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LOGISTIC - STOCHASTIC GROWTH AND PATCHY DISTRIBUTIONS IN THE SEA

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Abstract : Patchiness is the name given to a heterogeneous, unequal, spatial distribution of many populations. A simple mathematical model to explain the temporal evolution of patch sizes is built. The model introduces random parameters in a basic differential equation which rules the logistic growth of patch sizes thus obtaining a stochastic differential equation whose associated Fokker - Planck equation is solved afterwards.

1. INTRODUCTION .

Logistic growth is one of the most important tools in the modelling of various problems in Ecology. Roughly speaking, it is a variant of the classical malthusian growth law $y' = ky$ where k is dependent on the population size y and on some limiting factor. In general, this factor represents the maximum population that can survive on the available resources, although short periods can happen where the population is greater than this maximum. On the other hand, small-scale phenomena can modify the logistic growth path in such a way to render it difficult to recognize.

The contribution of these must be added to the logistic pattern, in order to obtain an equation where both large and small-scale factors are represented.This can be achieved by way of a stochastic differential equation, whose solution is the probabilistic distribution of the population size rather than its actual size.

2. PATCH SIZES IN THE OCEAN.

The study of patches of several substances or living beings is of foremost importance in the field of marine sciences, where estimation of patch sizes and of their spreading mechanisms is an active research field. When one deals with patches whose constituents are passive, logistic growth is a rather adequate modelling for the estimation of patch sizes. First of all, we shall suppose that patches are elongated bodies whose size can be described by the diameter $L(t)$, where time dependence shows the variability of L. Second, the size of the patch depends on how energy is fed into it. Two sources are available: a) large or medium-scale energy-containing eddies characteristic of the ocean zone where the patch appears. b) small-scale eddies responsible for very small variations in the patch size. The scale of these is so small with respect to the energy -containing eddies that they can be considered as noise. In any definite ocean area one can find a typical scale for the energy-containing eddies. It is natural to think that whenever the size of a patch is greater than this typical scale, the patch will break into smaller patches. Thus, the typical scale is a limiting factor for the patch size. Now we write E for this typical scale and find the logistic expression for $L(t)$:

$$\frac{dL}{dt} = \lambda\left(1 - \frac{L}{E}\right) L \qquad (1)$$

where λ models features of the ocean ambient. A typical interpretation for λ is the stress tensor given by the velocity gradient within the eddies. Now we turn to small-scale energy transfers. If the scale is very far from the typical scale, we find that fluctuations are much faster than those due to the general logistic pattern. Thus we model them as

$$\frac{dL}{dt} = \mu(t)\, L \qquad (2)$$

where exponential growth is prevented by the changing pattern of $\mu(t)$. This $\mu(t)$ is thought of as a stochastic process; thus equation (2) is a stochastic differential equation. Physical considerations allow us to write the expression

$$\mu(t) = \sqrt{2k}\;\; \phi(t) \qquad (3)$$

where $\phi(t)$ is white noise. The parameter k models the intensity of randomness and can be interpreted in various ways. One of them is the effect of shear stresses and of molecular viscosity. By adding equations (2) and (3) we find a simple model for patch size in the ocean:

$$\frac{dL}{dt} = \lambda\left(1 - \frac{L}{E}\right) L + (2k)^{1/2}\phi(t)\, L \qquad (4)$$

This is a Langevin-type stochastic differential equation whose solution is some stochastic process $L(t)$, instead of a deterministic function.

3. The Fokker–Planck equation for patch size.

Equation (4) is usually interpreted as shorthand for the physically formulated difference equation:

$$\Delta L = \lambda \left(1 - \frac{L}{E} \right) L \, \Delta t + \sqrt{2k} \quad L \, \Delta W(t)$$

where $\Delta W(t)$ is the increment of the Wiener process. This process (also called Brownian motion)is defined as the stochastic Ito integral of white noise. Equations (4) and (5) can be used to show that the solution process $L(t)$ is a Markov process.

In effect, we see that the formal solution to the Langevin equation (4), with the initial condition $L(0)=0$ is:

$$L(t) = e^{\sqrt{2k}} \; e^{\int_0^t \alpha(L(s))\, ds} \; e^{\int_0^t \phi(s)\, ds}$$

from which we obtain

$$L(t+\Delta t) = L(t) \; e^{\int_t^{t+\Delta t} \alpha(L)\, ds} \; e^{\int_t^{t+\Delta t} \phi(s)\, ds}$$

Since $\phi(t)$ is δ-correlated, its values in the interval $[t, t+\Delta t]$ are independent of the previous history and $L(t+\Delta t)$ does not depend on the history preceding $L(t)$. Therefore the process $L(t)$ is Markovian.

Homogeneity of the process can be assumed on physical grounds. Owing to the inertial system that prevails in the oceans, the joint probability densities depend only on the time difference between observations. In this way, time homogeneity expresses the invariance of the mechanism which generates fluctuations.

Thus, a Fokker-Planck equation (FPE) can be formulated for our process, whose drift and diffusion terms are calculated in the standard way:

$$\frac{\partial \rho(L,t)}{\partial t} = - \frac{\partial}{\partial L} \left[\lambda(1 - \frac{L}{E})L \rho\right] + k \frac{\partial^2}{\partial L^2}(L^2 \rho) \qquad (5)$$

The solution of this FPE is simply the probability distribution $\rho(L,t)$ of finding a size L at time t. As it is a continuous distribution, it must be interpreted as the probability of finding L at time t between some fixed values.

Now we proceed to find a solution of the FPE. It is reasonable to think that the steady state is natural in normal conditions, under the assumption that the environmental eddies are in statistical equilibrium within the tidal period.

The FPE (5) can be written as

$$\frac{\partial}{\partial t} \rho(L,t) = L_{fp}\, \rho(L,t) \qquad (6)$$

where

$$L_{fp} = - \frac{\partial}{\partial L} A(L) + \frac{\partial}{\partial L^2} B(L)$$

and $A(L)$ and $B(L)$ are time-independent drift and diffusion coefficients respectively.

Now equation (6) may be written in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial L} = 0 \qquad (7)$$

where

$$J(L,t) = \left[A(L) - \frac{\partial}{\partial L} B(L) \right] \rho(L,t) \qquad (8)$$

Because (7) is a continuity equation for a probability distribution, J may be interpreted as a probability current, i.e. probability flow.

Therefore, we concentrate on solving the stationary FPE with the supplementary normalization:

$$\int_0^\infty \rho(L)dL = 1$$

4. The Gamma Distribution.

Solving the stationary FPE, we obtain the following results. For a stationary procces the probability current must be a constant. Nevertheless, the stochastic variable L cannot reach values smaller than zero, so we require that the probability current be zero at $L = 0$. Thus, the probability current vanishes for all L. Setting $J = 0$, we rewrite equation (7) as

$$A(L)\, \rho_s(L) = \frac{\partial}{\partial L} B(L)\, \rho_s(L)$$

for which the solution is obtained by a single integration as

$$\rho_s(L) = \frac{N}{B(L)} \exp\left[\int_0^L \frac{A(S)}{B(S)}\, dS \right]$$

where N is a normalisation constant such that

$$\int_0^\infty \rho_s(L)\, dL = 1$$

We then obtain

$$\rho_s(L) = N \frac{1}{kL^2} L^{\frac{\lambda}{k}} e^{-\frac{\lambda}{kE} L}$$

with

$$N = \frac{\lambda}{\Gamma(\frac{\lambda}{k} - 1)\ E\ (\frac{Ek}{\lambda})^{\frac{\lambda}{k} - 2}}$$

Finally, we write the expression for stationary $\rho(L)$:

$$\rho_s(L) = \frac{1}{\Gamma(b)} \frac{1}{a} \left[\frac{L}{a}\right]^{b - 1} e^{-\frac{L}{a}}$$

where we have introduced $a = \frac{E\ k}{\lambda}$ and $b = \frac{\lambda}{k} - 1$

This happens to be a gamma distribution depending on two parameters: E and b. b models the relative incidence of large and small-scale eddies in the spreading of the patch. When b is large the energy-containing eddies dominate, and if b is small, the more chaotic behaviour of the small-scale eddies prevents the patch from attaining a size similar to E. This is shown in the accompanying graphs, where the evolution of $\rho(L)$ according to E and b is represented. It is interesting to note that the larger b is, the closer is the mode distribution to the typical scale E.

5. Acknowledgements.

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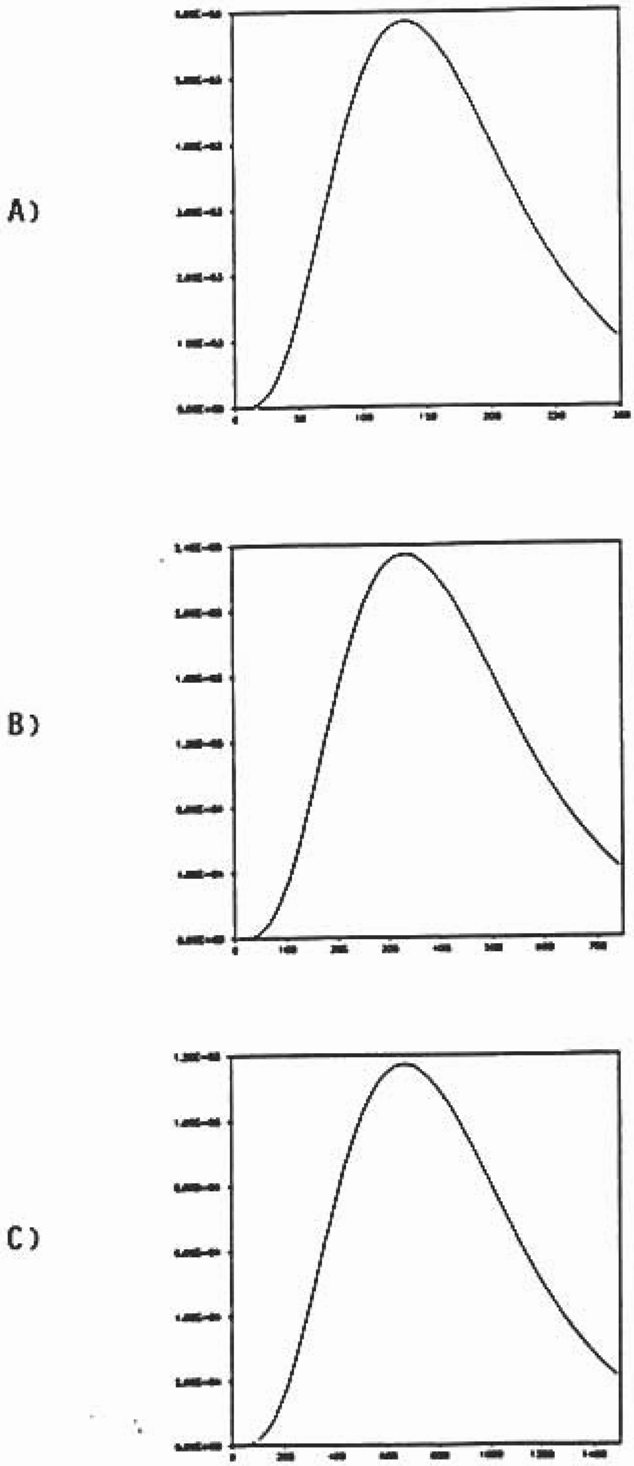

Figure 1: The Gamma distributions for b=5 and A) E=200, B) E=500, C) E=1000.

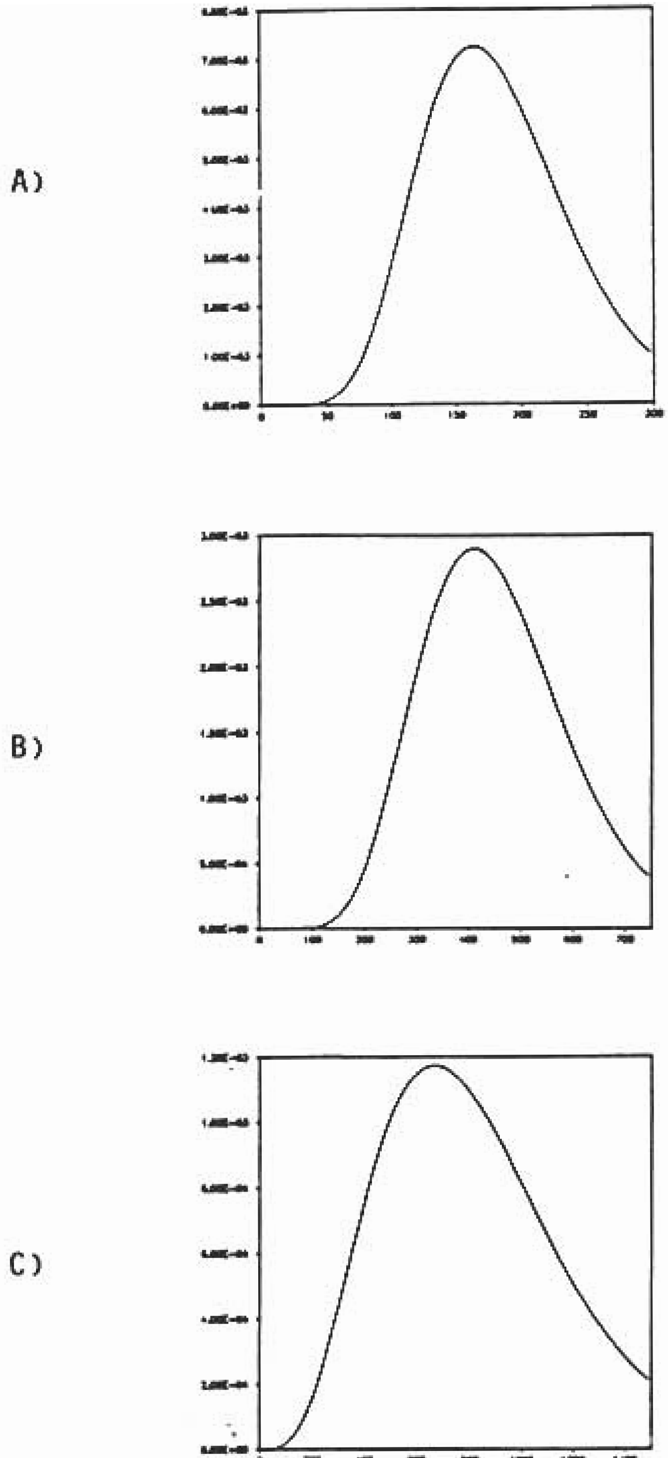

Figure 2: The Gamma distributions for b=10 and A) E=200, B) E=500, C) E=1000.

A)

B)

C)

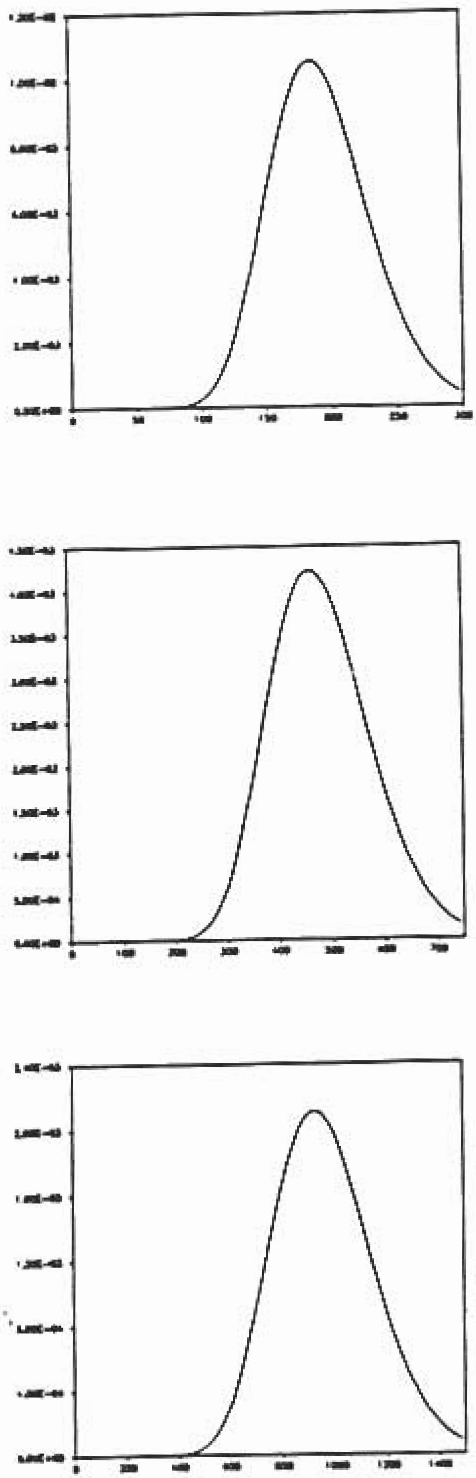

Figure 3: The Gamma distributions for b=25 and A) E=200, B) E=500, C) E=1000.

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