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Short Communication

A short note about a practical procedure for the calculus of the Sugeno integral

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In this note, we present a practical method for the computation of the Sugeno integral of a nonnegative measurable function with respect to a general monotone measure.

1. Introduction

The theory of fuzzy integration with respect to a fuzzy measure was introduced by Sugeno in [1] as an effective tool modeling for the treatment of non-deterministic problems. Shortly after, fuzzy integrals were used in the subjective evaluation patterns [2] and forecast evaluation [3].

Nowadays, the use of nonadditive measures (which take into account the interaction between measured objects) and nonlinear integrals has encouraged the study and application of Sugeno integrals and Choquet integrals. Since Sugeno introduced the fuzzy integrals a great number of papers have appeared in the literature studying their properties and applications. Concerning with general theory, in [4] the authors compare the Choquet and Sugeno integrals, in [5] the authors studied properties of the double Sugeno integral. Moreover, in [6] a new characterization of the Sugeno integral, viewed as a special aggregation function, is given. Ralescu and Adams [7] proved theorems of continuity of the fuzzy integral with respect to measure convergence and pointwise convergence for a continuous and subaddtive fuzzy measures.

Many techniques have been developed to compute a Sugeno integral. Specifically in a significant number of cases, a practical way to compute the fuzzy integral of a nonnegative measurable function translates to solving a fixed point problem which can be a challenging issue to address.

In this paper, we propose a practical method for the computation of the Sugeno integral of a nonnegative measurable function with respect to a general monotone measure.

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2. Preliminaries

In this section, we introduce some basic notation and properties of the fuzzy integral.
Suppose that Σ is a σ-algebra of subsets of a set X. A fuzzy (or monotone) measure is a mapping $\mu : \Sigma \to [0, \infty)$ satisfying the following two conditions:

(i) $\mu(\emptyset) = 0$,
(ii) for any $A, B \in \Sigma$, $A \subset B \Rightarrow \mu(A) \le \mu(B)$.

The triple (X, Σ, μ) is called a fuzzy measure space.
Now, given $f : X \to [0, \infty)$, Σ-measurable, for any $\alpha > 0$, by $F_\alpha(f)$ we denote the set

$$F_\alpha(f) = \{x \in X : \ f(x) \ge \alpha\}$$

and $F_0(f) = \{x \in X : f(x) > 0\} = supp f$ is the support of f.
For convenience, $\{f \ge \alpha\}$ and $\{f > 0\}$ will denote the sets $F_\alpha(f)$ and $F_0(f)$, respectively.
In the sequel, by $\mathcal{F}_+(X)$ denote the set of all non-negative measurable functions with respect to Σ.

Definition 1. Let (X, Σ, μ) be a fuzzy measure space, $f \in \mathcal{F}_+(X)$ and $A \in \Sigma$. The fuzzy integral (or Sugeno integral) of f on A with respect to the fuzzy measure μ is defined as

$$⨍_A f d\mu = \bigvee_{\alpha \ge 0} \left(\alpha \wedge \mu(A \cap F_\alpha)\right),$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

The following properties of the Sugeno integral are well known and can be found in [8].

Proposition 1. *Let (X, Σ, μ) be a fuzzy measure space, $A \in \Sigma$ and $f, g \in \mathcal{F}_+(X)$, then:*

(1) $⨍_A f d\mu \le \mu(A)$,
(2) $⨍_A k d\mu = k \wedge \mu(A)$, for any k nonnegative constant,
(3) If $f \le g$ on A then $⨍_A f d\mu \le ⨍_A g d\mu$,
(4) $\mu(A \cap F_\alpha) \le \alpha \Rightarrow ⨍_A f d\mu \le \alpha$,
(5) $\mu(A \cap F_\alpha) \ge \alpha \Rightarrow ⨍_A f d\mu \ge \alpha$,
(6) $⨍_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap F_\gamma) < \alpha$,
(7) $⨍_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap F_\gamma) > \alpha$.

In Theorem 3.5 of [9], it is proved the following result.

Theorem 1. *Suppose that there exists $\alpha_0 \in \mathbb{R}_+$ such that $\alpha_0 = \mu\left(A \cap \{f \ge \alpha_0\}\right)$, where $f \in \mathcal{F}_+(X)$ and $A \in \Sigma$. Then $\alpha_0 = ⨍_A f d\mu$.*

Remark 1. Theorem 1 appears in [8,10] where the authors used fuzzy measures more restrictive that our monotone measures. Moreover, it can be that the equation $\alpha = \mu(A \cap \{f \ge \alpha\})$ has not solution and in this case, the Sugeno integral must be computed in alternative manner (see Example 1 of [11] and Example 4.2 of [9]).

3. Main result

The main result of the paper in the following.

Theorem 2. *Let (X, Σ, μ) be a fuzzy measure space, $f \in \mathcal{F}_+(X)$, $A \in \Sigma$ and $\alpha_0 = \mu\left(A \cup \{f \ge \alpha_0\}\right)$ (this is, $⨍_A f d\mu = \alpha_0$). Let $g : [0, \infty) \to [0, \infty)$ be a strictly increasing function such that the equation $g(x) = x$ has as solution $x = \alpha_0$. Then*

$$⨍_A (g \circ f) d\mu = ⨍_A f d\mu = \alpha_0.$$

Proof. Firstly, we claim that

$$\{f \ge \alpha\} = \{(g \circ f) \ge g(\alpha)\}, \quad \text{for any } \ \alpha \ge 0.$$

In fact, the strictly increasing character of g gives us that if $f(x) \ge \alpha$ then $g(f(x)) \ge g(\alpha)$ and therefore,

$$\{f \geq \alpha\} \subset \{(g \circ f) \geq g(\alpha)\}.$$

For the reverse inclusion, we have that if $(g \circ f)(x) = g(f(x)) \geq g(\alpha)$ then, since g is strictly increasing, we infer $f(x) \geq \alpha$, this is $\{(g \circ f) \geq g(\alpha)\} \subset \{f \geq \alpha\}$.
This proves our claim.
Now, since $g(\alpha_0) = \alpha_0$, we deduce

$$\{f \geq \alpha_0\} = \{(g \circ f) \geq \alpha_0\}.$$

This gives us that

$$\alpha_0 = \mu(A \cap \{f \geq \alpha_0\}) = \mu(A \cap \{(g \circ f) \geq \alpha_0\}),$$

and, by Theorem 1, it follows

$$⨍_A (g \circ f) d\mu = \alpha_0.$$

This completes the proof. □

Next, we study some properties of the function g appearing in Theorem 2.

Proposition 2. *Suppose that $\gamma > 0$, we denote by C_γ the following class of functions*

$$C_\gamma = \{g : [0,\infty) \to [0,\infty) : \ g \text{ is strictly increasing and } Fix\ g = \{\gamma\}\},$$

where $Fix\ g$ is the set of fixed points of g (this is, those points satisfying $g(x) = x$).
For any $f, g \in C_\gamma$ it is satisfied:

a) $g \circ f \in C_\gamma$,
b) $f^n = f \circ \overset{n)}{\cdots} \circ f \in C_\gamma$, *for any* $n \in \mathbb{N}$,
c) any convex linear combination $\lambda f + (1-\lambda) g \in C_\gamma$, $\lambda \in [0,1]$,
d) if $r, s \in (0,\infty)$ *and* $r + s = 1$ *then* $f^r \cdot g^s \in C_\gamma$,
e) the maximum and minimum of the functions f, g *this is,* $h = \max\{f, g\}$ *and* $j = \min\{f, g\}$ *belong to* C_γ.

Proof. The proof of these results is straightforward and, therefore, we omit it. □

Taking into account Proposition 2 and Theorem 2, we can state the following corollary.

Corollary 1. *Suppose that (X, Σ, μ) is a fuzzy measure space, $f \in F_+(X)$, $A \in \Sigma$ and $\alpha_0 = \mu(A \cap \{f \geq \alpha_0\})$. Then*

$$⨍_A (g \circ f) d\mu = ⨍_A f\, d\mu = \alpha_0 \quad \text{for any } g \in C_{\alpha_0}.$$

In the sequel, we present some examples.
From now on, μ will be the usual Lebesgue measure.

Example 1. Suppose that $X = [0,1]$ and let $f : [0,1] \to [0,\infty)$ be the function defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x = 0, \\ \frac{1}{x}, & \text{if} \quad x \in (0,1]. \end{cases}$$

Then a straightforward calculus shows that $1 = \mu(x \in [0,1] : \ f(x) \geq 1) = ⨍_0^1 f \ d\mu$.

Now, we consider the function $g : [0,\infty) \to [0,\infty)$ defined by $g(x) = \sqrt{x}$, then

$$(g \circ f)(x) = \begin{cases} 0, & \text{if} \quad x = 0, \\ \frac{1}{\sqrt{x}}, & \text{if} \quad x \in (0,1]. \end{cases}$$

Since g is a strictly increasing function and the equation $g(x) = x$, this is, $\sqrt{x} = x$ has as positive solution $x = 1$, Theorem 2 says us that

$$\int_0^1 \frac{1}{\sqrt{x}}\, d\mu = 1.$$

Example 2. Consider $X = [0,1]$ and let $f : [0,1] \to [0,\infty)$ be the function defined by $f(x) = x^2$. It is easy to prove that

$$\mu\left(x \in [0,1] : f(x) \geq \tfrac{3-\sqrt{5}}{2}\right) = \tfrac{3-\sqrt{5}}{2},$$

and, from this, we deduce that $\int_0^1 x^2\, d\mu = \frac{3-\sqrt{5}}{2}$. On the other hand, we consider the function g defined on $[0,\infty)$ by

$$g(x) = x + \ln\left(x + \tfrac{\sqrt{5}-1}{2}\right).$$

Since $g'(x) = 1 + \dfrac{1}{x + \frac{\sqrt{5}-1}{2}} > 0$ in $[0,1]$, g is a strictly increasing function. Moreover, it is easily seen that the equation $g(x) = x$ has as unique solution $\frac{3-\sqrt{5}}{2}$.

Finally, by Theorem 2, we deduce that

$$\int_0^1 g(x^2)\, d\mu = \tfrac{3-\sqrt{5}}{2},$$

or, equivalently,

$$\int_0^1 \left(x^2 + \ln\left(x^2 + \tfrac{\sqrt{5}-1}{2}\right)\right)\, d\mu = \tfrac{3-\sqrt{5}}{2}.$$

Example 3. Let X be the interval $[0, \frac{\pi}{2}]$ and let μ be the usual Lebesgue measure on X. Consider the following integral

$$\int_0^{\frac{\pi}{2}} \left(\tfrac{\pi}{12} + \arctan\left(\tfrac{6}{\pi}x\right)\right) d\mu.$$

First, we note that the function $g : [0,\infty) \to [0,\infty)$ defined by

$$g(x) = \tfrac{\pi}{12} + \arctan\left(\tfrac{3}{\pi}x\right)$$

is strictly increasing since $g'(x) = \frac{3}{\pi + \frac{9}{\pi}x^2} > 0$. Moreover, as $g(\frac{\pi}{3}) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$, then $x = \frac{\pi}{3}$ satisfies the equation $g(x) = x$.

On the other hand, it is a routine calculation to prove that $\int_0^{\frac{\pi}{2}} 2x\, d\mu = \dfrac{\pi}{3}$, where this value has been obtained as a solution of the equation $\alpha = \mu\left(x \in [0, \frac{\pi}{2}] : f(x) \geq \alpha\right)$ where $f(x) = 2x$.

Now, by using Theorem 2, we infer that

$$\int_0^{\frac{\pi}{2}} (g \circ f)\, d\mu = \int_0^{\frac{\pi}{2}} \left(\tfrac{\pi}{12} + \arctan\left(\tfrac{6}{\pi}x\right)\right) d\mu = \frac{\pi}{3}.$$

Example 4. Let $X = [1,2]$. Then, it is a straightforward calculate gives us that

$$\int_1^2 x^2 d\mu = 1.$$

On the other hand, the functions $g, h : [0,\infty) \to [0,\infty)$ given by $g(x) = e^{x-1}$ and $h(x) = \frac{4}{\pi}\arctan x$ are strictly increasing, and the unique solution of equations $g(x) = x$ and $h(x) = x$ is $x = 1$. Therefore, $g, h \in C_1$. Applying Proposition 2 and Corollary 1, for any $\lambda \in [0,1]$ we have that

$$\fint_1^2 (\lambda g + (1-\lambda)h)(x^2)d\mu = \fint_1^2 \lambda e^{x^2-1} + \frac{4(1-\lambda)}{\pi}\arctan x^2 \; d\mu = 1.$$

Remark 2. As we see in the Examples 2 and 3, this is in the following fuzzy integrals $\fint_0^1 \left(x^2 + \ln\left(x^2 + \frac{\sqrt{5}-1}{2}\right)\right) \; d\mu$ and $\fint_0^{\frac{\pi}{2}} \left(\frac{\pi}{12} + \arctan\left(\frac{6}{\pi}x\right)\right) d\mu$, its computation by using the equation $\alpha = \mu([0,1] \cap \{f \geq \alpha\})$ or $\beta = \mu\left([0, \frac{\pi}{2}] \cap \{g \geq \beta\}\right)$, where $f(x) = x^2 + \ln\left(x^2 + \frac{\sqrt{5}-1}{2}\right)$ and $g(x) = \frac{\pi}{12} + \arctan\left(\frac{6}{\pi}x\right)$ is a very complicated question since to give an analytic expression of the sets $\{f \geq \alpha\}$ and $\{g \geq \beta\}$ is not trivial or it does not exist as in Example 3. These examples make sense to the result obtained in Theorem 1.

4. Concluding remarks

In the present note, we give a sufficient condition in order to compute Sugeno integral. Our method is based in a fixed point technique. Moreover, we present some examples of Sugeno integrals where our result can be applied and without knowledge of this result, the value of these integrals would be a difficult question.

CRediT authorship contribution statement

J. Caballero: Writing – review & editing, Writing – original draft, Investigation. **B. López:** Writing – review & editing, Writing – original draft, Investigation. **K. Sadarangani:** Writing – review & editing, Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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