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Optimization of Feed Ration Size in Aquatic System According to the Optimal Control Approach: Implications of Using the von Bertalanffy Growth Model

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Rationing aquaculture farming feed is challenging for producers due to high feed costs, representing 30%–60% of the total operating costs. Therefore, optimal timing of feeding could lead to improved economic returns from an aquatic system. An optimal dynamic feeding model has been determined considering the von Bertalanffy growth model. A bioeconomic model of tilapia production in Mexico for specific markets was used for numerical illustration. The von Bertalanffy growth model was parameterized with experimental data from tilapia (*Oreochromis niloticus*) fed four ration sizes (50%, 80%, 100%, and satiety), in order to determine different optimal rationing for different fish sizes (200, 300, and 400 g), market prices (Monterrey, Cancún, Mexico City, and On site), and optimal harvesting times (OHT), considering the time value of money. The results of the modeled optimal feeding trajectories show a continuous decrease from stocking to reach a minimum value and then slightly approaching the harvest size. This result contrasts with the recommendations of the feed suppliers and with those found when a potential growth model was used. The results in the case study showed that the Monterrey market presented the highest present value of the benefits in the OHT and the different market sizes. The implications of the Bertalanffy model for optimal rationing trajectories are presented in the discussion.

Keywords: bioeconomic model; growth model; optimal control theory; optimal feeding trajectory; tilapia

1. Introduction

In aquaculture systems, it is important to analyze feed efficiency and determine the most profitable feeding strategies in the production process [1], as they also contribute to sustainable aquaculture development [2, 3]. Food intake by organisms is influenced by multiple factors, both biotic and abiotic, including temperature, weight of the organism, food quality, feeding frequency, and ration size, among others [3, 4]. Feed suppliers often recommended ration sizes (feeding charts) to fish farmers based on the size or weight of organisms. It is also common for producers to feed fish empirically by observing feed consumption on the water surface, which can lead to overfeeding and waste of commercial feed [5, 6]. However, in practice, feed costs represent between 30% and 60% of total costs [7–9]. Overfeeding can further increase production costs and water pollution, and underfeeding can result in

lower-than-expected growth, with consequent economic losses [1, 9–11].

Dynamic growth models are used to analyze the effect of factors such as diet on the growth of individuals. Several models have been used to represent the growth of cultured organisms, which have also been adapted from models of populations [12–17]. These models have been modified to estimate the effects of diverse biological, physiological, and environmental variables [18–20]. The von Bertalanffy growth model, a special case of the Richards model based on the Pütter model [13, 21, 22], has been widely used to model the growth of numerous species, such as fish, mammals, birds, and invertebrates, among others [21], as it conforms to the biological principles of fish growth, described as the metabolic processes of organisms in terms of energy, formulated as the positive difference between anabolism (building up) and catabolism (breaking down), with a growth rate that slows weight gain as the organism approaches maximum weight [21–23]. It has also been widely used in tilapia culture because it successfully represents fish growth in captivity [12, 24, 25].

The main characteristic of the Bertalanffy model is that it initially shows a growth rate with exponential behavior until it reaches a maximum and then decreases asymptotically to zero. This model has been modified to increase the reliability of its predictions by including factors such as fish size/weight and environmental factors such as temperature, dissolved oxygen, unionized ammonia, and feeding rate [15, 18, 26, 27]. More recently, it has been used in modeling the growth of tilapia juvenile with density effect [24], in fattening with temperature effect [18], and the size of the ration [5]. The importance of growth models in aquaculture lies in the design and management of stocking and harvesting plans, assessment of feasibility, costs, and benefits [16, 28, 29]. Of course, inaccurate model predictions could lead to large economic losses [12, 30].

Tilapia remains the fourth most farmed species in the world, with 5.3 million tons. In Mexico, production in 2022 was around 59,000 tons, although it was 16.9% lower than in 2020 [2]. However, over the last 10 years, it has recorded an average annual growth rate of 3.1% [31], as it is considered an excellent substitute for other white fish [25, 32, 33]. Most of the national production has been destined for subsistence consumption or local sale. Currently, it continues to be one of the activities with the greatest potential, yielding social and economic benefits, as a source of food with high nutritional value and accessible costs. However, despite this, development has been insufficient [34], and one of the factors that could contribute to this could be the management of a constant food rationing; however, so far, only the work of Domínguez-May et al. [5] has attempted to determine optimal nonconstant feeding trajectories in tilapia using a potential growth model.

To determine the optimal choice of a growth factor, such as ration size, throughout the growth cycle, mathematical optimization tools, such as optimal control models, are necessary. They have been used in several aquaculture systems and have been considered time-dependent because of their flexibility and simplicity. For example, using a hypothetical model of time-dependent logarithmic farm growth, they considered profit maximization using two control variables: feeding schedule and harvest timing. Similarly, Mistiaen and Strand [35] and Esmaeili [36] followed the same line as Arnason [37]'s work and used a similar growth model for gilthead sea bream (*Sparus aurata*) and the shrimp (*Penaeus indicus*), including the effect of size-dependent prices as a linear step function and a linear function, respectively. León, Hernández, and León-Santana [38] and Hernández, León-Santana, and León [19] in studies with the gilthead sea bream (*S. aurata*) considered a modified potential-type growth model that included a multiplicative effect for the effects of fish size, ration size, and temperature to determine optimal harvesting strategies. Karimanzira et al. [20] applied optimal control methods in an aquaponic farm to maximize the benefits of optimal resource use based on a bioenergetic growth model and taking into account metabolite constraints. Domínguez-May et al. [5] determined optimal trajectories of ration size for different target markets and harvest sizes based on optimal control theory. These authors used a potential-type growth model.

The growth models that have been used in the optimization of ration or feed quantity in aquatic systems within the framework of optimal control theory have been of the potential type, dependent on size, ration, and temperature, and of the logarithmic type, dependent on time. Although these models have claimed to show favorable results, the metabolism, which plays an important role in growth dynamics, such as the von Bertalanffy model, has not been used. In addition, solving the control problem which became a system of equation problem with boundary values is relatively new in aquaculture. Based on the above, this paper proposes to determine the dynamics of optimal rationing using a bioeconomic model when the growth model is of the von Bertalanffy type modified by a rationing function. von Bertalanffy is a growth model that is governed by physiological principle and is the most used in the representation of the growth of different organisms [1, 12, 15, 21–23, 39]. For numerical illustration of this work, a bioeconomic model of tilapia (*Oreochromis niloticus*) farmed in Mexico was used, considering different market sizes.

2. Material and Methods

2.1. Experimental Systems. The experiment lasted 182 days, and the semiopen aquaculture recirculation system consisted of eight indoor circular fiberglass tanks with a capacity of 0.75 m^3 per tank with a diameter of 0.8 m and a height of 1 m. The experiment was conducted with two replicates for each treatment. The distribution of the tanks was randomized, and the initial stocking density was 44.0 fish/m^3, in agreement with the tilapia farms in the region. Culture at high fish densities affects zootechnical performance, metabolism, and digestion and induces stress in the aquatic animal [40, 41]. To avoid the accumulation of nitrogenous residues in the tanks, feces siphoning, daily water replenishment (25%), and total water replacement once a week were performed. Cleaning and general maintenance were performed every 14 days to eliminate biota deposited on the tank walls.

Tilapia juveniles were fed with commercial feed containing 27.8% protein (Agribrands Tilapia Chow). Male fish with an average initial weight of 14.23 ± 0.23 g, provided by the aquaculture laboratory of Cinvestav, Mérida, were used. Treatments included three rations and one at satiety (control). The ration recommended by the feeding chart (100%) and the 80% and 50% rations were considered. The restricted rations were adjusted every 14 days, according to the average weight of the fish, according to the feeding table provided by the feed suppliers. The treatment for satiety was designed according to the maximum amount of feed that a fish can consume per unit of time, avoiding wastage [8], and was higher than the 100% recommended by the feed suppliers (feeding table). The feed was divided into four daily rations (9, 12, 15, and 18 h, each of 5 min), evenly distributed. Higher feeding frequency improves fish growth and feed efficiency [42–44].

2.2. Model Description and Optimization.
The growth model $g(\cdot)$ considered in this work assumes that individual growth is influenced by the size of the organism $x(t)$ and the size of the ration $r(t)$ supplied at time t. The ration size takes normalized values between 0 and 1; i.e., it is $r = 0$ in the case of not feeding, and $r = 1$ is feeding to satiety. The normalized $r(t)$ values for the four treatments were 0.42 (50%), 0.67 (80%), 0.84 (100%), and 1 (satiety). The growth equation determined by the effects of size and ration is given by the following:

$$\dot{x}(t) = g(x(t), r(t)), \quad x(0) = x_0, \tag{1}$$

where $x_0 > 0$ is the initial weight of the individual, $x(t)$ is the size of the individual, and $r(t)$ is the ration standardized over time t. It is assumed that $g(x, r(t))$ is continuously differentiable and positive within. If the interval is $[0, \omega] \times [0, 1]$, it follows that $g(0, r(0)) = g(\omega, r(\omega)) = 0$, where $\omega > 0$ is the maximum weight of the fish.

Although there is evidence that organisms in a cultured population tend to grow heterogeneously in size [45–47], to simplify this analysis, heterogeneity will not be taken into account. The instantaneous daily mortality rate is assumed to be $(\mu > 0)$, is constant, and independent of the size of the individual. However, this value of μ is in accordance with the fish farms in the study region. Therefore, the number of organisms over time will be given by the following exponential model:

$$N(t) = N_0 e^{-\mu t}, \tag{2}$$

where N_0 represents the number of initial organisms in $t = 0$.

For simplicity, a single culture cycle is considered. The evaluation of the economic conditions also depends on the specific model. The accumulated costs are given by the following:

$$C(t) = \int_0^T e^{-it} [f(x, r(t)) N(t)] dt + c_0, \tag{3}$$

where i is the diary discount rate, T is the cultivation period, $f(x, r)$ represents the function of operating costs depending on fish size and ration size, and c_0 is the fixed cost in the cultivation period (fingerling cost [US$]).

The operating cost function $f(x, r)$ considered variables with higher incidence, such as energy cost $C_e(x, r)$, feed cost $C_f(x, r)$, and maintenance cost $C_m(x, r)$. Therefore,

$$f(x, r) = C_e(x, r) + C_f(x, r) + C_m(x, r). \tag{4}$$

The cost of energy consumed on the farm $C_e(x, r)$ (cost of energy for water exchanges and aeration per gram of tilapia per day [24 h]) $(E(r))$ was calculated from the direct relationship between fish size and the different satiety feeding regimes and subsequently calibrated for each ration from the amount of ammonia, nitrogen, and water replacement which was required [25]. This function is defined by the following:

$$C_e(x, r) = xE(r). \tag{5}$$

Feeding costs were calculated from the feed conversion rate (FCR) $\xi(\cdot)$, which indicates the amount of food necessary to increase the individual's weight by one unit. Thus,

$$C_f(x, r) = c_f \xi(x, r) g(x, r), \tag{6}$$

where c_f is the cost per kilogram of feed. The FCR is influenced by several factors, such as individual weight, rationing, temperature, and density. For simplicity, the model only incorporates individual daily weight and rationing. A logistic functional expression for the feed conversion ratio was considered:

$$\xi(x, r) = \xi_1 (1 + \xi_2 e^{-\xi_3 xr})^{-1}, \tag{7}$$

where ξ_1, ξ_2, and ξ_3 are parameters of the feed conversion model.

For simplicity, the maintenance cost $C_m(x, r) = c_m x$ was assumed to be linearly dependent on size. The coefficient of c_m was calibrated according to the local market and represents the maintenance cost per individual.

2.2.1. Analytical Solution.
For aquaculturists, it is important to know the relationship between ration size and fish growth to manage a better feeding program for the culture system. The unit revenue from the sale of the product depends on the individual size, classified by size. This variable price is represented by the function $p(x)$, which is assumed to be increasing with size $p_x(x) > 0$ and convex $p_{xx}(x) \le 0$. Considering Equation (2) $N(t) = N_0 e^{-\mu t}$ and assuming a single culture cycle, the objective function of the tilapia farmer to maximize profit is given by the difference between revenue and costs discounted over time:

$$\pi(T) = p(x(T)) x(T) N_0 e^{-\mu t} e^{-rT} - c_M N_0 e^{-\mu t} x(T) e^{-rT} \cdots,$$

$$\cdots - \int_0^T e^{-rt} \{x(t) E(r) + c_f \xi(x, r) g(x, r) + c_m x\} N_0 e^{-\mu t} dt - c_0,$$

where $x(T)$ is the harvest size, $p(x(T))$ is the price size of the harvest, e^{-rt} is the discount factor, c_0 is the fixed cost, c_M is the market cost, T is the culture's horizon, and $x(t)$ is the

tilapia growth over time (from sowing until it reaches commercial size at harvest time). $N(t)$ represents the number of individuals over time (Equation (2)). For simplicity, the previous objective function has been divided by N_0 (the individual's initial number) and then simplified. Therefore, assuming a culture cycle and considering aquaculture's purpose of finding the best (trajectory) daily ration $(r(t))$, the ability to optimize the present value of the benefits over time is given by the following:

$$\underset{T,x(T),r(t)}{Max} \quad \pi(T) = \left[p(x(T)) - \frac{c_M}{N_0}\right]x(T)e^{-hT} - \frac{c_0}{N_0} - \cdots, \tag{8}$$

$$\cdots - \int_0^T e^{-ht}\left[x(t)E(r) + c_f\xi(x,r)g(x,r) + c_mx\right]dt, \tag{9}$$

$$s.t. \ \dot{x}(t) = g(x(t), r(t)), \ x(0) = x_0, \tag{10}$$

$$x(T)\text{free}; \ T \text{ free}, \tag{11}$$

$$0 \le r(t) \le 1, \tag{12}$$

where e^{-ht} is the discount factor and $h = \mu + r$ is the sum of the discount and mortality rate. According to Leonard and Long [48], the generalized Lagrangian with current values for the problem (Equations (8)–(12)) is defined as follows:

$$L_C = -\left[xE(r) + c_f\xi(x,r)g(x,r) + c_mx\right] + \lambda_C(t)g(x,r)$$
$$+ \mu_{1|}(t)r + \mu_2(t)(1-r), \tag{13}$$

where $\lambda_C(t)$ is the costate variable (also called adjoint variable or shadow price in economics), while $\mu_1(t)$ and $\mu_2(t)$ are nonnegative Lagrange multipliers, satisfying that $\mu_1(t)r = 0$ and $\mu_2(t)(1-r) = 0$. In Equation (10), $\dot{x}(t)$ represents the state variable of the problem (growth model) and depends on the decision of the size of the ration $r(t)$ (control) and fish size $x(t)$. Using the Pontryagin's maximum principle to the problem (Equations (13) and (8)–(12)) the instant ration rate $r(t)$ and instant growth $x(t)$ are obtained (Appendix); both allow a better understatement of the optimal feeding paths and fish growth dynamics [48, 49]). Thus, we have the following system:

$$\dot{x}(t) = g, x(0) = x_0, \tag{14}$$

$$\dot{r}(t) = \frac{g_r\{g_r\theta - \varphi(g_x - h)\} - \dot{x}(\phi_1 g_r - \varphi g_{rx})}{\phi_2 g_r - \varphi g_{rr}}, \tag{15}$$
$$\text{con } \phi_2 g_r - \varphi g_{rr} \neq 0.$$

The transversality conditions are according to Leonard and Long [48] and Grass et al. [49], which define the optimal solutions. In other words, it analyzes the effect of changes in some parameters over the optimal span time T and $x(T)$ [19]:

$$\frac{\dot{x}(T)}{p(x(T))x(T)}\left\{\frac{\partial p(x(T))}{\partial x(T)}x(T) + p(x(T))\right\} = h + \frac{c_M}{N_0}\left(\frac{g}{p(x(T))x(T)} - \frac{h}{p(x(T))}\right)$$
$$+ \frac{E(r(T))}{p(x(T))} + \frac{c_f\xi(x(T),r(T))g(x(T),r(T))}{p(x(T))x(T)} + \frac{c_m}{p(x(T))}, \tag{16}$$

$$\lambda_C(T) = \frac{\partial p(x(T))}{\partial x(T)}x(T) + p(x(T)) - \frac{c_M}{N_0}. \tag{17}$$

When $x(T)$ and T are not free, the problem (Equations (8)–(12)) is governed by the same equations [48, 49], without the conditions of transversality (Equations (16) and (17)). $\lambda_C(T)$ is the current costate variable. Transversality conditions were found by using a similar procedure to that of Hernández, León-Santana, and León [19] and Arnason [37]. Equation (16) indicates that terminal time T, marginal increase, or individual growth of 1 g of additional weight $(\lambda_C(T))$ is directly proportional to the increase in income due to size gain $(p_{x(T)}(x(T))x(T) + p(x(T)))$. The optimal bioeconomic condition is found in the second transversality condition (Equation (17)) [19, 35, 50, 51], which can be interpreted as the marginal increase in value, by delaying the harvest, which must then be equal to the sum of opportunity costs, mortality, energy, feed, and a proportion of maintenance costs. From this, dependence on harvest size $x(T)$ and other biological and economic factors is observed.

2.2.2. Frontier Value r_T.
As the marginal of the Lagrangian is equal to $\frac{\partial L_C}{\partial r} = 0$, $\forall t \in [0, T]$ (Equation (13)) along the optimal trajectory, it is also valid for $t = T$. Therefore, substituting this inEquation (A.2) (Appendix), we have the following:

$$\lambda_C(T) = \frac{\varphi(x(T), r(T))}{g_r(x(T), r(T))}. \tag{18}$$

Equating the above equation with the transversality condition (Equation (17)), we obtain the following:

$$\frac{\partial p(x(T))}{\partial x(T)}x(T) + p(x(T)) - \frac{c_M}{N_0} = \frac{\varphi(x(T), r(T))}{g_r(x(T), r(T))}. \tag{19}$$

The system (Equations (14) and (15)) was resolved by the shooting method and the secant method to solve nonlinear border value problems [52], setting boundary conditions exogenously. The boundary condition $r(T) = r_T$ was found using the *fsolve* routine from the Optimization Toolbox of

TABLE 1: Mathematical expression of the bioeconomic model.

Description	Mathematical expression
Fish growth model	$g(x, r) = 0.1599R(r)x^{2/3} - 0.0172x$
Ration size effect function	$R(r) = r^{0.6369}e^{0.6369(1-r)}$
Number of fishes in time	$N(t) = 11000e^{-0.00044t}$
Biomass value	$V(t) = p(x)x(t)N(t)$
Price function	$p(x) = p_{\max}(1 - e^{-kx})$
Total costs	$C(t) = \int_0^T e^{-ht}[f(x, r(t))N(t)]dt + c_0$
Cost function	$f(x, r) = C_e(x, r) + C_f(x, r) + C_m(x, r)$
Energy cost according to ration size	$C_e(x, r) = x(0.00056r^2 + 0.00182r - 0.00044)$
Feeding cost	$C_f(x, r) = c_f\xi(x, r)g(x, r)$
FCR function according to ration size and fish weight	$\xi(x, r) = 2(1 + 1.32e^{-0.0175xr})^{-1}$
Maintenance costs per fish	$C_m(x, r) = 0.00001x$

Note: $h = r + \mu$, where $r = 0.00019$ and is the daily discount rate and $\mu = 0.00044$.
Abbreviation: FCR, feed conversion rate.

TABLE 2: Bioeconomic parameters of tilapia farming.

Parameter	Description	Value	Source
r	Daily discount rate	0.00019	CETES[a]
μ	Instantaneous mortality rate	0.00044	—
$h = r + \mu$	The sum of discount rate and mortality	0.00063	—
c_0	Fixed costs: cost of fingerlings (US$)	1668.02	Local farm data
c_f	Feed cost (US$/kg)	0.86	Local market
	Maximum price per kilogram (US$/kg):	—	—
	Mexico City	4.08	—
	Monterrey	4.64	—
$p_{\max}$	Cancún	3.56	—
	On site	3.25	—
	Instantaneous rate of change of prices:	—	—
	Mexico City	0.0052	—
	Monterrey	0.0057	—
[a]k	Cancún	0	—
	On site	0	—
	Market cost (US$/kg):	—	Local transport
	Mexico City	0.0979	—
	Monterrey	0.1449	—
c_M	On site	0	—
	Cancún	0.0469	—

[a]$k = 0$ indicates constant prices (Cancún and On site).

MATLAB R2010b via the transversality equation (Equations (16) and (19)) from the Appendix, respectively.

2.2.3. Numerical Analysis Data and Assumptions. For numerical application, we use biological and economic data from tilapia farming in Yucatan, Mexico [5, 53], for different target markets in Mexico (Mexico City, Monterrey, On site, and Cancún). Currently, fish farmers in the region sell their production at the On site or to Cancún through intermediaries. The markets of Mexico City and Monterrey are important aquaculture markets in Mexico and are potential markets for these producers. For the bioeconomic model, circular tanks of a 10 m diameter were used, similar to the tilapia farms in Yucatan where the experimental trial was carried out. Tilapia

is an important aquaculture species and has a high economic value to the industry due to its rapid growth, high survival, captive breeding, high meat quality, and market value [2, 54]. Table 1 shows the specific mathematical expressions of the bioeconomic model, and Table 2 shows the parameters related to costs and prices (fixed and variable) that were used for the numerical study.

Since rationing is the most sensitive factor in the cost function, different target markets were compared: Mexico City (latitude, 19°29′52″ North; longitude, 99°7′37″ West), Monterrey (latitude, 25°40′17″ North; longitude, 100°18′32″ East), On site (latitude, 22°35′10″ North and 19°33′04″ South; longitude, 87°32′00″ East and 90°24′26″ West), and Cancún (latitude, 21°09′38″ North; longitude, 86°50′51″ West). Market

TABLE 3: Results of bioeconomic analysis of optimal rationing and optimal harvesting time in tilapia culture.

Parameters	Units	Mexico City	Monterrey	On site	Cancun
Optimal harvesting time	Days	310.5	339.9	280.0	257.5
Initial ration	—	0.842	0.859	0.831	0.711
Final ration	—	0.696	0.742	0.646	0.525
Size	g	484.43	533.69	366.00	397.69
Biomass	kg	4640.88	5048.87	3553.24	3899.01
Consumed feed	kg	8036.85	9096.41	5341.16	6282.76
FCR	—	1.53	1.54	1.38	1.47
Marketing cost (transportation and insurance costs)	US$	454.43	731.68	166.72	0
Total cost	US$	10,322.38	12,322.13	6522.04	7446.53
Income	US$	13,390.70	16,093.19	7639.46	10,488.33
Net profit	US$	2781.97	3396.50	980.35	2827.21

Abbreviation: FCR, feed conversion rate.

price models (fixed and size-dependent) were considered to determine their effects on optimal rationing trajectories in the culture (Table 2). Since tilapia producers face market size constraints, in this work, we determined different optimal rations for market sizes of 200, 300, and 400 g, respectively.

3. Results

3.1. Experimental Data. The final weights in the treatments 100% (314.40 ± 65.5 g), 80% (292.16 ± 42.4 g), and satiation (334.10 ± 70.7) did not show statistically significant differences (analysis of variance (ANOVA), $p > 0.05$); however, 50% (206.85 ± 27.2 g) did present statistically significant results with concerning the above three. With respect to survival, even when the treatment with the 50% ration presented the smallest percentage, no significant differences were found between treatments ($p > 0.05$), remaining above 90%. The FCR showed a minimum value of 1.34 for 50% and a maximum of 1.92 for satiety.

3.2. Optimal Rationing and Harvesting. Table 3 presents the results of the bioeconomic model. The Monterrey market generated the maximum present value of the profit (PVP) with US$3396.50 and an optimal harvesting times (OHT) of 339.9 days with an optimal size of 533.97 g. On the contrary, the On site market generated the lowest present value of profits with US$980.35 and an optimum harvest time of 280.0 days for an optimum harvest of 366.00 g.

The trend of optimal rationing in this study indicates that, from stocking to harvest, the trajectory decreases until reaching a minimum value (model-based value) and then increases until approaching the optimal harvest size (Figure 1). The optimal trajectory presented a lower (normalized) ration for tilapia in the On site market (fixed price) with initial and final ranges of 0.525 and 0.711, respectively, and with a minimum of 0.427. The Monterrey market had a higher optimal rationing, with initial and final ranges of 0.859 and 0.742, respectively, and a minimum of 0.59.

3.3. Optimum Rationing Trajectories for 200, 300, and 400 g sizes. In this section, we analyzed optimal rationing when culture sizes are predetermined. In that case, optimal
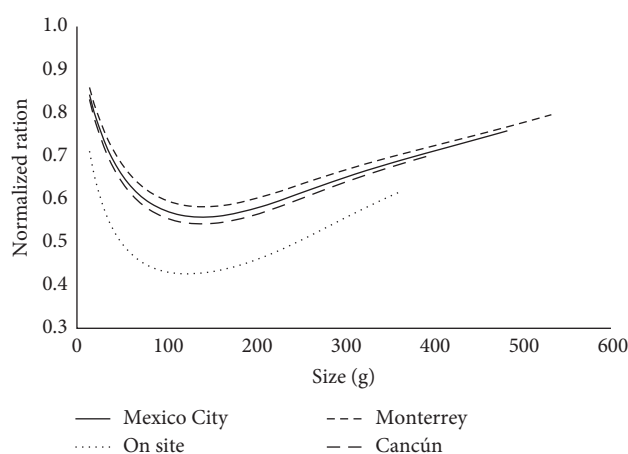

FIGURE 1: Optimal rationing trajectories for tilapia considering different market prices.

rationing for market sizes of 200, 300, and 400 g showed similar trends to those of OHT, shown above. Optimal rationing curves, however, were less concave for the size of 200 g (Figure 2), and in this case, the Monterrey and Mexico City markets showed similar optimal trajectories, although the Monterrey market was slightly larger in size of 400 g. The market that generated the lowest rationing in all the sizes analyzed was the On site, and the highest rationing occurred in the Monterrey market.

3.4. Production, Consumed Feed, FCR, and Net Profits for 200, 300, and 400 g Sizes. According to each size, biomass production (kg) was similar in all markets. The Monterrey market, however, recorded the highest with 3958.28 kg for sizes of 400 g, and the smallest was in the On site market with 2057.37 kg for a size of 200 g (Figure 3a). Optimal rationing in tilapia shows a difference in feed consumption per size under different market prices. There was a higher consumption of balanced feed in the 400 g size in the Monterrey market with 6578.28 kg, while the smallest was in size 200 g in On site market with 2510.60 kg in all markets (Figure 3b). FCR, following the same trend of results, was the largest in the Monterrey market at a size of 400 g and was smaller in
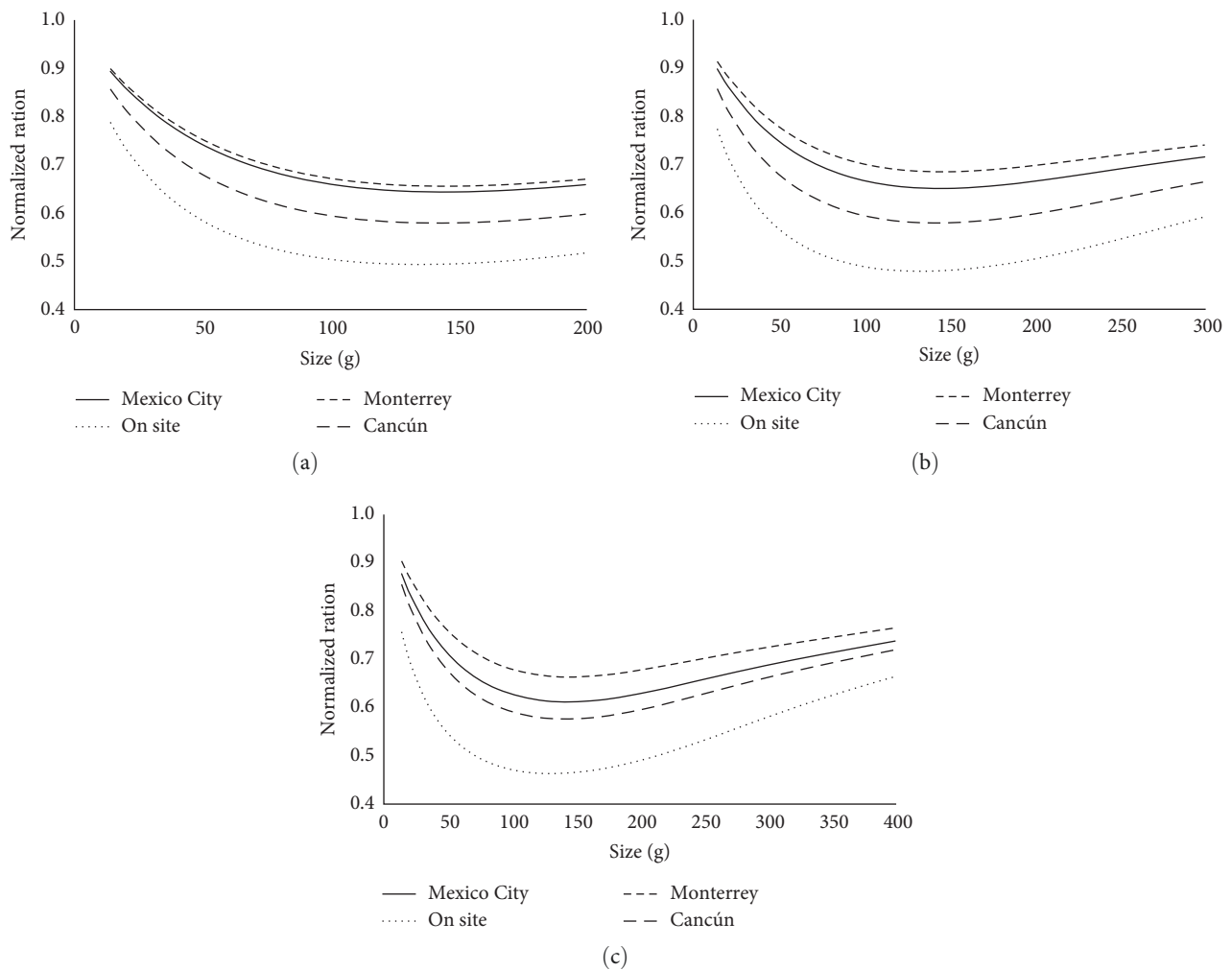
FIGURE 2: Optimal rationing trajectories for different market sizes with the von Bertalanffy growth model for 200 g (a), 300 g (b), and 400 g (c).

the On site market with 1.27 (Figure 3c). Finally, the results of the present value of the profit (Figure 3d) show that the best market to sell tilapia of 200 and 300 g is Cancún (fixed price) with net profits of US$1943.68 and US$2803.35 (Figure 3d) and the Monterrey market for 400 g of tilapia with US$3974.91. The On site market generated the lowest net profit of all market sizes (200, 300, and 400 g).

4. Discussion

Rationing in fish farming can have economic and environmental implications, considering that any feed inefficiency can increase production costs and increase negative externalities related to input use [55–58]. An oversupply of feed can increase waste, causing greater environmental impact and monetary losses, while an undersupply of feed can reduce digestive efficiency due to feed competition and size nonuniformity [59]. From a productive point of view, the optimal ration promotes the best growth and FCR [6, 54, 59, 60]. Although the final weights of the treatments were not statistically different ($p > 0$), the tilapia growth data were modeled. The predictions of the different models could translate into

significant differences in fish weights and could have an impact on profits [12]. In this work, the von Bertalanffy growth model, widely used in aquaculture and, in particular, in tilapia culture, was used [12, 15, 39, 61–63], to determine the dynamics of optimal rationing. There is evidence that this model can represent fish growth affected by environmental and management factors such as food composition, food availability, and water temperature [21, 64, 65].

Under this growth model and for tilapia culture, the optimal rationing found in this work decreases as the fish grows from stocking, reaches a minimum ration, and then increases until harvest (concave rationing curve). This could be due to the physiological principle of metabolism in von Bertalanffy's growth model [1, 22, 23], which includes all the processes by which an organism absorbs and transforms energy and materials for its growth, development, maintenance, and reproduction [66]. Although the metabolic interpretation of the von Bertalanffy growth model is poorly understood [64]. In the case of nonconstant rationing, it had not been reported previously. However, these authors found that the weight-specific growth rate only decreases after birth, with constant food availability. And, absolute growth in weight
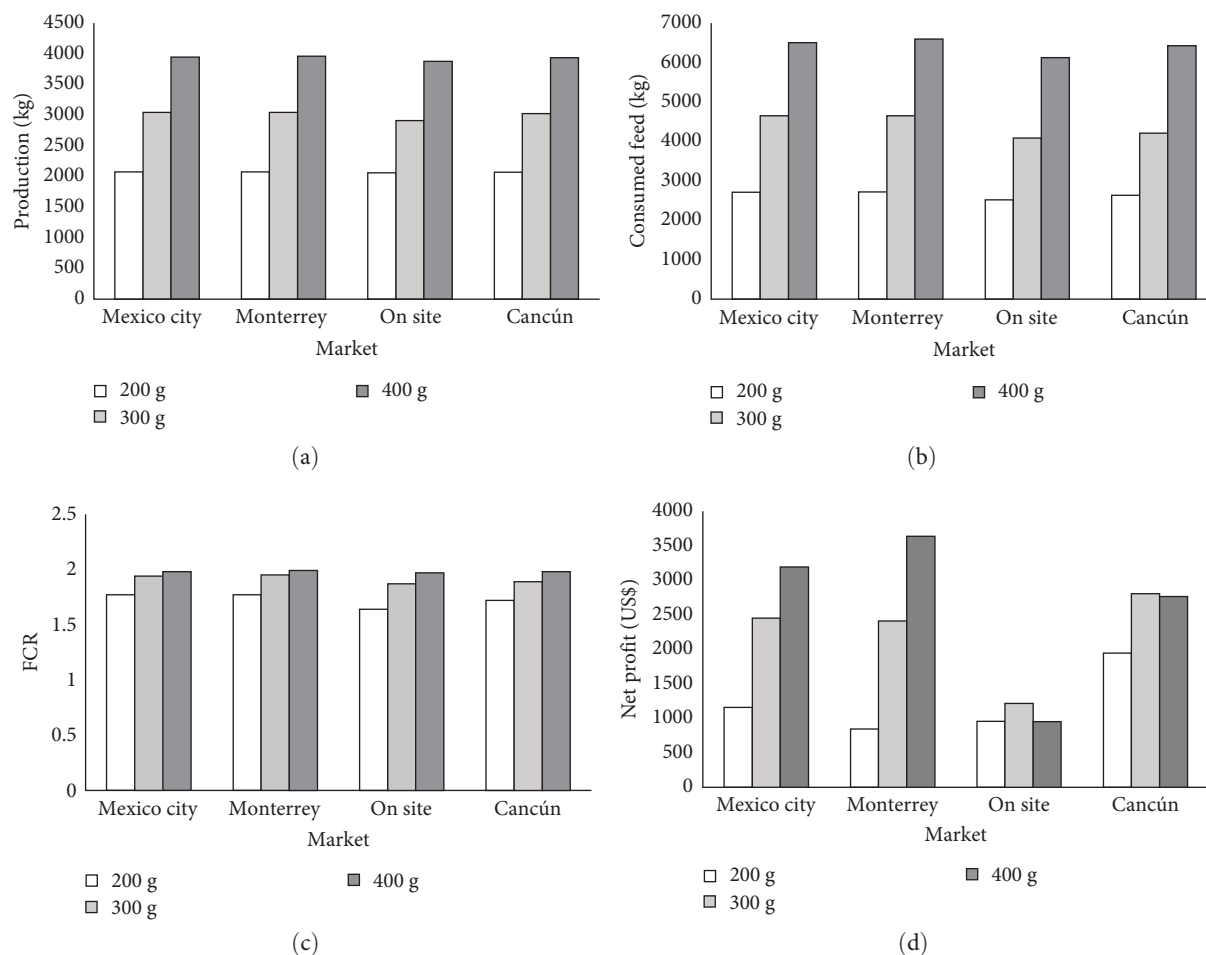
FIGURE 3: Production (a), consumed feed (b), feed conversion rate (FCR) (c), and net profit (d) in optimal rationing by market.

increases first after birth and then decreases. In the case, the effect of this principle was reflected in the size of the ration. That is, when organisms are stocking, they are initially more efficient in converting the food consumed, which could indicate a possible decrease in ration size. However, when the organisms reach maturity, for example, due to the reproductive phase, the organisms become less efficient in growth, which would indicate a higher requirement for the amount of feed consumed [23, 63, 64, 66]. In the case of tilapia, sexual maturity occurs between 80 and 100 g [67], and in this work, the minimum sizes reached in preadult tilapia in the minimum ration, using the von Bertalanffy model, were in the range of 67.39–174.96 g. Obtaining the lowest in the Cancún market and the highest in the Monterrey market, in market sizes of 200 g and 400 g, respectively.

Several methods are used to calculate ration size in aquatic systems. These include estimations based on the use of feeding tables, feeding equations, and growth predictions. The methods used vary among species and aquatics systems. The results of this work could be useful for producers of aquatic systems to improve their feeding strategies, since it finds an optimal dynamic ration size depending on the fish size, similar to Domínguez-May et al. [5]. However, the bioeconomic results of this study contrast with those reported by these authors, for the same species and ration sizes, where a potential growth model was used to determine the optimal ration. By example, the dynamics of the optimal ration found by these authors show a decreasing trend as the size of the organisms increases. This behavior is in agreement with the recommendations of feed suppliers [11]. This study considered the von Bertalanffy model, which takes into account the decreasing and increasing trend of the ration derived from the anabolism and catabolism parameters and the ration effect function, respectively. Some controversies were found in these studies, as OHT, optimal size, and feed consumed were higher than those shown by Domínguez-May et al. [5], while FCR and net profit were lower. However, marketing costs in that work were calculated after OHT, whereas in this study, they were directly integrated into the farm cost structure. Hernández, León-Santana, and León [19] and León, Hernández, and León-Santana [38] used a potential-type growth model (without the catabolism parameter) and determined long-term decreasing and increasing trends of optimal rationing of sea bream (S. aurata) while considering the effect of temperature. This work is similar to that of Arnason [37], although this author stressed that optimal feeding paths that maximize net benefits can take any form, depending on the growth function and the conditions of the problem. However, considering an autonomous growth

function and free terminal conditions, one would expect non-decreasing optimal feeding trajectories.

An important contribution of this work in aquatic systems is to show that the benefit generated by using a potential growth type model could possibly underestimate the benefit. This could be because the feed consumed in the optimal rationing trajectories using the sigmoidal-type model in this work was slightly higher than that obtained using a potential-type model [5] in similar conditions, in which the increase was between 0.62% and 13.80% at the Monterrey and On site markets, at 200 and 400 g sizes, respectively, and this can also be verified with the slightly higher FCRs. However, it is confirmed that providing optimal levels of rationing to individuals could be beneficial in terms of food cost savings [5, 25, 68], since organisms fed at suboptimal ration levels, such as satiety or higher, generate a high FCR [6, 23, 60]. In this case, these discrepancies can be attributed to the different markets and optimal sizes of fish encountered. However, FCR may also be due to feed composition, genetic potential, fish size, life stages, and nutrient efficiency, among other aspects [60]. Mortality rate or heterogeneous growth due to reduced access to feed or low feed conversion efficiency can also reduce productivity and profits [69]. In addition, in a satiating feeding regime, fish show low feed efficiency, as occurs in other species such as turbot, drum, rainbow trout, and carp [6, 60, 70]. In the latter case, it has also been shown in other work on commercial-sized production tilapia [5, 25, 53], and these authors showed that the 80% ration size generated the highest economic returns in tilapia farms.

The study also revealed different optimal rationing trajectories depending on final selling prices, increasing the intensity of rationing for higher prices and extending the growing season if the market permits. This result is consistent with Domínguez-May et al. [53] that the target market for tilapia production influences the optimal management of a tilapia farm. Specifically, selling price or discount factors directly influence OHT and sizes [35, 37, 71]. In addition, an optimal feed management strategy could result in farmed fish playing an important role in reducing feed use and improving sustainability through reduced costs and environmental impact, e.g., lower total ammonia nitrogen (TAN) production [5, 72–75].

5. Conclusion

From a bioeconomic model of an aquaculture culture, considering a modified von Bertalanffy growth model, dependent on size and ration, and adjusted to tilapia culture in Mexico, it was found that a convex rationing curve throughout the production cycle obtained the highest economic returns. This functional form shows that fish can feed according to their size. In bioeconomic terms, this work makes an important contribution to efficient feed management in tilapia culture for different destinations and market sizes. This can be used by producers to obtain higher profits and make the culture more environmentally sustainable by generating less waste per feed. In the case study, the best market for tilapia farmers in Yucatan, Mexico, in economic terms, is the Monterrey market at 400 g, and in terms of reducing feed use, the On site farm market at 200 g size (less beneficial in economic terms). The growth model can include factors such as temperature or stocking density to measure their effect on the optimal feeding route of cultured organisms in aquatic systems. Also, size heterogeneity is an issue to be analyzed in future research.

Appendix A: Instant Ration Rate

By ignoring the t from x and r for Equation (13), we obtained that

$$\frac{\partial L_C}{\partial r} = -xE_r - c_f\xi_r g + g_r(\lambda_C(t) - c_f\xi) + \mu_{1|} - \mu_2 = 0. \tag{A.1}$$

Assuming that $\mu_1(t)r = 0$ and $\mu_2(t)(1 - r) = 0$ are fulfilled, costate variable $\lambda_C(t)$ is cleared out from the Langrangian function, thus turning to

$$\lambda_C(t) = \frac{\varphi}{g_r}, \tag{A.2}$$

where $\varphi = xE_r + c_f(\xi_r g + \xi g_r)$, and responding $\frac{d\varphi}{dt}$, we obtain

$$\frac{d\varphi}{dt} = \dot{x}E_r + xE_{rr}\dot{r} + c_f\left\{\frac{d\xi_r}{dt}g + \xi_r\frac{dg}{dt} + \frac{d\xi}{dt}g_r + \xi\frac{dg_r}{dt}\right\}. \tag{A.3}$$

Finding $\frac{d\xi_r}{dt}, \frac{dg}{dt}, \frac{d\xi}{dt}$, and $\frac{dg_r}{dt}$ and replacing them in Equation (A.3), we obtain

$$\frac{d\varphi}{dt} = \dot{x}E_r + xE_{rr}\dot{r} + c_f\{(\xi_{rx}\dot{x} + \xi_{rr}\dot{r})g + \xi_r(g_x\dot{x} + g_r\dot{r}) + (\xi_x\dot{x} + \xi_r\dot{r})g_r + \xi(g_{rx}\dot{x} + g_{rr}\dot{r})\}. \tag{A.4}$$

From this latter Equation (A.4), we factorize the terms $\dot{x}$ and $\dot{r}$, and then, we have

$$\frac{d\varphi}{dt} = \dot{x}\{E_r + c_f[\xi_{rx}g + \xi_r g_x + \xi_x g_r + \xi g_{rx}]\} + \dot{r}\{xE_{rr} + c_f[\xi_{rr}g + \xi_r g_r + \xi_r g_r + \xi g_{rr}]\}. \tag{A.5}$$

Now denoting
$\phi_1 = E_r + c_f\xi_{rx}g + c_f\xi_r g_x + c_f\xi_x g_r + c_f\xi g_{rx}$ and $\phi_2 = xE_{rr} + c_f\xi_{rr}g + c_f\xi_r g_r + c_f\xi_r g_r + c_f\xi g_{rr}$,

we rewrite Equation (A.5) as

$$\frac{d\varphi}{dt} = \dot{x}\phi_1 + \dot{r}\phi_2. \qquad (A.6)$$

Substituting the derivatives $\frac{d\varphi}{dt}$ (Equation (A.6)) and $\frac{dg_r}{dt}$ from Equation (A.2), we obtain

$$\frac{d\lambda_C(t)}{dt} = \frac{(\dot{x}\phi_1 + \dot{r}\phi_2)g_r - \varphi(g_{rx}\dot{x} + g_{rr}\dot{r})}{(g_r)^2}$$
$$= \frac{\dot{x}(\phi_1 g_r - \varphi g_{rx}) + \dot{r}(\phi_2 g_r - \varphi g_{rr})}{(g_r)^2}. \qquad (A.7)$$

Furthermore, the costate variable defined by the maximum principle through L_C [48, 49, 76] and denoted as

$\theta = E(r) + c_f(\xi_x g + \xi g_x) + c_m$, we have

$$\dot{\lambda}_C(t) = -\frac{\partial L_C}{\partial x} = \theta - \lambda_C(t)(g_x - h). \qquad (A.8)$$

Substituting the costate variable from Equation (A.2) to Equation (A.8) and equaling the two expressions' costate variable, Equations (A.7) and (A.8), we have

$$\frac{\dot{x}(\phi_1 g_r - \varphi g_{rx}) + \dot{r}(\phi_2 g_r - \varphi g_{rr})}{(g_r)^2} = \theta - \frac{\varphi}{g_r}(g_x - h). \qquad (A.9)$$

From this latter, the instant optimal ration rate $\dot{r}(t)$ is cleared.

From this latter, the instant optimal ration rate $\frac{dr}{dt}$ is cleared. Therefore, the optimal trajectories for x and r follow the following equations:

$$\frac{dx}{dt} = g, x(0) = x_0 \quad \frac{dr}{dt} = \frac{g_r\{g_r\theta - \varphi(g_x - h)\} - \frac{dx}{dt}(\phi_1 g_r - \varphi g_{rx})}{\phi_2 g_r - \varphi g_{rr}} \text{ with } \phi_2 g_r - \varphi g_{rr} \neq 0. \qquad (A.10)$$

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Ethics Statement

The authors confirm that the ethical policies of the journal have been followed, in accordance with the official Mexican standard (NOM-062-ZOO-1999) on technical specifications for the production, use, and care of laboratory animal.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

Roger Domínguez-May contributed to conceptualization, formal analysis, and writing original draft; Gaspar R. Poot-López contributed to methodology and conceptualization; Juan M. Hernández contributed to methodology and formal analysis, and Iván Velázquez-Abunader contributed to writing, review and editing. Roger Domínguez-May should be considered the first author.

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