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# Finite number of similarity classes in Longest Edge Bisection of nearly equilateral tetrahedra

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ABSTRACT

In 1983 Adler [1] pointed out that if a tetrahedron is nearly equilateral (edge lengths within 5% of each other) and the first and second longest edges are opposite, then the iterative Longest Edge Bisection (LEB) method produces $\leq 37$ similarity classes. The importance of nearly equilateral tetrahedra is that they generate a finite number of similarity classes during the iterative LEB, a desirable property in Finite Element computations.

We prove the conjecture given by Adler and improve the bound of 5% to 22.47%. A new algorithm is introduced for the computation of similarity classes in the iterative Longest Edge Bisection (SCLEB) of tetrahedra using a compact and efficient edge-based data structure.

## 1. Introduction

The LEB method began to be applied only to triangles in a series of works around three decades ago. First, Rosenberg and Stenger [11] proved that the method does not degenerate the smallest angle in 2D. Then Kearfott [5], Adler [1], and Stynes [18] gave a bound and proof for the behavior of the triangle diameters (the length of the longest edge). From their proofs, they also deduced that the number of similarity classes of triangles generated is finite, although they give no bound.

Algorithms for bisection (not necessarily bisecting the longest edge) were developed in 3D by Bansch [3], Maubach [8], Liu Joe [6,7], and Arnold et al. [2]. The stability of these methods is proved for any targeted tetrahedral mesh. All are equivalent and affine invariant when applied iteratively to any element, generating at most 36 similarity classes [4].

The LEB method of tetrahedra has been studied by Plaza and Rivara [9,10], among others. Although it is known that different types of tetrahedra behave computationally "well" or "bad", especially in meshes used in Finite Element computations, no systematic study is known [14,15]. Stability has not been proved yet for the LEB of tetrahedra.

A bisection method is stable if the tetrahedra generated in the iterative refinement does not degenerate. Degeneration can be prevented if the interior angles are always greater than zero.

A finite number of similarity classes is not only sufficient but also necessary for stability, [4]. In practical situations, the number of similarity classes should be finite and as small as possible. For example, there are data depending on the element's similarity classes and refinement level only. This means that a small and bounded number of similarity classes would lead to fast executions of programs if data of this type are computed and stored only once.

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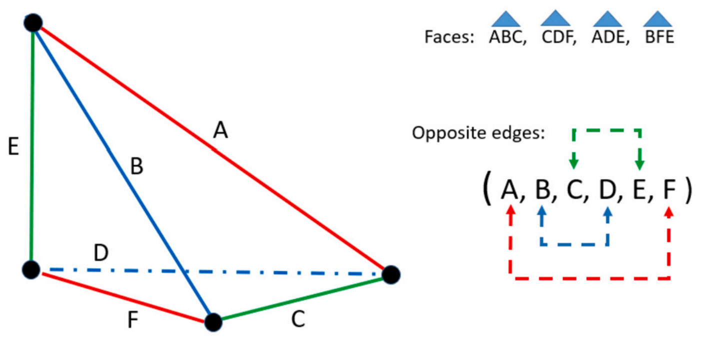


**Fig. 1.** Sextuple Representation of an arbitrary tetrahedron.

A problem that remains open is the question of whether the iterative LEB of tetrahedra produces a finite number of new shape classes or not. In 1983, Adler [1] conjectured that there exist certain classes of tetrahedra that produce up to 37 similarity classes when LEB is iterated. Adler did not provide any proof of his statement although he mentioned that the tetrahedra with this relevant property should hold: (1) to be nearly equilateral and (2) the longest edge and the second longest edge are opposite. Adler did not give any precise definition of "nearly equilateral" and indicated the complexity of describing it briefly. However, Adler pointed out that it is satisfied if all edge lengths are within 5% of each other. Since this contribution by Adler, no other work has studied the LEB of nearly equilateral tetrahedra. Experimental results and the algorithm presented in [13] show that, in practice, the Adler statement was correct, although no rigorous proof of his ideas is given.

Indeed, the LEB of a regular tetrahedron has been the focus of interest in some papers such as [12,16,17]. The complex nature of LEB of a regular tetrahedron is analyzed in [17] where it is claimed the difficulty of the theoretical analysis as the bifurcation branches grow exponentially if a random criterion is chosen for the bisection edge. In [16] a Branch-and-Bound algorithm is given to obtain the minimum number of classes, eight, of a regular tetrahedron where the critical point is how to select the longest edge to be bisected. They find the rule by computing the smallest sum of all angles with the neighbor edges, $LEB\alpha$. This way the bigger angles are split avoiding needle-shaped simplices. The selection rule $LEB\alpha$ is optimal for the algorithm, although evaluating the rule for every tetrahedron generated is computationally more complex than a rule just selecting the first longest edge for subdivision, [16].

In this paper, we study the family of nearly equilateral tetrahedra that Adler roughly described in [1].

Section 2 introduces an efficient edge-based data structure for tetrahedra class representation, and an algorithm for the computation of similarity classes in the Longest Edge Bisection (SCLEB) of tetrahedra is proposed. In Section 3 we study the regular tetrahedron and verify that, using our proposed data structure, the iterated SCLEB produces only 8 similarity classes. Section 4 presents a precise definition of the concept of a nearly equilateral tetrahedron, and we study the conditions for the convergence into a finite number of similarity classes. In Section 5 we give a proof and rigorous result with tetrahedra shape classes that indeed shows that the iterated SCLEB applied to nearly equilateral tetrahedra introduces at most 37 similarity classes. Section 6 studies the particular case of the SCLEB of nearly equilateral tetrahedra with repeated edges, which converge in less than 37 similarity classes. Finally, in section 7 we improve the Adler bound of 5% for nearly equilateral tetrahedra, expanding up to 22.47%.

## 2. Tetrahedra representation for efficient LEB

Let us represent a tetrahedron as a sextuple with the square lengths of its 6 edges in a certain order as in [13], called here, sextuple representation of a tetrahedron. In this way, the position and orientation of a tetrahedron are not taken into account, and only its geometric shape is represented. The order of these 6 values is very important. The first position in the sextuple can be any of the six edges, let's call $a$ (see Fig. 1). The second, $b$, is any edge connected to $a$. The third edge $c$ is the edge that closes the triangle $\widehat{abc}$. The fourth edge $d$ is the edge connected to $a$ and $c$. The fifth edge $e$ is the closing edge of the $\widehat{ade}$ triangle. The sixth edge $f$ is the last edge opposite to the initial edge $a$. Then an arbitrary tetrahedron can be represented by the sextuple $(A, B, C, D, E, F)$, where the capital letters represent the square lengths of the edges $a, b, c, d, e, f$. It is also important to note that this representation guarantees that the pairs of opposite edges in the tetrahedron are always $A$-$F$, $B$-$D$, and $C$-$E$, as seen in Fig. 1. As an example, the regular tetrahedron with all its edges of length 1 is given by the sextuple $R = (1, 1, 1, 1, 1, 1)$, and the trirectangular tetrahedron given by the coordinates $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ is represented by the sextuple $(2, 2, 2, 1, 1, 1)$.

There are 24 different sextuples to represent the same tetrahedron: there are 6 possibilities depending on the edge chosen as the first value of the sextuple. For each of them, 4 different alternatives depending on which edge, neighbor of the first, is taken as the second value.

**Definition 1.** We call the normalized sextuple representation of a tetrahedron, as defined in [13], the sextuple that places the longest edge as the first value, and the neighboring edge with the longest length as the second value. In the case of repeated edges, we choose the sextuple that places the highest possible values in the first positions of the sextuple.

**Remark 1.** Note that following Definition 1, the normalized representation of a similarity class guarantees that $A \geq B, C, D, E, F$, and $B \geq C, D, E$.

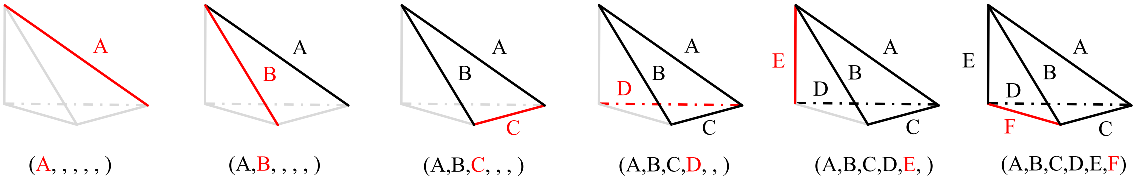


**Fig. 2.** The six steps to obtain the normalized sextuple representation for a tetrahedron $(A, B, C, D, E, F)$.

Fig. 2 shows how the edges are picked up to set the normalized sextuple representation for a tetrahedron $(A, B, C, D, E, F)$. The longest edge $A$ is always taken as the first value of the sextuple. Of the 4 edges connected to $A$, $B$ is the longest. $C$ closes the triangle $\widehat{ABC}$. $D$ is the edge that connects to $A$ and $C$. The edge closing the triangle $\widehat{ADE}$ is $E$. $F$ is the last edge, opposite to $A$.

A similarity class is a set of tetrahedra with the same geometric shape, regardless of their specific position, orientation, and scale. So any two tetrahedra in the same family are similar to each other after applying any affine transformation. The following definition relates this concept with our representation.

**Definition 2.** A similarity class is represented by $k(A, B, C, D, E, F)$ in the normalized sextuple representation, for $k \in \mathbb{R}^+$ being a scale factor.

Without loss of generality, from now on we can omit the factor $k$ and use brackets to represent a similarity class, $[A, B, C, D, E, F]$, and parenthesis to represent a single tetrahedron. In this manner, $[A, B, C, D, E, F] = k(A, B, C, D, E, F), \forall k \in \mathbb{R}^+$.

**Definition 3.** Given two tetrahedra, $T_1$ and $T_2$, both belong to the same similarity class if $T_1 = kT_2$, being $k \in \mathbb{R}^+$.

**Remark 2.** The normalized sextuple representation is a suitable data structure for representing similarity classes of tetrahedra. Moreover, the cost of comparing two tetrahedra classes $T_1$ and $T_2$ given by their sextuple representation implies only a sextuple comparison ($T_1 = kT_2$). This is considerably cheaper computationally than other methods involving vertex-based representation, as in [16,17] that use matrix computations as determinants, transpose, scaling, and translation in high-precision arithmetic.

**Definition 4.** The LEB of a tetrahedron $T = (A, B, C, D, E, F)$ is the subdivision of $T$ by its longest edge ($A$) to produce two new tetrahedra $T_1$ and $T_2$ given by:

$$T_1 = \frac{1}{4}(A,\ 4B,\ 2B + 2C - A,\ 2E + 2D - A,\ 4E,\ 4F) \tag{1}$$

$$T_2 = \frac{1}{4}(A,\ 2B + 2C - A,\ 4C,\ 4D,\ 2E + 2D - A,\ 4F) \tag{2}$$

**Remark 3.** As in this paper, we are dealing with similarity classes of tetrahedra, the expression for the classes of $T_1$ and $T_2$ can be simplified by multiplying both Equations (1), (2) by 4. Thus, for example, the class $R = [1, 1, 1, 1, 1, 1]$ produces the new classes $T_1 = [1, 4, 3, 3, 4, 4]$ and $T_2 = [1, 3, 4, 4, 3, 4]$.

To continue applying the LEB to the newly generated similarity classes, it is necessary to convert them to their normalized representation previously. This representation guarantees that the longest edge is always at the first position in the sextuple. For example, the two new classes $T_1$ and $T_2$ generated by the LEB of the regular $R$ are normalized as $[4, 4, 4, 3, 3, 1]$, so it follows that both belong to the same similarity class. The process of subdividing tetrahedra by the Longest Edge Bisection that only obtains similarity classes of tetrahedra leads us to define a particular algorithm called here Similarity Classes Longest Edge Bisection (SCLEB):

**Definition 5.** The Similarity Classes Longest Edge Bisection (SCLEB) of a similarity class $T = [A, B, C, D, E, F]$ can be defined as the following two-step process:

1. Subdivision of T to get:

$$T_1 = [A,\ 4B,\ 2B + 2C - A,\ 2E + 2D - A,\ 4E,\ 4F] \tag{3}$$

$$T_2 = [A,\ 2B + 2C - A,\ 4C,\ 4D,\ 2E + 2D - A,\ 4F] \tag{4}$$

2. Normalize representation of T1 and T2.

Definition 5 allows us to roughly determine an algorithm for the iterative SCLEB. Given an input tetrahedron by its normalized sextuple representation, an output is generated with the list of all the similarity classes obtained through the application of the iterative SCLEB to all of the descendants of the input tetrahedron. The output is a list of normalized sextuples representing each one of the similarity classes obtained.

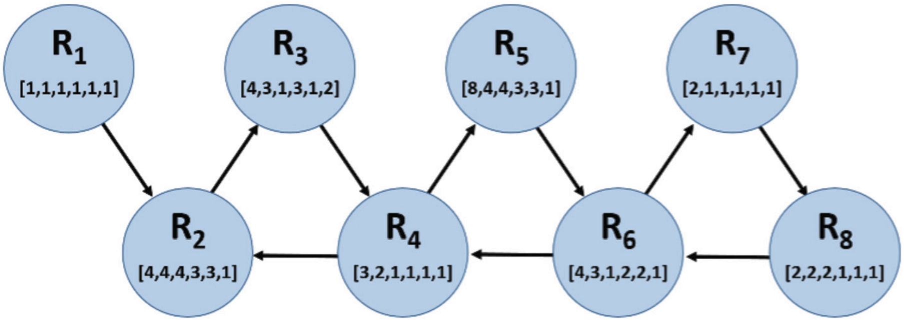


**Fig. 3.** Graph of similarity classes generated in the iterative SCLEB of class $R_1 = [1,1,1,1,1,1]$.

**Remark 4.** The iterative SCLEB algorithm discards those similarity classes that already appeared. In the case that after several iterations no new classes appear, then the algorithm stops and returns a finite list of similarity classes. Furthermore, we say that the iterative SCLEB algorithm converges.

## 3. SCLEB of the regular tetrahedron

Adler in [1] mentioned that the LEB of certain classes of tetrahedra generate at most 37 similarity classes. He focused on tetrahedra that are nearly equilateral, this is, tetrahedra with a shape very close to the regular. This suggests first studying the SCLEB of a regular tetrahedron.

**Lemma 1.** *The iterative SCLEB of the regular similarity class produces exactly 8 similarity classes.*

**Proof.** Let $R_1 = [1,1,1,1,1,1]$ be the similarity class of the regular tetrahedron. If we apply iteratively the SCLEB to $R_1$ using the Equations (3), (4), a binary tree is constructed where each level in the tree is a level of SCLEB subdivision. We remark here that the SCLEB algorithm always selects the first edge in the sextuple as the subdivision edge.

In the first SCLEB subdivision, two child tetrahedra of the same class are generated, $R_2 = [4,4,4,3,3,1]$. In the second level, SCLEB of $R_2$ produces two children of the same class, $R_3 = [4,3,1,3,1,2]$. Again, SCLEB of $R_3$ produces two children of the same class, $R_4 = [3,2,1,1,1,1]$. At this point, we have obtained 4 different similarity classes.

Continuing this bisection process from the $R_4$ class, two different classes are produced. One class is again $R_2$ already appeared previously in level 2, and a new class, $R_5 = [8,4,4,3,3,1]$, is appeared. It should be noted that at each SCLEB level, we only continue subdividing new classes generated, discarding classes that previously appeared. From $R_5$, we iterate the SCLEB as before, and in the next levels of subdivision, we obtain classes $R_6 = [4,3,1,2,2,1]$, $R_7 = [2,1,1,1,1,1]$ and $R_8 = [2,2,2,1,1,1]$, together with other classes already appeared before. This last class $R_8$, marks an interesting point in the SCLEB, as no new classes appeared after it. This is, if we continue the SCLEB process from the $R_8$ class, the classes obtained will have already appeared in previous levels ($R_2$ to $R_7$). We then conclude that there are no more than 8 similarity classes in the iterative SCLEB of a regular tetrahedron. □

Fig. 3 shows the graph indicating the eight similarity classes generated as the iterative SCLEB method is being applied to the regular tetrahedron $R_1$. Each node depicts a similarity class, and a directed edge represents the new classes produced in the bisection. In Fig. 4 we show the tetrahedra shapes representing the eight tetrahedra similarity classes.

**Remark 5.** In the SCLEB of the regular tetrahedron, $R_1$, $R_2$ and $R_8$ have more than one longest edge, unlike the other 5 classes $R_3$, $R_4$, $R_5$, $R_6$ and $R_7$. In the case of $R_1$, it does not matter by which longest edge we subdivide, since all six edges are equal. In the same way, in the case of $R_8 = [2,2,2,1,1,1]$, the three longest edges of length $\sqrt{2}$ form a face of the tetrahedron, and it does not matter by which edge we subdivide because it is also symmetric on all three sides.

**Remark 6.** Note that in the SCLEB normalization process, the first edge in the sextuple is always one of the longest edges. In the case of $R_2 = [4,4,4,3,3,1]$, it holds that the opposite edge to the initial one in the sextuple is the one with the smallest length (given by 1, the last value of the sextuple). This last fact is essential because it defines in itself the rule to guarantee the convergence of the number of similarity classes. If we had subdivided any of the other two longest edges, opposite to the edges of length $\sqrt{3}$, the newly generated classes would be completely different, and the subdivision process would not converge.

Following the same idea by Adler to focus on the family of nearly equilateral tetrahedra, we are interested in studying the conditions for which a nearly equilateral tetrahedron follows the same graph of Fig. 3. In the next section, we study the conditions for this family of tetrahedra.

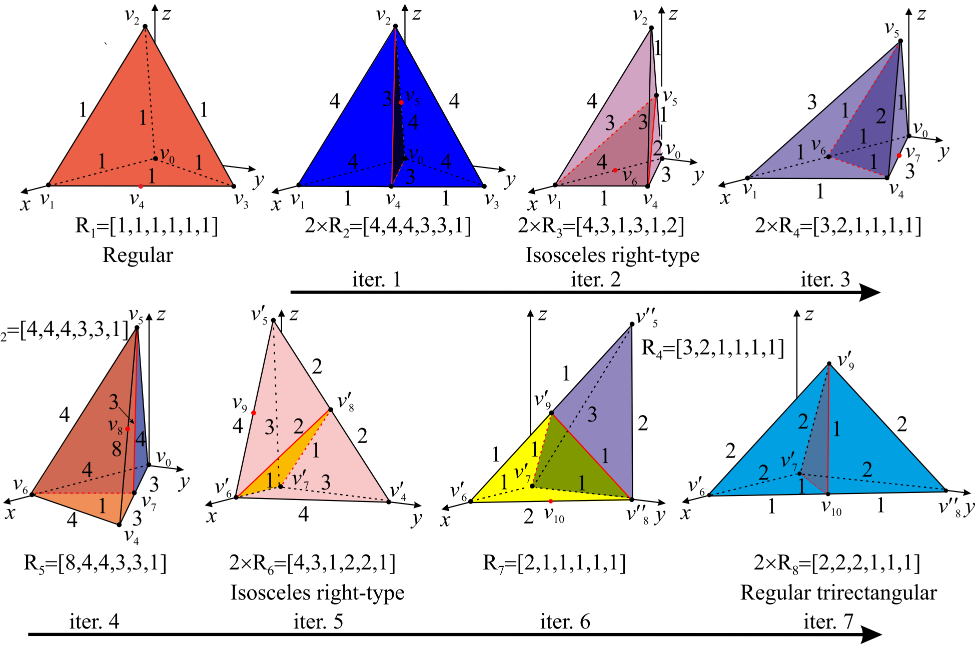


**Fig. 4.** Shapes representing the eight tetrahedra similarity classes that are generated in the SCLEB of the regular class $R_1$.

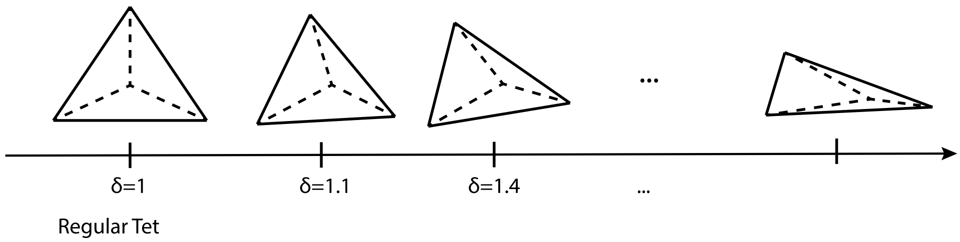


**Fig. 5.** Sequence of delta values starting from the regular tetrahedron class.

## 4. Revisiting the Adler family of nearly equilateral tetrahedra

In [1] Adler mentioned that a tetrahedron can be considered nearly equilateral if all its edge lengths are within 5% of each other, even though he advertised to be slightly complicated to describe briefly. It is interesting to note that to our knowledge, nearly equilateral tetrahedra have not been studied concerning the LEB since 1983. The first question to answer is whether the SCLEB of a tetrahedron whose geometric shape is very close to the regular $R_1$ converges to a finite number of similarity classes or not. The answer is that it only happens with some tetrahedra, and therefore the concept of nearly equilaterality of a tetrahedron must be studied rigorously.

**Definition 6.** Let $\delta(T)$ be the ratio between the square of the longest edge and the square of the shortest edge in any tetrahedron T.

Let us note that $\delta(R_1) = 1$, and $\delta(T) > 1$ for all tetrahedra $T$ different to $R_1$. Fig. 5 shows tetrahedra and their respective $\delta(T)$ values.

It should be noted that as $\delta$ deviates further from 1, the tetrahedron becomes more irregular, and it means that the ratio between the lengths of the longest and shortest edges is increasing. The greater the value of $\delta(T)$ (greater than 1), the more irregular the tetrahedron becomes. Therefore, $\delta(T)$ provides a means to assess the extent to which a tetrahedron deviates from being a regular tetrahedron.

**Definition 7.** A similarity class $T$ can be said to be **nearly equilateral**, expressed as $T \approx R_1$, if $\delta(T) \in [1, \delta_{max}]$, where $\delta_{max}$ is a predefined threshold. For convenience, we call $R_1^*$ to the set of nearly equilateral classes.

The threshold $\delta_{max}$ is a predefined value that demarcates the maximum allowable difference in edge lengths for a tetrahedron to be considered nearly equilateral. The choice of $\delta_{max}$ is crucial as it sets the strictness of the criteria. A tetrahedron with $\delta$ value significantly less than $\delta_{max}$ would closely resemble a regular tetrahedron. A tetrahedron with $\delta$ approaching $\delta_{max}$ would be at the limit of being classified as "nearly equilateral", with its edge lengths exhibiting the maximum permissible disparity under the defined criterion. In summary, the "nearly equilateral" designation quantifies how closely a tetrahedron approaches the uniform edge length characteristic of a regular tetrahedron, within the bounds set by $\delta_{max}$.

Let us note that the idea of nearly equilateral for Adler was based on the criterion that edge lengths are within 5% of each other. That means that the ratio between the lengths of the maximum and minimum edges is less than 1.05. Therefore, as $\delta$ represents the ratio between its square values, this is equivalent to saying that $\delta(T) \leq 1.05^2 = 1.1025$. Let us call this value $\delta_A$, the Adler nearly equilateral threshold. For example, the class $[11, 10, 10, 10, 10, 10]$ with $\delta(T) = 1.1 < \delta_A$ is nearly equilateral under the definition of Adler.

The value of $\delta(T)$ helps us to measure the equilaterality of a tetrahedron. To compare the geometric shape between two arbitrary tetrahedra, we need a new definition.

**Definition 8.** Let $T = [A, B, C, D, E, F]$ and $Q = [A_Q, B_Q, C_Q, D_Q, E_Q, F_Q]$ be two arbitrary classes. We define T./Q as:

$$T./Q = \left[\frac{A}{A_Q}, \frac{B}{B_Q}, \frac{C}{C_Q}, \frac{D}{D_Q}, \frac{E}{E_Q}, \frac{F}{F_Q}\right]. \tag{5}$$

Note that $\delta(T./Q)$ gives a measure to compare the shapes of $T$ and $Q$. Moreover, if $\delta(T./Q) < \delta_{max}$ then $T \approx Q$ and we can say that $T \in Q^*$. For example, for $T = [42, 41, 40, 32, 31, 10]$ and $R_2 = [4, 4, 4, 3, 3, 1]$ and $\delta_{max} = 1.1$ then $\delta(T./R_2) = 1.066 < \delta_{max}$. Then it follows that $T \in R_2^*$.

Not all tetrahedra in $R_1^*$ converge into a finite number of similarity classes. The convergence affects only those tetrahedra following the graph of Fig. 3 during the generation of the SCLEB. Let us see two examples:

**Example 1.** Let $T = [105, 103, 102, 101, 100, 104]$ be a class belonging to $R_1^*$. This class converges after 7 iterations into 37 similarity classes. $T$ follows a subdivision pattern very similar to the regular $R_1$ (see Fig. 3). The first step of the SCLEB for $T$ produces the classes $T_1 = [416, 412, 400, 297, 305, 105]$ and $T_2 = [416, 408, 404, 297, 305, 105]$, both belonging to $R_2^*$, since $\delta(T_1./R_2) = \delta(T_2./R_2) = 1.06$. Let us note that, in this step, there is a first difference with the regular case: two new different classes appear, both very similar to $R_2 = [4, 4, 4, 3, 3, 1]$, but not of the same similarity class. In the next step of SCLEB, these two classes produce four new different classes, all similar to $R_3$. Continuing this process, we get new different classes in each subdivision level, following the same pattern that the SCLEB of $R_1$, until reaching the convergence.

**Example 2.** Let $Q = (105, 104, 103, 102, 101, 100)$ be another tetrahedron belonging to $R_1^*$. The iterative SCLEB of $Q$ does not converge in a finite number of similarity classes. In the first subdivision step, the new classes $Q_1 = [416, 404, 400, 309, 105, 301]$ and $Q_2 = [412, 408, 400, 309, 105, 301]$ appear. The main difference of the new classes concerning those produced in Example 1 ($T_1$ and $T_2$) is that these new classes do not belong to $R_2^*$ (they are not similar, due to the order of the edges in the similarity class). In this case, we have that $\delta(Q_1./R_2) = \delta(Q_2./R_2) = 8.6$. Therefore, they do not follow the same pattern subdivision as in the regular case. Continue this process iteratively to see that $Q$ does not converge in a finite number of similarity classes.

Examples 1 and 2 show that the SCLEB are quite sensible to the precision of edge lengths. This may cause the sextuple normalization for tetrahedra with very close edge lengths can result in different sextuple representations. For example, in $R_2 = [4, 4, 4, 3, 3, 1]$ we see that having its 3 longest edges exactly equal, normalization always puts in the first place to the edge opposite to the one of length 1. However, in tetrahedra $T_1$, $T_2$, $Q_1$, and $Q_2$, whose geometric shapes are very similar to that of $R_2$, it happens that depending on which is the largest edge, it produces a different sextuple order. Therefore, we may find a criterion that allows us to detect which tetrahedra in $R_1^*$ converge in the SCLEB and which do not.

The second condition announced by Adler for the convergence of nearly equilateral tetrahedra concerns whether the second longest edge is opposite to the longest edge. We confirm in Lemma 2 this important condition for the convergence in terms of the normalized sextuple representation of tetrahedra.

**Lemma 2.** *Let $T = [A, B, C, D, E, F]$ an arbitrary similarity class belonging to $R_1^*$. Then the SCLEB of $T$ converges into a finite number of similarity classes if and only if $F \geq B$.*

**Proof.** An alternative form for $T$ is $T = [1, B', C', D', E', F']$, extracting the value of A as a scale factor. In this way, we know that $B', C', D', E'$, and $F'$ are values close to or equal to 1, but never higher.

The SCLEB of $T$ produces two new classes $T_1$ and $T_2$. According to Equation (3), we obtain that $T_1 = [1, 4B', 2B' + 2C' - 1, 2E' + 2D' - 1, 4E', 4F']$. It is easy to see that $T_1$ will have values very similar to $[1, 4, 3, 3, 4, 4]$. Therefore to impose that $T_1 \approx R_2 = [4, 4, 4, 3, 3, 1]$, the longest edge must be $4F'$, whose opposite edge is the one of value 1. Therefore, it must be true that $4F'$ is greater than the other two high values in the sextuple (second and fifth positions). This means that $F' \geq B', E'$.

$T_2$ follows the expression $[1, 2B' +' 2C' - 1, 4C', 4D', 2E' + 2D' - 1, 4F']$, which is similar to $[1, 3, 4, 4, 3, 4]$. Proceeding in the same way, $T_2$ will belong to $R_2^*$ if $4F' \geq 4C', 4D'$. That means that, $F' \geq C', D'$.

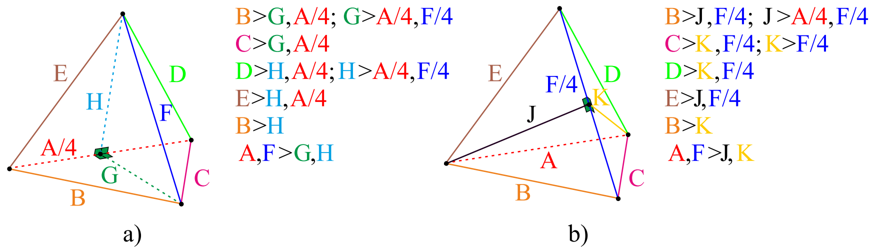


**Fig. 6.** Inequalities of the original lengths of $T$.

Since $B \geq C, D, E$ (see Remark 1), it is easy to see that the condition of convergence can be simplified as $F \geq B$. As in the normalized sextuple representation it is also fulfilled that A and F are opposite, we can conclude that this statement is equivalent to the one conjectured by Adler, who indicated that the second longest edge must be opposite to the longest edge. □

**Definition 9.** We call $R_1^+$ to be the subset of $R_1^*$ of all the tetrahedra that converge into a finite number of similarity classes.

**Lemma 3.** *For all tetrahedra in $R_1^+$ it holds that $A \geq F \geq B \geq C, D, E$.*

**Proof.** The sextuple normalization implies that $A \geq B, C, D, E, F$ and $B \geq C, D, E$. And according to Lemma 2, the tetrahedron converges if $F \geq B$. Therefore, $A \geq F \geq B \geq C, D, E$. □

## 5. Convergence of nearly equilateral tetrahedra into a finite number of similarity classes

We have just derived the Adler main condition that indicates whether the SCLEB of nearly equilateral tetrahedra converges into a finite number of classes. But the reason why it never converges on more than 37 classes can also be deduced. In this section, we just focus on those tetrahedra where all their edge lengths are different, and in the following section, we will study the case of repeated edges.

**Lemma 4.** *Each new similarity class generated by the SCLEB applied to a class $T$ can be determined as linear combinations of the original values of $T$.*

**Proof.** It can be seen from Equations (3) and (4) that the expressions for the new two classes that appear in the SCLEB are linear combinations of their parent. Therefore, the expressions of each new class appearing in any level of the iterated SCLEB of a tetrahedron $T$ can be obtained as linear combinations of the original values of $T$. □

**Lemma 5.** *Let $T = (A, B, C, D, E, F)$ be a tetrahedron in $R_1^+$ with all edges having different lengths. The SCLEB applied to $T$ produces exactly 37 new similarity classes.*

**Proof.** Since $T \in R_1^+$, and all its edges are of different lengths, it holds that $A > F > B > C, D, E$ and $C \neq D \neq E$. For this proof, we use the Equations (3) and (4) from Definition 5, and Fig. 2 from Definition 1, to build the normalized sextuples of tetrahedra generated in each step of bisection. Finally, to check the new classes we use the Definition 3.

In the SCLEB process, new mediatrices appear that become edges in the newly generated classes. These mediatrices, $G, H, J, K$, can be computed with the following expressions:

$$4G = 2B + 2C - A \tag{6}$$

$$4H = 2D + 2E - A \tag{7}$$

$$4J = 2B + 2E - F \tag{8}$$

$$4K = 2D + 2C - F \tag{9}$$

Fig. 6 shows the relations between these mediatrices and the original edges of $T$. These inequalities come from the fact that when the SCLEB is applied to a nearly equilateral triangle, the two sub-triangles are nearly right-angled triangles and the largest angles located around the midpoints edges $A$ and $F$ are nearly right angles, as it happens with the equilateral triangle.

Fig. 7 shows the 7 steps of the full iterated SCLEB of $T = [A, B, C, D, E, F]$. It is important to take into account that in each step of bisection, all the similarity classes that have appeared previously are not subdivided again, and only new similarity classes will be bisected. It can also be seen that all the sextuples of new tetrahedra generated are linear combinations of the original values of $T$ according to Lemma 4. Let $\{T_1, T_2, \ldots, T_{37}\}$ be the sequence of 37 similarity classes generated by the SCLEB of $T$, with $T_1 = T$. Next, we study each step of SCLEB.

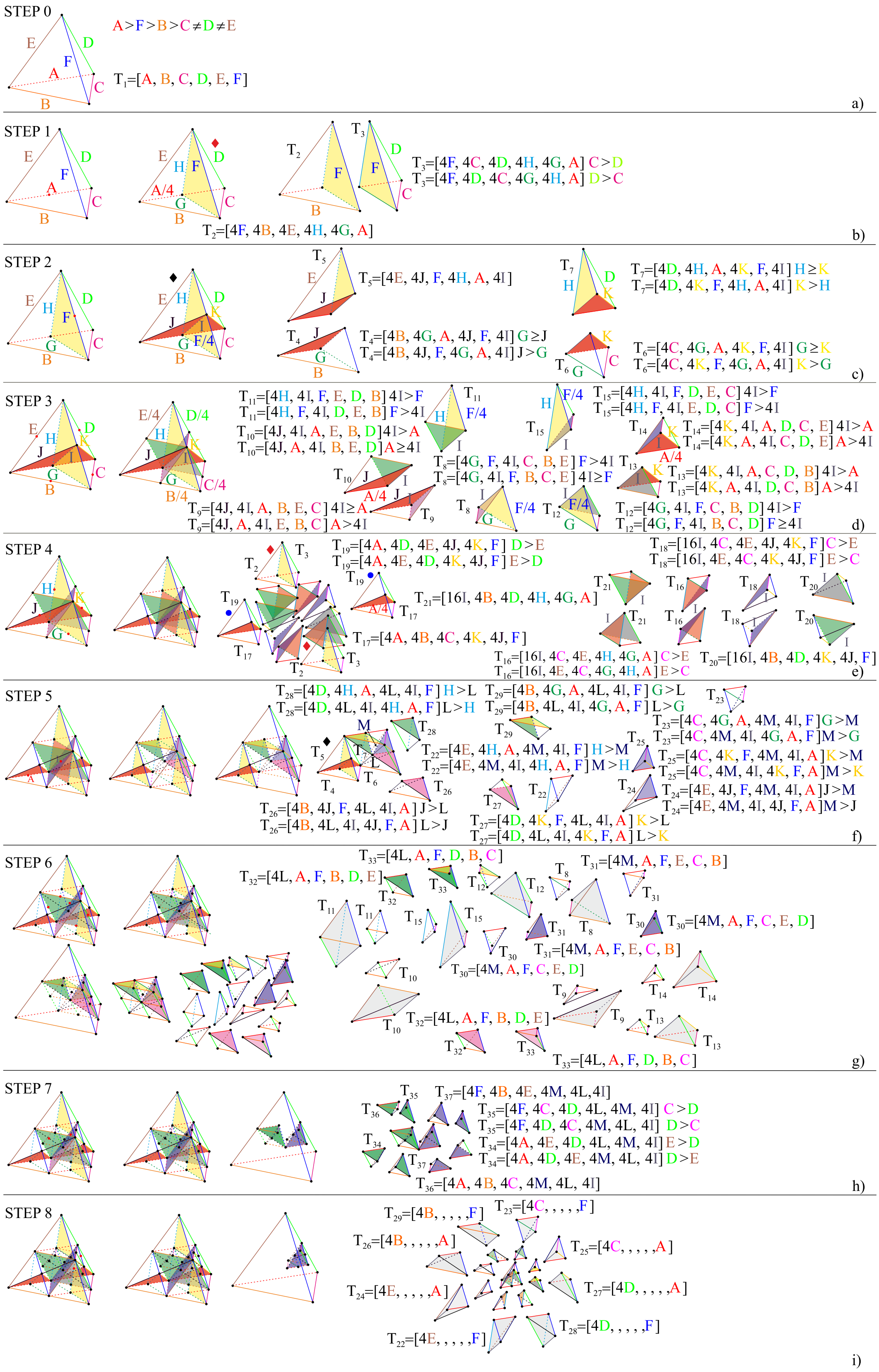


**Fig. 7.** The sequence of the 37 similarity classes for $T_1 = [A, B, C, D, E, F]$.

**Step 1.** The SCLEB of $T_1$ generates $T_2$ and $T_3$, with mediatrices $H$ and $G$ (see Fig. 7 b)). The longest edge for $T_2$ and $T_3$ is $F$. The longest edge connected to $F$ is $B$ for $T_2$, and $C$ or $D$ for $T_3$, depending on which of the two is the longest one.

**Step 2.** Four new classes $T_4$, $T_5$, $T_6$ and $T_7$ are generated (see Fig. 7 c)). In addition to mediatrices $J$ and $K$ that appear in this step, a new interior edge $I$ which connects the midpoints of edges A and F appears, where

$$4I = -A + B + C + D + E - F \tag{10}$$

From Fig. 6, it can be seen that $B, C > G > A/4$ and $B, E > J > A/4$. Besides, $G, J > I$ since

$$\begin{aligned} G > I &\Longrightarrow B + C + F > D + E \\ J > I &\Longrightarrow B + E + A > C + D \end{aligned} \tag{11}$$

then, the longest edges of $T_4$, $T_5$ and $T_6$ are $B$, $E$ and $C$, respectively.

For $T_4$ the longest edge connected to $B$ can be $G$ or $J$, and for $T_6$ the longest edge connected to $C$ can be $G$ or $K$. Therefore, the second edge in the sextuple for both tetrahedra depends on which of the two mediatrices is the longest one in each case. It is important to remark that when $G = J$ or $G = K$ the sextuple chosen for $T_4$ and $T_6$ requires to have $A$ as the third edge because $A > F$. For $T_5$, $J > H$, then the longest edge connected to $E$ is $J$, since

$$4J = 2B + 2E - F > 4H = 2E + 2D - A \Longrightarrow 2B + A > F + 2D$$

For $T_7$, $D > A/4$ since $2D + 2C > A + F$ so $D$ is the longest one. Besides, $K, H > I$ since

$$\begin{aligned} K > I &\Longrightarrow C + D + A > B + E \\ H > I &\Longrightarrow E + D + F > B + C. \end{aligned} \tag{12}$$

Therefore, $D$ becomes the first edge in the sextuple. The longest edge connected to $D$ can be $H$ or $K$, depending on which of the two is the longest one. When $H = K$, $A$ becomes the third edge in the sextuple.

**Step 3.** Eight new classes $T_8, T_9, ..., T_{15}$ are generated (see Fig. 7 d)). Note that all original edges have already been subdivided. It is clear that for $T_8$ and $T_{12}$, the longest edge is $G$, for $T_9$ and $T_{10}$ is $J$, and for $T_{11}$ and $T_{15}$ is $H$ (see Fig. 6 and Equations (11) and (12)). Besides, for $T_8$ and $T_{12}$ the longest edge connected to $G$ is either $F$ or $I$, for $T_9$ and $T_{10}$ the longest edge connected to $J$ is either $A$ or $I$, and for $T_{11}$ and $T_{15}$ the longest edge connected to $H$ is either $F$ or $I$. The second and third edges in the sextuples $T_9$ and $T_{10}$ is either $A$ or $I$, and in the sextuples $T_8$ and $T_{12}$ is either $F$ or $I$. Note that, when $A = 4I$ or $F = 4I$, $B$ becomes the fourth edge in the sextuple (see Fig. 7 d)). Finally for $T_{13}$ and $T_{14}$ the longest edge is $K$, and the longest edge connected to $K$ is either $A$ or $I$.

It is important to remark on the next cases when $4I = A$ or $4I = F$:

$T_{11}$: If $4I = F$, we take $T_{11} = [4H, 4I, F, E, D, B]$ if $E > D$, or $T_{11} = [4H, F, 4I, D, E, B]$ if $D > E$.
$T_{13}$: If $4I = A$, we take $T_{13} = [4K, 4I, A, D, C, B]$ if $D > C$ or $T_{13} = [4K, A, 4I, C, D, B]$ if $C > D$.
$T_{14}$: If $4I = A$, we take $T_{14} = [4K, 4I, A, D, C, E]$ if $D > C$ or $T_{14} = [4K, A, 4I, C, D, E]$ if $C > D$.
$T_{15}$: If $4I = F$, we take $T_{15} = [4H, 4I, F, D, E, C]$ if $D > E$ or $T_{15} = [4H, F, 4I, E, D, C]$ if $E > D$.

**Step 4.** Up to step 3, the number of new similarity classes generated in each step is $2^n$ with $n = 0, 1, 2, 3$. In the fourth step, 16 classes are generated but only 6 are new. The four classes marked with a red diamond in Figs. 7 b) and e) have already appeared in the first step. Normalized sextuples for these four classes can be written as $kT_2$ and $kT_3$ with $k = 0.25$ (see Definition 3). Note that $T_{17}$ and $T_{19}$ are generated twice with $k = 1$ (see blue circle in Fig. 7 e)).

The eight classes sharing a common edge $I$ are $T_{16}$, $T_{18}$, $T_{20}$ and $T_{21}$, generated twice (see Fig. 7 e)). Therefore, the new 6 similarity classes are $T_{16}$, $T_{17}$, $T_{18}$, $T_{19}$, $T_{20}$ and $T_{21}$.

The sextuple $T_{17}$ is straightforward (see Fig. 7 e)). For $T_{19}$, it was proved that $E > J$ and $D > K$ (see Fig. 6 b)), then it follows that $B > J, K$. Also, $B > H$ since it has already proved that $J > H$ and $B > G$ from Fig. 6 a). The longest edge for $T_{19}$ is $A$, and the longest edge connected to $A$ is either $E$ or $D$.

For the remaining classes, we study if $4I > B$. Obviously when $4I \geq A$ or $4I \geq F$, then $4I > B$, because $A, F > B$. Fig. 8 shows the case where $T_1$ has already been subdivided into eight tetrahedra. We focus on diagonals of parallelograms with edges $B, D$ and $C, E$ (see Figs. 8 b) and c)). Notice that these edges are opposite each other in $T_1$, and both parallelograms are nearly squares since $T_1$ is nearly equilateral. Diagonal $I$ is already known, and the new ones are $L$ and $M$, which can be written as a linear combination of the values of $T_1$, $4L = A + B - C + D - E + F$ and $4M = A - B + C - D + E + F$. The shortest diagonal for both parallelograms is $I$ since,

$$\begin{aligned} L > I &\Longrightarrow F + A > C + E \\ M > I &\Longrightarrow F + A > B + D. \end{aligned} \tag{13}$$

All these diagonals cut off at the same point, and $I$ is the diagonal common to both parallelograms. Fig. 8 b) shows that the diagonal $I > B/4, D/4$, since the parallelogram is nearly square. Therefore, for $T_{16}$, $T_{18}$, $T_{20}$ and $T_{21}$ the longest edge is $I$. For $T_{20}$ and $T_{21}$ the longest edge connected to $I$ is $B$, and for $T_{16}$ and $T_{18}$ is either $C$ or $E$. All these sextuples are depicted in Fig. 7 e). Notice

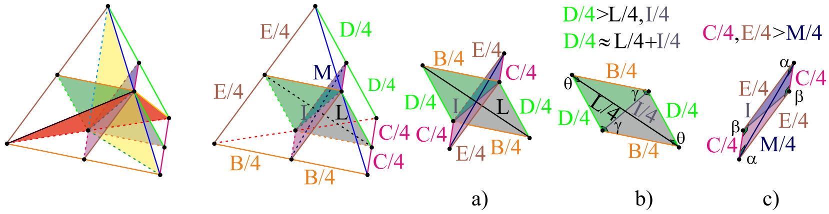


**Fig. 8.** Relation among original edges of $T_1$ and interior edges.

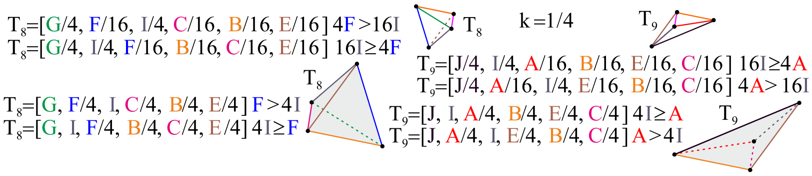


**Fig. 9.** Similarity classes between two pairs of sub-tetrahedra belonging to 6 and 3 bisection steps.

that when this step is finished, all mediatrices have already been subdivided, and also this is the first time where some similarity classes have already appeared.

**Step 5.** In this step, 12 classes are generated but only 8 are new, because 4 have already appeared. These 4 repeated classes are labeled with a black diamond in Fig. 7 f), which are the same classes that already appeared in Fig. 7 c) in the second step. Note that these classes can be written as $kT_4$, $kT_5$, $kT_6$ and $kT_7$ with $k = 0.25$ (see Definition 3).

From Fig. 8 b) and c) it is clear that $B, D > L$ and $E, C > M$. Note that the longest edge of $T_{22}$, $T_{23}$, $T_{24}$, $T_{25}$, $T_{26}$, $T_{27}$, $T_{28}$ and $T_{29}$, is one of the edges $B$, $C$, $D$, or $E$. For these classes, the longest edges connected to the longest ones are either the mediatrices $H, G, J, K$ or the interior edges $L, M$ (see Fig. 7 f).

We next study the cases where the inequalities become equalities:

$T_{22}$: If $H = M$, $T_{22} = [4E, 4H, A, 4M, 4I, F]$ if $A > 4I$, or $T_{22} = [4E, 4M, 4I, 4H, A, F]$ if $4I > A$. Both classes are equivalent if $4I = A$.
$T_{23}$: If $G = M$, $T_{23} = [4C, 4G, A, 4M, 4I, F]$ if $A > 4I$ or $T_{23} = [4C, 4M, 4I, 4G, A, F]$ if $4I > A$. Both classes are equivalent if $4I = A$.
$T_{24}$: If $J = M$, $T_{24} = [4E, 4J, F, 4M, 4I, A]$ if $F > 4I$ or $T_{24} = [4E, 4M, 4I, 4J, F, A]$ if $4I > F$. Both classes are equivalent if $4I = F$.
$T_{25}$: If $K = M$, $T_{25} = [4C, 4K, F, 4M, 4I, A]$ if $F > 4I$ or $T_{25} = [4C, 4M, 4I, 4K, F, A]$ if $4I > F$. Both classes are equivalent If $4I = F$.
$T_{26}$: If $J = L$, $T_{26} = [4B, 4J, F, 4L, 4I, A]$ if $F > 4I$ or $T_{26} = [4B, 4L, 4I, 4J, F, A]$ if $4I > F$. Both classes are equivalent if $4I = F$.
$T_{27}$: If $K = L$, $T_{27} = [4D, 4K, F, 4L, 4I, A]$ if $F > 4I$ or $T_{27} = [4D, 4L, 4I, 4K, F, A]$ if $4I > F$. Both classes are equivalent if $4I = F$.

**Step 6.** In this step 16 classes are generated but only 4 are new. The 8 classes which have already appeared in the third step, are depicted in Fig. 7 g) by pairs with their respective partners, $T_8$, $T_9$, $T_{10}$, $T_{11}$, $T_{12}$, $T_{13}$, $T_{14}$ and $T_{15}$. Fig. 9 shows two examples of pairs of classes where their respective sextuples can be obtained by multiplying each other by $k = 0.25$. Notice that the new classes $T_{30}$, $T_{31}$, $T_{32}$ and $T_{33}$ are repeated twice in this step.

To build the sextuples of the new classes, we only need to study if $L, M > A/4$.

$$4L > A \Longrightarrow B + D + F > C + E$$

$$4M > A \Longrightarrow C + E + F > B + D.$$

It is clear that the second edge is $A$, and the four sextuples are built straightforwardly (see Fig. 7 g)).

**Step 7.** In the seventh step, the last four similarity classes are generated $T_{34}$, $T_{35}$, $T_{36}$ and $T_{37}$ (see Fig. 7 h)). The longest edges of these classes are either $A$ or $F$. For $T_{36}$ and $T_{37}$, $B$ is the longest edge connected to $A$ and $F$ respectively. For $T_{34}$ and $T_{35}$, the longest edges connected to $A$ and $F$ are either $C$ or $D$.

**Step 8.** In the eighth step, no more new classes are generated by the SCLEB, since all of them have already appeared in step 5. These classes are $T_{22}$, $T_{23}$, $T_{24}$, $T_{25}$, $T_{26}$, $T_{27}$, $T_{28}$ and $T_{29}$. They have been depicted in pairs in Fig. 7 i) with their respective partners from this step. Notice that these sextuples can be obtained by multiplying each other by $k = 0.25$. □

**Remark 7.** The number of unique similarity classes appearing in the SCLEB of $R_1^+$ follows the sequence $\{1, 2, 4, 8, 6, 8, 4, 4, 0\}$, as can be seen in Fig. 10, where the genealogy tree for the 37 different similarity classes is shown.

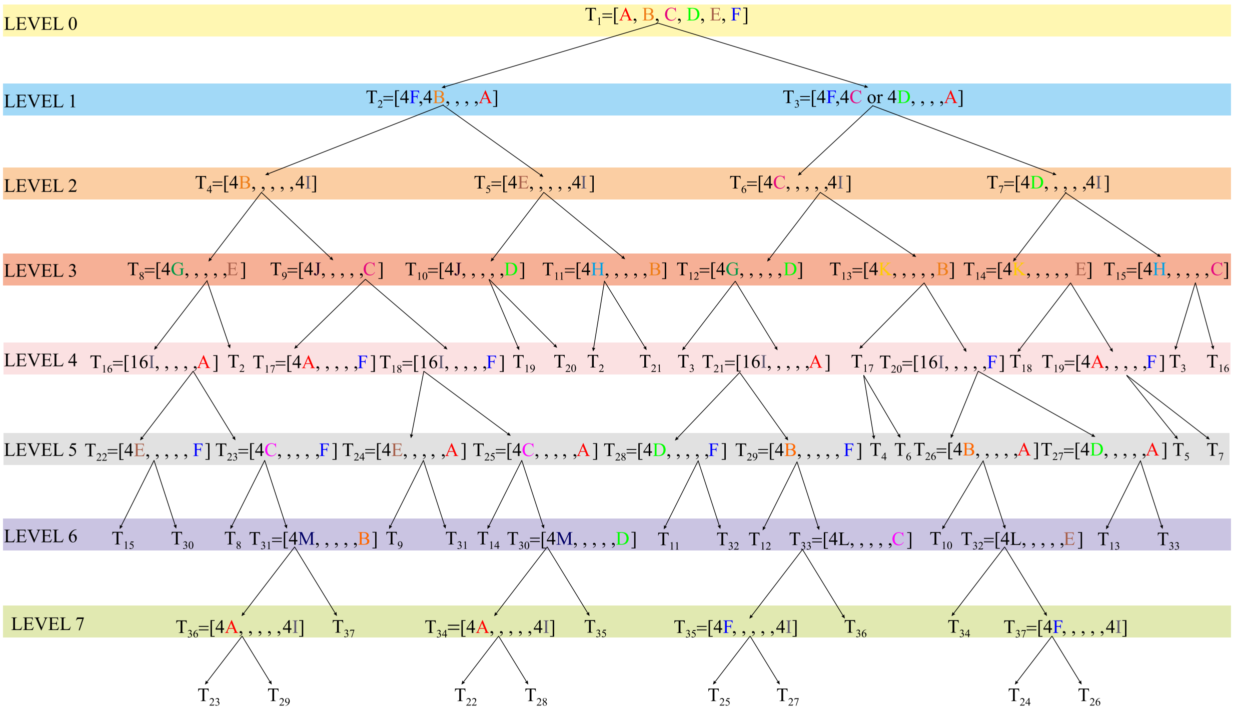


**Fig. 10.** Binary tree for the 37 different similarity classes.

## 6. Study of $R_1^+$ tetrahedra with repeated edge lengths

We have proved that all the tetrahedra $T = (A, B, C, D, E, F)$ belonging to the family $R_1^+$ that have all their edges different, always generate exactly 37 similarity classes, which correspond to the 37 different expressions depicted in Fig. 10. However, when tetrahedra have some equal edges, it happens that some of these expressions produce the same values in the sextuples, resulting in repeated similarity classes. This means that we can find less than 37 similarity classes in many cases. In practice, when we work with tetrahedral meshes, especially when some refinement strategies based on the LEB are used, edge lengths may differ from one tetrahedron to another, even in very small amounts. The cases where there are equal edges represent some minor cases. This leads to a scenario where most cases in a tetrahedral mesh containing nearly equilateral tetrahedra may converge into 37 similarity classes.

**Lemma 6.** *Let $T = [A, B, C, D, E, F]$ be a class in $R_1^+$. There can be up to 37 sextuples representing the different tetrahedra classes when the SCLEB is applied to T.*

**Proof.** Note that for the case that all lengths are different, we apply Lemma 5, and then there are exactly 37 similarity classes. For the case of repeated edge lengths, the number of similarity classes can be less than 37. In this case, some of the classes produce the same values in sextuples, resulting in repeated similarity classes. □

A clear example of a similarity class with repeated edges in $R_1^+$ is the regular $R_1 = [1, 1, 1, 1, 1, 1]$, with all its edges equal. We already saw in section 3 (Lemma 1) that this class converges exactly in 8 classes, given by the graph of Fig. 3. By using the diagram of Fig. 7 we can determine the 8 similarity classes for the SCLEB of the regular as follows: the two classes generated in step 1 becomes the class $R_2 = [4, 4, 4, 3, 3, 1]$, all the 4 classes generated in step 2 becomes the class $R_3 = [4, 3, 1, 3, 1, 2]$, and so on until reaching 8 similarity classes in 7 steps.

Other examples with repeating edges, which can be tested using the diagram in Fig. 7, are:

- Example 1: Classes converging into 21 similarity classes. These are for example classes with a pair of repeated interior edge lengths, as $[12, 10, 8, 8, 9, 11]$ and $[15, 12, 10, 10, 11, 13]$.
- Example 2: Classes converging into 13 similarity classes. This happens when the interior edges are equal to their opposite edges. See for example $[4, 3, 3, 3, 3, 3]$ and $[6, 5, 4, 5, 4, 6]$.
- Example 3. Classes converging into 9 similarity classes. This is the case when $A = F$ and three of the four interior edges are equal, as $[7, 6, 5, 5, 5, 7]$ and $[9, 7, 7, 7, 6, 9]$.
- Example 4. Classes converging into 8 similarity classes, like those with four equal interior edges as $[5, 4, 4, 4, 4, 5]$ and $[7, 5, 5, 5, 5, 6]$, or the regular $[1, 1, 1, 1, 1, 1]$.
- Example 5. The class $[4, 3, 3, 3, 3, 4]$ converges only into 4 similarity classes.

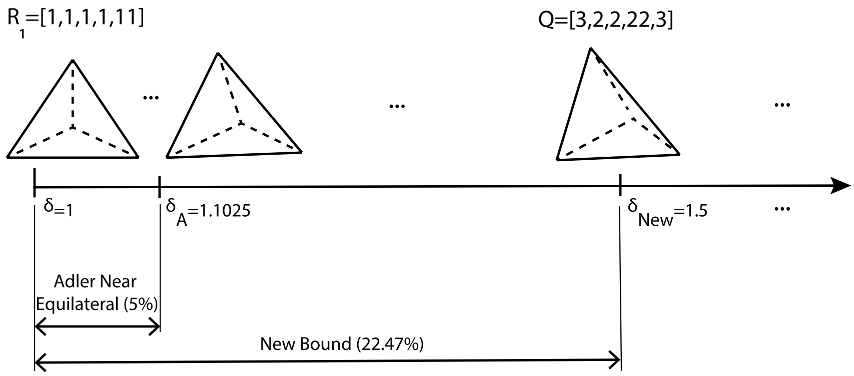


**Fig. 11.** Near equilateral Adler tetrahedra with 5% is remarked and also the tetrahedra with new bound of 22.47%.

## 7. An improved bound for the $R_1^+$ family

In his original paper [1] Adler studied the family of tetrahedra that converged into 37 or fewer classes to be those nearly equilateral tetrahedra whose edge lengths are within 5%. As seen in Section 4 this is equivalent in terms of $\delta$ to $\delta(T) < \delta_A = 1.1025$. Next lemma gives the maximum threshold of $\delta$ that guarantees that a tetrahedron of $R_1^*$ converges into 37 or fewer classes, and then we state the condition to belong to $R_1^+$.

**Lemma 7.** *Let $T = [A, B, C, D, E, F]$ being $F \geq B$ and $\delta(T) \leq \frac{3}{2}$. Then, $T$ converges into 37 or fewer classes, and therefore $T \in R_1^+$.*

**Proof.** We want to find what is the maximum threshold value for delta which guarantees the convergence of $T$. To do that, we must check that all the 37 classes generated in the SCLEB of $T$ according to Fig. 7 are correctly normalized, fulfilling the conditions of Remark 1: the first value of the class must be greater or equal to the rest of values, and the second value must be greater than the 3rd, 4th and 5th values. Exploring which is the worst scenario for all the classes, will lead to compute the threshold value of $\delta(T)$ that guarantees the convergence of $T$.

For example, in the case of $T_2 = [4F, 4B, 4E, 4H, 4G, A]$, it would be necessary to verify that the first value ($4F$) is greater than the rest of the values of the sextuple, and the second value ($4B$) is greater than the third, fourth and fifth value. It is clear that $F \geq B \geq E$ using Lemma 3. From definitions of $H, G$ (see Equations (7) and (6)) it is easy to see that $F$ and $B$ are greater than $H$ and $G$. To deduce that $4F \geq A$, let $m$ be the minimum of the six values of the class $[A, B, C, D, E, F]$. Since $F \geq m$, we can deduce that $4F \geq 4m \geq A$ if $\delta(T) = A/m \leq 4$.

Using a similar reasoning for all similarity classes $T_3, T_4, \dots, T_{37}$ in the diagram, we can find that the most restrictive case that produces the smallest bound for $\delta(T)$ holds for $T_{20} = [16I, 4B, 4D, 4K, 4J, F]$. In this class, to guarantee that $16I \geq 4D$, it can be deduced (see Equation (10)) that $A + F \leq B + C + E$ must be fulfilled. And therefore $A + F \leq 2A \leq 3m \leq B + C + E$, from which it follows that $\delta(T) = A/m \leq \frac{3}{2}$. □

**Remark 8.** Considering that the six values of a class are the square of the edges, the condition $\delta(T) \leq \frac{3}{2}$ means that the ratio between the lengths of the maximum and minimum edges is less than or equal to $\sqrt{\frac{3}{2}}$, which is equivalent to saying that its edge lengths are within 22.47%.

Lemma 7 considerably improves the condition given by Adler of the 5% as a criterion of convergence for the nearly equilateral tetrahedra. Fig. 11 shows the bound by Adler $\delta_A$, and the new bound, $\delta_{new} = \frac{3}{2}$ that improves the criterion to belong to $R_1^+$. The new bound of 22.47%, clearly extends the family of tetrahedra originally called “nearly equilateral”.

We remark here the interesting fact that, as we improve the bound of $\delta$ to a higher value than Adler, a tetrahedron class like $Q = [3, 2, 2, 2, 2, 3]$ with $\delta(Q) = \frac{3}{2}$ holds a geometric shape very different to the regular $R_1$. Indeed, $Q$ has two obtuse dihedral angles, unlike $R_1$ whose angles are approximately 70°. Then, our new bound $\delta_{new}$ can cause a misleading situation as we may be calling near equilateral to those tetrahedra whose shapes are far away from being equilateral, as $Q$. To avoid this scenario, instead of using the term near equilateral class, we will refer as $R_1^+$ to those tetrahedra with $\delta \in [1, \frac{3}{2}]$.

## 8. Conclusions

The SCLEB of nearly equilateral tetrahedra, first introduced by Adler [1] is revisited here with a complete study of important properties that concern the convergence of the number of similarity classes. Nearly equilateral and regular tetrahedra are important as they give many keys to the convergence of the SCLEB method into a finite number of similarity classes. Some difficulties in the study of those families of tetrahedra arise with the equality of edge lengths and the selection of the bisection edge, where the regular case represents a singular example. To overcome the difficulties of the representation issues, shape comparison, and tetrahedra subdivision, we have used a sextuple edge-based representation.

We prove that there is a family of tetrahedra whose SCLEB introduces a finite number of classes less than or equal to 37. We determine the conditions for the convergence up to 37 similarity classes. We also improve the bound of 5% to 22.47% of the family of nearly equilateral tetrahedra satisfying the convergence into a finite number of similarity classes and give a proof of such a new bound.

Note that the bound $\delta(T) \le \frac{3}{2}$ with $F \ge B$ guarantees that tetrahedra $T$ converges and therefore belongs to $R_1^+$. However, the opposite is not true. There are tetrahedra with $\delta(T) > \frac{3}{2}$ that converge, and others that do not. In a forthcoming study, we can completely delimit the $R_1^+$ region for which all tetrahedra converge in the SCLEB.

As a future work, it is very interesting to study the other seven families of tetrahedra derived from the graph of the iterative SCLEB of the regular tetrahedron. This comprises the families $R_2^+, R_3^+, \dots, R_8^+$ together with the bounds of $\delta$ for assuring the convergence into a finite number of similarity classes.

## Data availability

No data was used for the research described in the article.

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