$ are the shortest medians of faces BEF and CDF respectively. In this case, we conclude that both classes coincide numerically if $B = E$ and $G = H$. The latter condition is equivalent to $C = D$, and

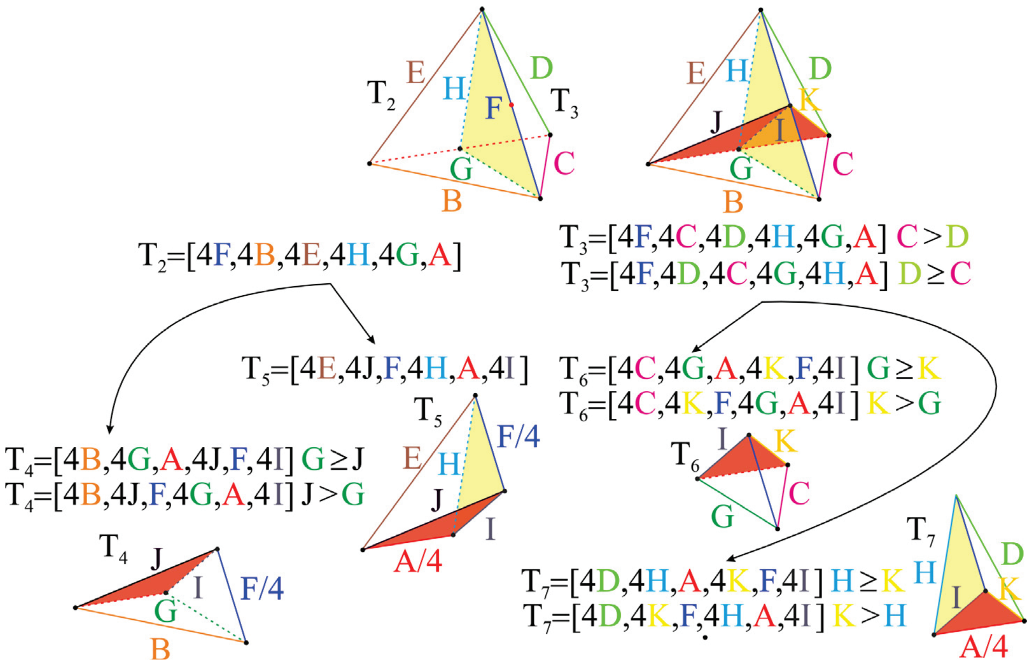

Fig. 2. Second step of the SCLEB. Subdivision of T_2 and T_3.

therefore, $T_4 = T_5$ if $B = E$ and $C = D$. In the opposite case, $J \leq G$, we would have $B = E$, $G = J$, $A = F$ and $J = H$, which can be simplified to $A = F$ and $B = C = E$.

Note that, to fully study the second step in the SCLEB, it is needed to analyze the remaining 5 cases $T_4 = T_6$, $T_4 = T_7$, $T_5 = T_6$, $T_5 = T_7$ and $T_6 = T_7$, to see all possible numerical coincidences. However, we should not limit ourselves to comparisons within the same iteration, since in subsequent iterations, the new similarity classes produced could numerically coincide with classes that appeared in previous iterations. Thus, we should compare all possible pairs within the 37 similarity classes produced by the SCLEB of a class in R_1^+. This gives us a total of $\sum_{k=1}^{n-1}(n-k) = 666$ cases with $n = 37$.

3.2. Reducing the number of cases to study

To reduce the total number of cases to compare, we take into account the fact that not all 37 generated classes by the SCLEB can be similar to one another. They can all be grouped into 8 distinct families, making it unnecessary to perform comparisons between classes from different families. Fig. 3 shows the subdivision graph produced in SCLEB from the regular tetrahedron class, where the different families appearing in each iteration can be seen. This same subdivision sequence occurs with any class in R_1^+, as demonstrated in [13]. The arrows in Fig. 3 have been colored to indicate the subdivision process: Eq. (1), labeled *child #1* is shown in red, while Eq. (2) labeled *child #2* is shown in blue.

In [13], we proved that all classes holding the conditions to belong to the R_1^+ family follow the same subdivision graph. Thus, let $T_1 = [A, B, C, D, E, F] \in R_1^+$ with all its edges distinct. The first iteration generates 2 very similar children $\in R_2^+$ but different, called T_2 (child #1 in red) and T_3 (child #2 in blue). In turn, each of them produces 2 new children $T_4, T_5, T_6, T_7 \in R_3^+$, giving a total of 4 new classes in the second iteration. Likewise, 8 different new classes emerge in the third iteration $T_8, T_9, \dots, T_{15} \in R_4^+$.

The fourth iteration does not produce classes similar to each other. On the one hand, using the child #1 formula, we obtain the classes $T_{17}, T_{19} \in R_2^+$. On the other hand, using the child #2 formula, we obtain the classes $T_{16}, T_{18}, T_{20}, T_{21} \in R_5^+$. It should be noted that there are not 16 new classes in this iteration, because the repeated ones are discarded. From this point, the subdivision process continues until the eighth iteration, so that all the generated classes have already appeared. In this way, it is possible to identify the family to which each of the 37 classes belongs.

From this result, it is not necessary to compare all 37 classes with each other, but only those belonging to the same family need to be compared. Table 1 shows the 37 classes, indicating which family they belong to and in which iteration they have been produced. In this way, for a family with m classes, the number of comparisons required is $\sum_{k=1}^{m-1} k = \frac{(m-1)m}{2}$. The number of cases is given by the sum of the entire right column, thus, a total number of 86 cases must be studied.

3.3. Edge conditions for convergence in less than 37 classes

Proceeding to study these 86 cases in the same manner as in Section 3.1, where the comparison between classes T_2 and T_3 was performed, we obtained a final set of 9 edge conditions, grouped into three categories, such that at least 2 conditions from the same category must always hold. Fig. 4 shows the three categories enclosed in grey boxes, each containing three conditions. The value I represents the interior segment connecting the midpoints of the 2 primary edges, A and F, where $4I = -A + B + C + D + E - F$.

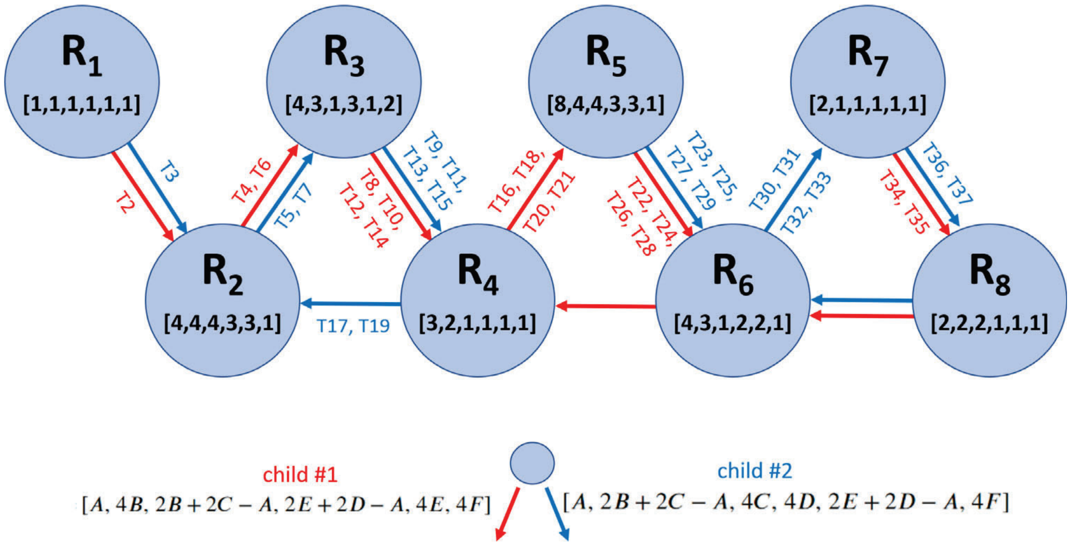

Fig. 3. The generation graph of the families emerging from the SCLEB of $R_1 = [1, 1, 1, 1, 1, 1]$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Similarity classes organized into families to be compared.

Family	Iter.	Classes	Number of classes (m)	Cases to compare
R_1^+	0	T_1	1	0
R_2^+	1	T_2, T_3	4	6
	4	T_{17}, T_{19}		
R_3^+	2	T_4, T_5, T_6, T_7	4	6
	5	No new classes generated		
R_4^+	3	$T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}$	8	28
R_5^+	4	$T_{16}, T_{18}, T_{20}, T_{21}$	4	6
R_6^+	5	$T_{22}, T_{23}, T_{24}, T_{25}, T_{26}, T_{27}, T_{28}, T_{29}$	8	28
R_7^+	6	$T_{30}, T_{31}, T_{32}, T_{33}$	4	6
R_8^+	7	$T_{34}, T_{35}, T_{36}, T_{37}$	4	6
			37	86

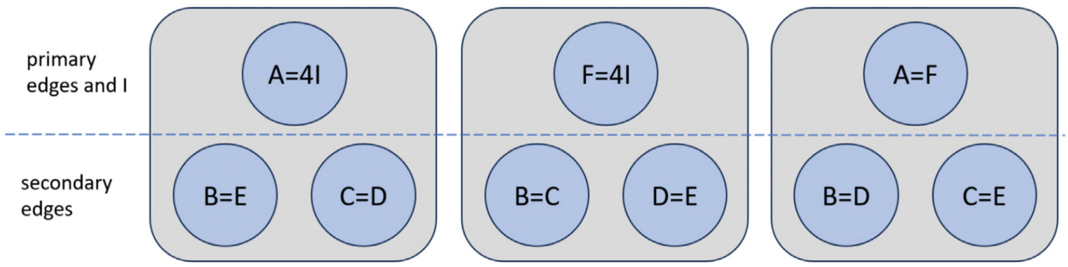

Fig. 4. The 9 edge conditions for convergence in less than 37 classes are grouped into three categories.

This result is summarized in the following Lemma.

Lemma 2. *Let $T = [A, B, C, D, E, F]$ be a class $\in R_1^+$. The SCLEB of T produces less than 37 classes as long as 1 of the following conditions holds:*

1. *At least 2 of the following three equalities: $A = 4I$, $B = E$ and $C = D$.*
2. *At least 2 of the following three equalities: $F = 4I$, $B = C$ and $D = E$.*
3. *At least 2 of the following three equalities: $A = F$, $B = D$ and $C = E$.*

Proof. We aim to determine all the necessary conditions for a class $\in R_1^+$ to converge in less than 37 classes. From Lemma 1, we know that this is equivalent to identifying the conditions under which some of the 37 classes coincide numerically. Furthermore, from Section 3.3 we know that 86 cases need to be analyzed.

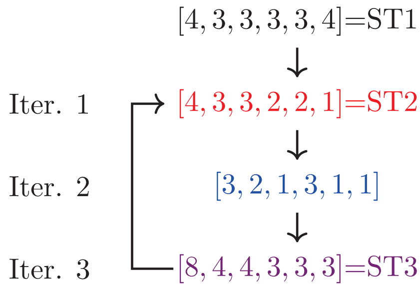

Fig. 5. Generation tree of the Sommerville tetrahedron ST1, converging in 4 classes. Nine conditions are satisfied: $A = F = 4I$ and $B = C = D = E$.

In Section 3.1, it was proved that the needed conditions for the cases $T_2 = T_3$ and $T_4 = T_5$. Using similar reasoning for the remaining 84 cases, and grouping the necessary conditions in each case, we arrive at the indicated result. □

Thus, for example, the class $[10, 8, 8, 7, 7, 10]$, which satisfies $B = C$ and $D = E$, converges in 13 classes, while the class $[9, 7, 7, 7, 6, 9]$, which satisfies $A = F$ and $B = D$, converges in 9 classes. On the other hand, the class $[13, 11, 10, 10, 10, 11]$, despite having some repeated edges, $C = D = E$ and $B = F$, converges in 37 classes because it does not satisfy Lemma 2.

4. Study of the subfamilies that converge in less than 37 classes

Lemma 2 shows the set of minimum conditions that R_1^+ classes must satisfy to converge in less than 37 classes. However, depending on how many of these conditions hold, the SCLEB will converge to a different number of classes. The more conditions are satisfied, the fewer classes the SCLEB will produce.

Theorem 1. *A class of* R_1^+ *converges in* n *classes with* $n \in \{4, 8, 9, 13, 21, 37\}$.

In the remainder of this section we present a proof of Theorem 1.

Let us define the *subfamily* $R_1^+C_n$ as the set of R_1^+ classes that converges in n classes. In the following subsections, we study the different subfamilies.

Here, we recall the expressions to clarify Figs. 5 to 15:

$$\begin{aligned} 4G &= 2B + 2C - A \\ 4H &= 2D + 2E - A \\ 4J &= 2B + 2E - F \\ 4K &= 2C + 2D - F \end{aligned} \qquad \begin{aligned} 4I &= -A + B + C + D + E - F \\ 4L &= A + B - C + D - E + F \\ 4M &= A - B + C - D + E + F \end{aligned} \tag{3}$$

4.1. Subfamily that converges in 4 classes $R_1^+C_4$

This family consists of the classes that simultaneously satisfy 9 conditions of Lemma 2 and Fig. 4. Seven out of 9 conditions require that the class to be $[A, B, B, B, B, A]$, since $A = F$ and $B = C = D = E$. The 2 remaining conditions arise from $A = 4I$ and $F = 4I$. Given that $4I = 4B - 2A$, the condition $A = F = 4I$ implies $3A = 4B$. This produces a single class that fulfills this condition, $[4, 3, 3, 3, 3, 4]$. This class is known as the Sommerville tetrahedron ST1, which is relevant in numerical mathematics [15].

Fig. 5 shows the result of the SCLEB for the class $[4, 3, 3, 3, 3, 4]$. In the first three iterations, a single new class appears in each, and in the fourth iteration, the class obtained in the first iteration reappears, thus converging in only 4 classes. The convergence is achieved in three iterations, and in this process, Sommerville's tetrahedra ST2 and ST3 [15] are also generated. Note that, when the SCLEB generates 2 equal children, only 1 is shown in the tree.

We remark that this subfamily is composed of only one class.

4.2. Subfamily that converges in 8 classes $R_1^+C_8$

In the previous subsection, we studied the case that satisfies the 9 conditions of Fig. 4. It is easy to see that no class satisfies exactly 8 conditions: if at least 2 conditions related to the primary edges hold, see top row in Fig. 4, the third one becomes mandatory. Similarly, if 5 out of the 6 conditions related to the secondary edges hold, see bottom row in Fig. 4, the sixth one also becomes mandatory.

Focusing on the case that satisfies 7 conditions, 6 of them require $B = C = D = E$ for the secondary edges, while the remaining condition comes from 1 of the 3 conditions related to the primary edges, $A = F$, $4I = A$ or $4I = F$. Then, there are four possible scenarios:

1. The primary edges are equal to each other and also to the secondary edges, so $[A, A, A, A, A, A]$ is the only possible class, which corresponds to the regular class. The convergence is achieved in 7 iterations, see Fig. 6.

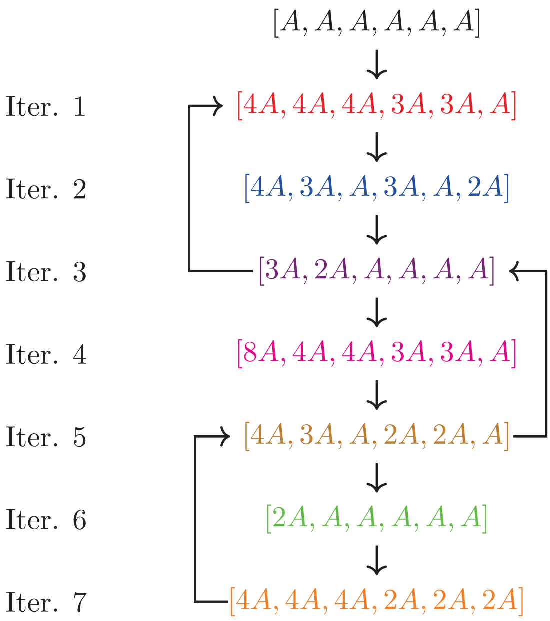

Fig. 6. Generation tree of the class $[A, A, A, A, A, A]$ converging in 8 classes. Seven conditions are satisfied: $A = F$ and $B = C = D = E$.

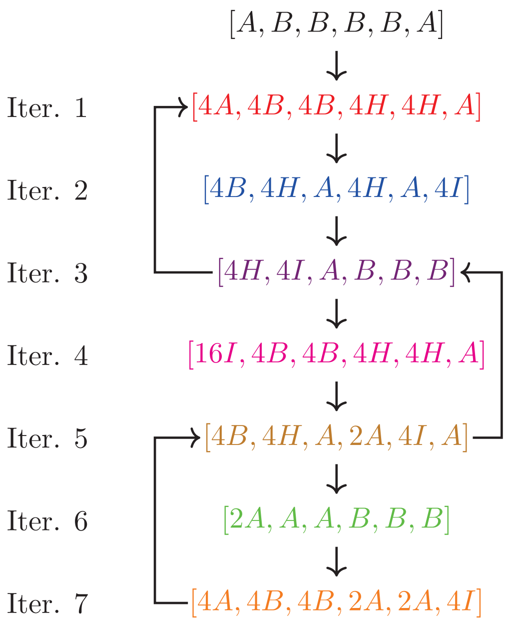

Fig. 7. Generation tree of the class $[A, B, B, B, B, A]$ converging in 8 classes. Seven conditions are satisfied: $A = F$, and $B = C = D = E$.

2. The secondary edges are equal to each other but different to the primary edges, so $[A, B, B, B, B, A]$ is the generated class. The convergence is achieved in 7 iterations, see Fig. 7.
   At this point, it is interesting to highlight how the Sommerville tetrahedron ST1, already studied in the previous section, arises from this scenario. In the fourth iteration of Fig. 7, if $4I = A$ is satisfied, then the 2 generated classes are similar to each other and to the class generated in the first iteration, thus converging to only 4 classes.
3. The primary edges differ from each other and from the secondary edges, which are equal, so $[A, B, B, B, B, F]$ is the generated class. This class satisfies 6 conditions, and the seventh condition follows from the equations $4I = A$ or $4I = F$. In both cases, the convergence is achieved in 5 iterations. Fig. 8 shows one such case when $4I = A$.
4. The last case occurs when the secondary edges are also equal to the primary edge F, so $[A, B, B, B, B, B]$ is the produced class. However, the seventh condition requires $A = 4I$. The only class that satisfies these conditions is $[3, 2, 2, 2, 2, 2]$, and it converges in 5 iterations, see Fig. 9.

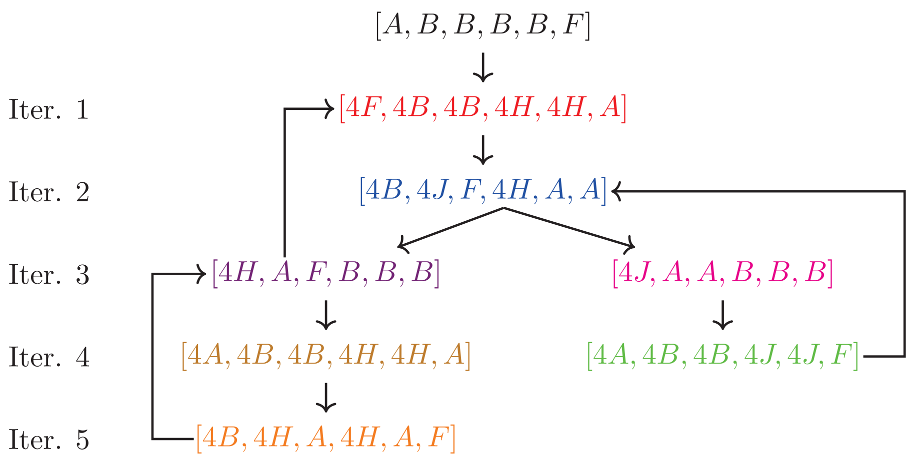

Fig. 8. Generation tree of the class $[A, B, B, B, B, F]$ with $4I = A$ converging in 8 classes. Seven conditions are satisfied: $A = 4I$ and $B = C = D = E$.

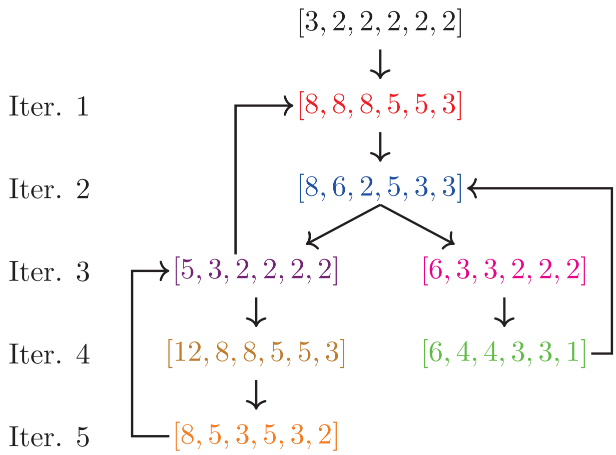

Fig. 9. Generation tree of the class $[3, 2, 2, 2, 2, 2]$ converging in 8 classes. Seven conditions are satisfied: $A = 4I$, and $B = C = D = E$.

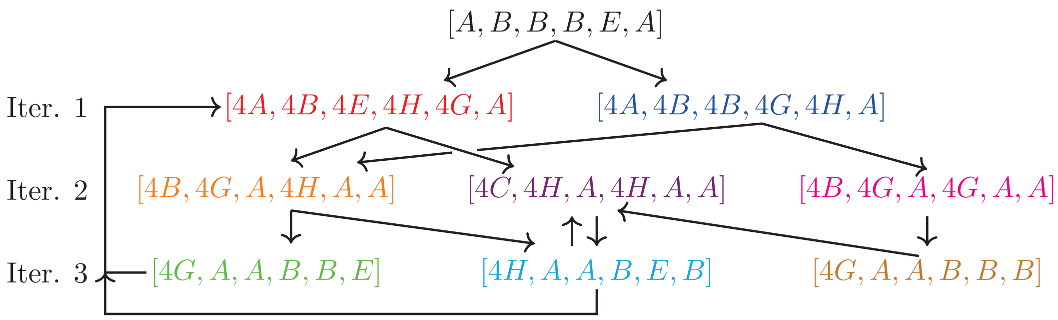

Fig. 10. Generation tree of the class $[A, B, B, B, E, A]$ converging in 9 classes. Six conditions are satisfied: $A = F = 4I$, and $B = C = D$.

4.3. Subfamily that converges in 9 classes $R_1^+ C_9$

These classes satisfy 6 out of 9 conditions. Therefore, the 3 conditions related to the primary edges are held, while the remaining 3 conditions involve secondary edges. Thus, the primary edges are equal to each other and among the secondary edges, 3 out of 4 are equal. The classes $[A, B, B, B, E, A]$ and $[A, B, C, C, C, A]$ belong to this subfamily. Fig. 10 shows the subdivision of $[A, B, B, B, E, A]$. The convergence of these classes is achieved in 3 iterations.

4.4. Subfamily that converges in 13 classes $R_1^+ C_{13}$

This subfamily is classified into three scenarios, depending on the number of satisfied conditions: 6, 5 or 3.

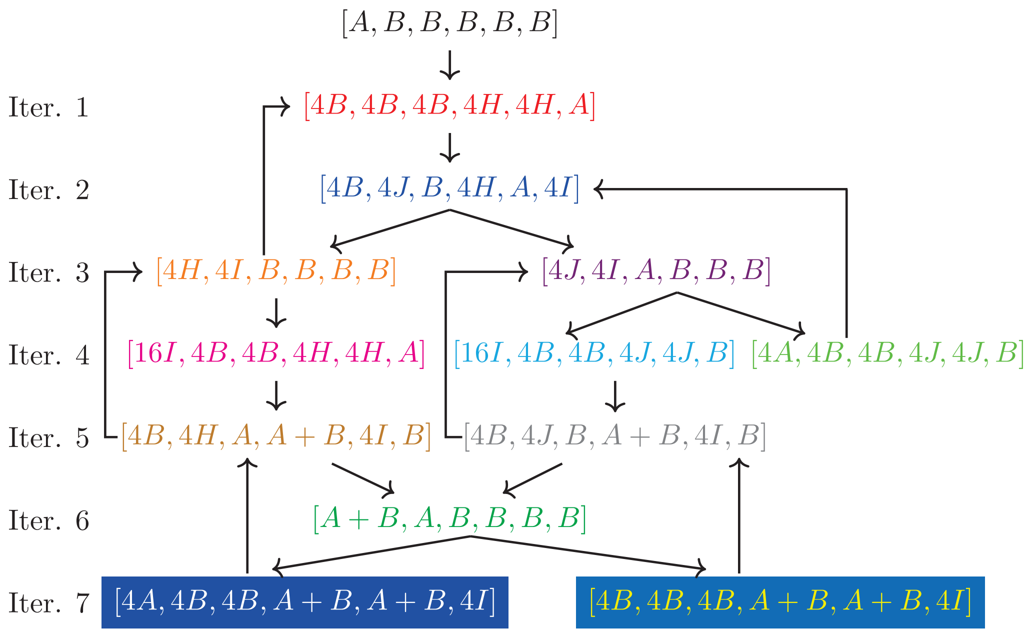

Fig. 11. Generation tree of the class $[A, B, B, B, B, B]$ converging in 13 classes. Six conditions are satisfied: $B = C = D = E$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

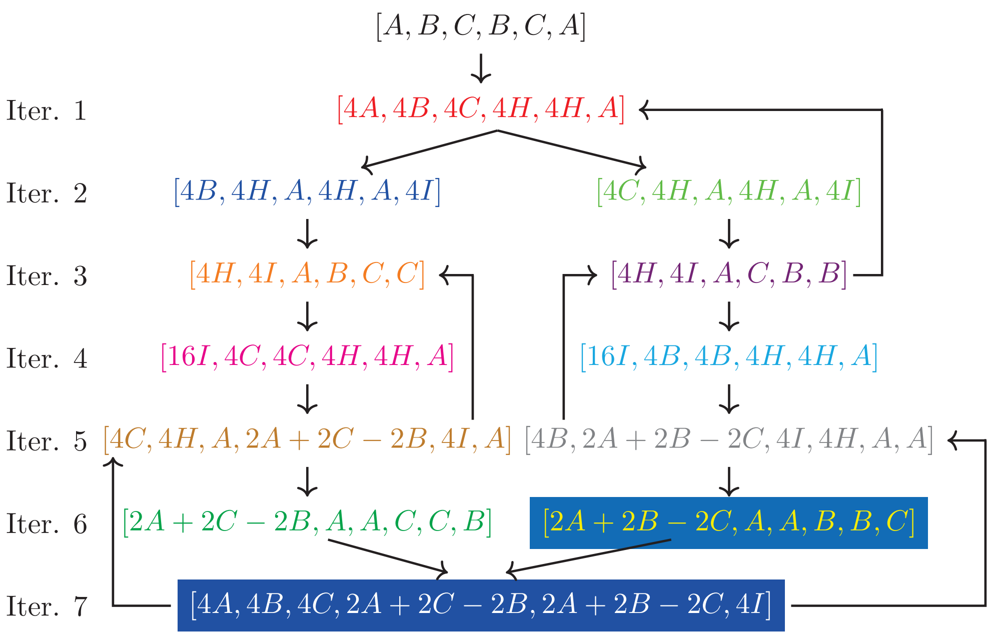

Fig. 12. Generation tree of the class $[A, B, C, B, C, A]$ converging in 13 classes. Five conditions are satisfied: $A = F = 4I$, $B = D$, and $C = E$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

1. Only 2 classes satisfy 6 conditions. In these cases, their secondary edges are either equal to each other and to the primary edge F, or equal to each other but not to F. Therefore, these classes are as follows $[A, B, B, B, B, B]$ and $[A, B, B, B, B, F]$. Here, it is important to highlight that $4I \neq A$ and F, otherwise both classes would converge in 8 classes. This means that, the 6 conditions related to the secondary edges hold. Fig. 11 shows the subdivision of $[A, B, B, B, B, B]$.
   At this stage, it is relevant to note that when $16I = 4A$ is satisfied in the fourth iteration, the green and blue light classes become the same class. This particular case converges in 8 classes, as previously studied in Section 4.2. A similar behavior occurs for the class $[A, B, B, B, B, F]$ when $4I = A$ or $4I = F$, in which case this class also converges in 8 classes.
2. Classes satisfying 5 conditions require that the 3 conditions related to the primary edges hold, while the remaining 2 conditions involve the secondary edges. Thus, these classes are $[A, B, C, B, C, A]$ and $[A, B, B, D, D, A]$. The first one converges in 7 iterations, while the second one converges in 5 iterations. The class $[A, B, C, B, C, A]$ is shown in Fig. 12.

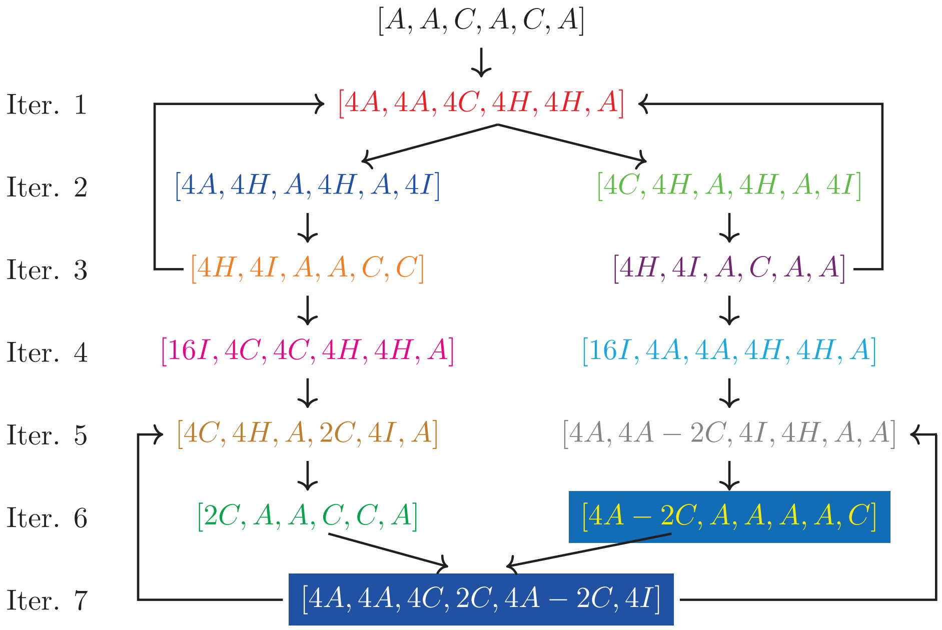

Fig. 13. Generation tree of the class $[A, A, C, A, C, A]$ converging in 13 classes. Three conditions are satisfied: $A = F$, $B = D$, and $C = E$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. For classes satisfying 3 conditions, 1 condition is related to the primary edges, while the remaining 2 conditions involve the secondary edges. However, these 3 conditions must belong to the same category, see Fig. 4. For example, the class $[A, A, C, A, C, A]$ which satisfies $A = F$, $B = D$ and $C = E$, or the class $[A, B, C, C, B, F]$ with $4I = A$, $B = E$ and $C = D$. Fig. 13 illustrates an example.

4.5. Subfamily that converges in 21 classes $R_1^+C_{21}$

This subfamily has three scenarios depending on how many conditions are satisfied: 4, 3 or 2.

1. Classes that satisfy 4 conditions are obtained in 2 different ways. In the first case, 1 condition is related to the primary edges, while the remaining 3 conditions involve the secondary edges. This means that 3 out of 4 secondary edges are equal, as in the classes $[A, B, C, C, C, F]$ or $[A, B, B, B, E, A]$, see Fig. 14.
   In the second case, the 3 conditions related to the primary edges hold, while the remaining condition involves secondary edges. This means that the primary edges are equal, as well as 2 of the secondary edges, as in the classes $[A, B, B, D, E, A]$ or $[A, B, C, D, C, A]$. Note that when $4I = A$, the classes $[A, B, C, C, C, A]$ and $[A, B, B, B, E, A]$ converge in 9 classes.
2. Classes satisfying 3 conditions include the condition $4I = F$ along with 2 additional conditions involving secondary edges. This means that 2 pairs of secondary edges are equal. Some examples are the classes $[A, B, B, D, D, F]$ and $[A, B, C, C, B, F]$ with $4I = F$, see Fig. 15.
3. Finally, classes satisfying 2 conditions are found in 2 different cases. In the first case, the primary edges are different while the remaining 2 conditions come from the secondary edges, which implies that 2 pairs of secondary edges must be equal, as in the classes $[A, B, B, D, D, F]$ or $[A, B, C, C, B, F]$, see Fig. 15. Notice that for this class, if $4I = A$, it converges in 13 classes. In the second case, the primary edges are equal and 2 secondary edges are also equal, as in the classes $[A, B, C, D, D, A]$ and $[A, A, C, A, E, A]$.

4.6. Subfamily that converges in 37 classes $R_1^+C_{37}$

This subfamily includes all other R_1^+ classes that satisfy less than 4 conditions, where each condition falls into a different category in Fig. 4. This includes the following scenarios:

1. Classes that satisfy 3 conditions where 3 out of 4 secondary edges are equal. These classes are $[A, B, B, B, E, F]$, $[A, B, B, D, B, F]$, $[A, B, C, B, B, F]$ and $[A, B, C, C, C, F]$.
2. Classes that satisfy 2 conditions are as follows: $[A, B, B, D, E, A]$, $[A, B, C, C, E, A]$ and $[A, B, C, D, D, A]$ where only 2 secondary edges are equal, and the primary edges are identical. Other classes include $[A, B, C, D, C, F]$ and $[A, B, C, D, D, F]$ where 2 secondary edges equal and $4I = A$ or $4I = F$.
3. Classes satisfying 1 condition come from either 2 primary edges being equal, or 2 secondary edges being equal, such as $[A, B, C, D, E, A]$ and $[A, B, C, D, D, F]$.

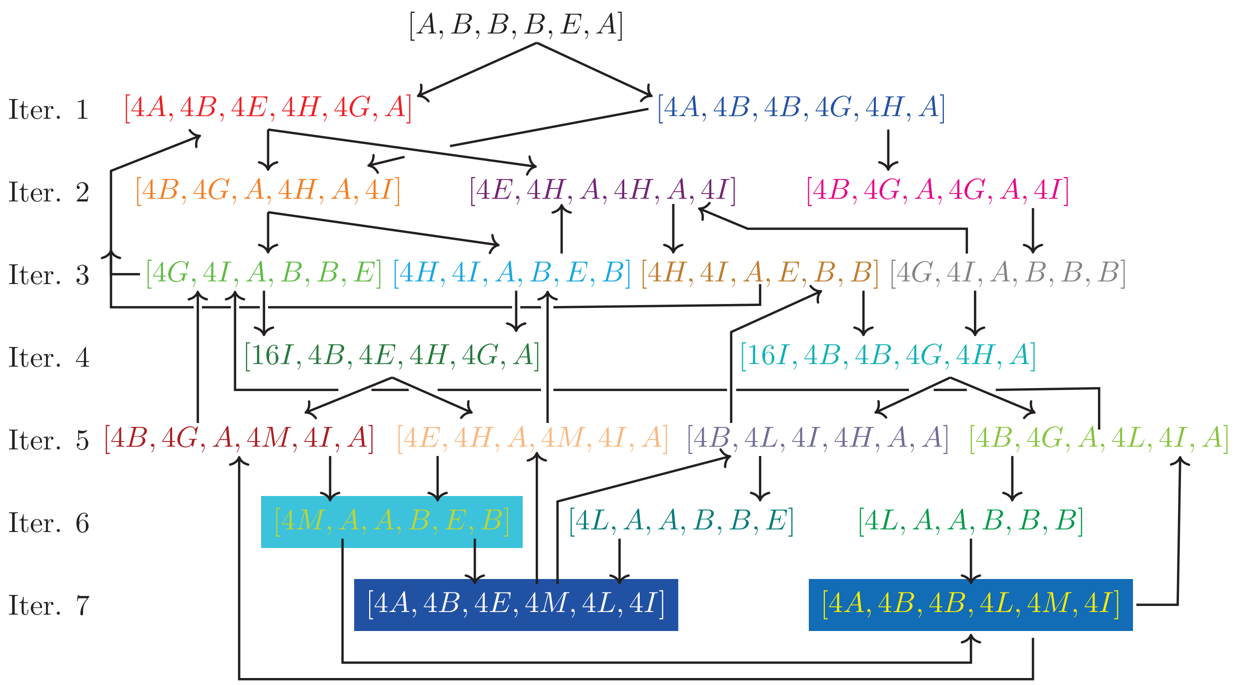

Fig. 14. Generation tree of the class $[A, B, B, B, E, A]$ converging in 21 classes. Four conditions are satisfied: $A = F$, and $B = C = D$.

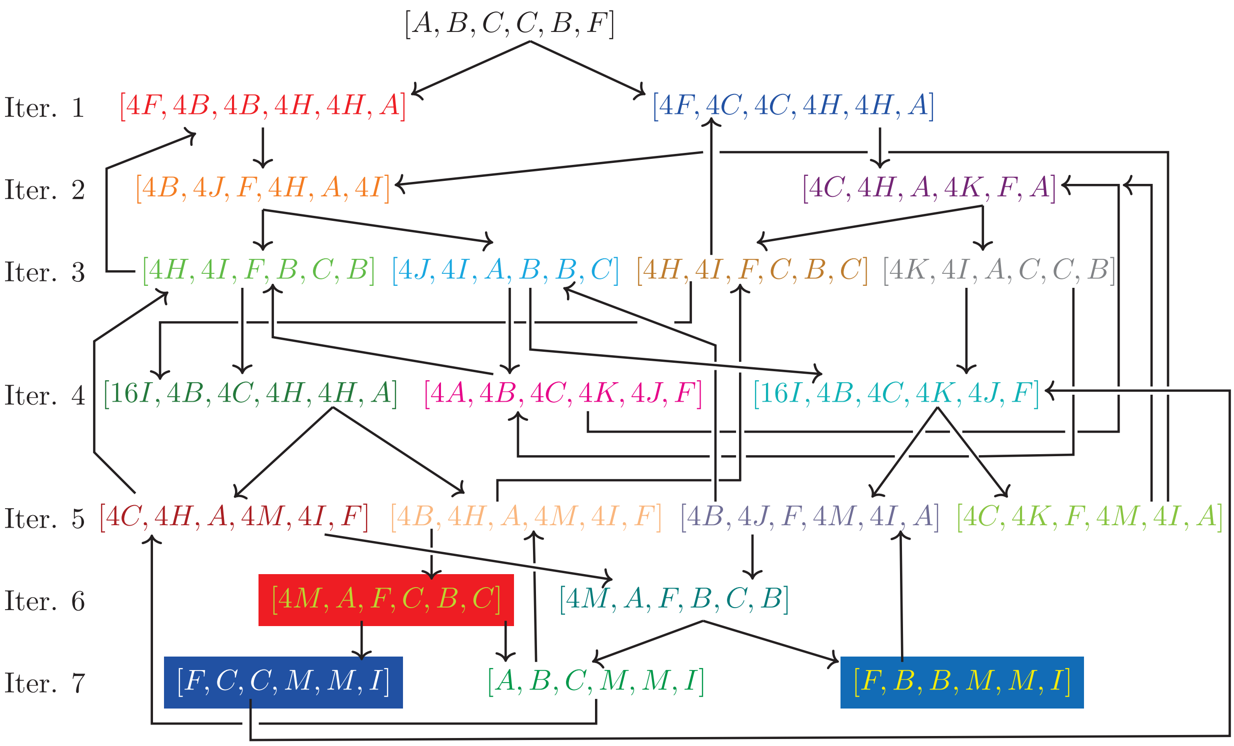

Fig. 15. Generation tree of the class $[A, B, C, C, B, F]$ converging in 21 classes and satisfying 2 conditions, $B = E$ and $C = D$.

4. Classes that do not satisfy any condition but have at least 2 equal edges. In this case, the single class that satisfies these requirements is $[A, B, C, D, E, B]$.
5. Finally, classes that do not hold any condition and have all their edges different include $[A, B, C, D, E, F]$, as demonstrated in [13].

We also remark that some classes exhibit many edge coincidences, but they still do not satisfy the minimum conditions of Lemma 2 to converge in less than 37 classes. This means that not all edge coincidences are relevant for convergence to a fewer number of classes. For example, the class [13,11,10,10,10,11] satisfies that $C = D = E$ and $B = F$, which implies a high number of coincidences.

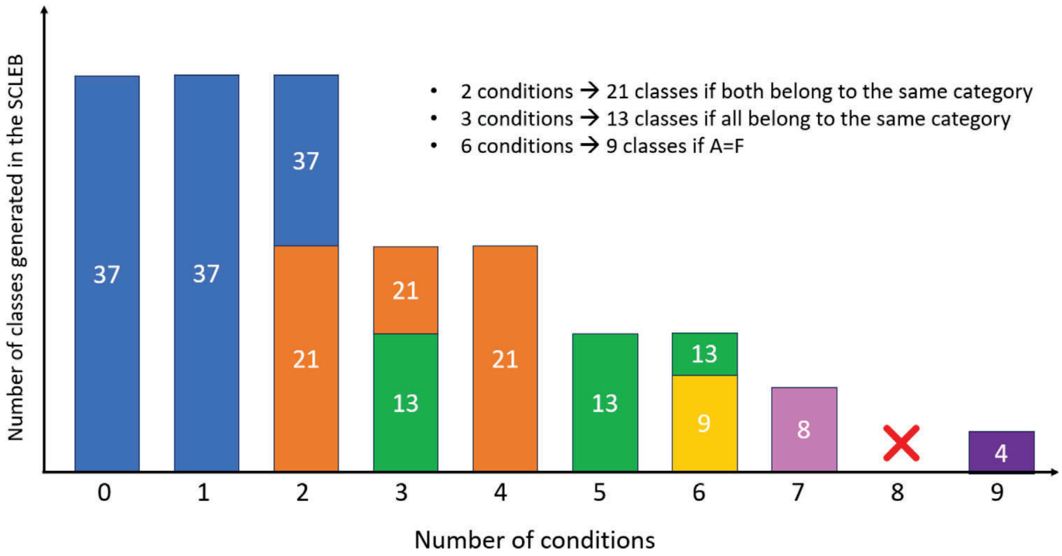

Fig. 16. Number of generated classes by the SCLEB for the R_1^+ class as a function of the number of satisfied conditions.

However, the equality of the 3 secondary edges, $C = D = E$, actually represents 3 conditions, but one in each different category in Fig. 4. On the other hand, the equality $B = F$ is not one of the relevant conditions for convergence. Therefore, this class converges in 37 classes.

Fig. 16 shows the number of generated classes for the R_1^+ family vs. the number of satisfied conditions.

5. Conclusions

The study of similarity classes arising from the LEB applied to tetrahedra is of great importance in the Finite Element Method. The efficiency of the method depends on the accurate and efficient assembly of the stiffness matrix, since each different similarity class require time-consuming computations, increasing the computational cost as the number of classes grows and reducing efficiency.

In this paper we study the convergence of the SCLEB in ≤ 37 classes for R_1^+, of an improved family of near equilateral tetrahedra. By applying the sextuple representation of any tetrahedron class, we analyze the conditions that lead to the convergence in $\{4, 8, 9, 13, 21, 37\}$ classes. Although the three dimensional nature of tetrahedra makes the study of their geometry and shape somewhat complicated, we precisely analyze the generated classes by the SCLEB applied iteratively to the tetrahedra subdivision using clear representations, such as graphs and sextuple trees. Then, we provide a systematic study of the similarity classes in the LEB for a family of tetrahedra that generalizes the near equilateral family mentioned by Adler [12].

The results of this paper open a series of new challenges regarding the convergence of SCLEB applied to other families of tetrahedra such as, the families $R_2^+, R_3^+, \ldots, R_8^+$, obtained through the iterative SCLEB process applied to the regular tetrahedron.

CRediT authorship contribution statement

Miguel A. Padrón: Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Agustín Trujillo-Pino:** Writing – original draft, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Jose Pablo Suárez:** Writing – original draft, Validation, Supervision, Resources, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

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