Now, let $x_k = F_k$, so that

$$\mathcal{S} = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

Thus,

$$\frac{(n-1)^2}{n} \sum_{k=1}^{n} \left(\frac{F_k}{F_{n+2} - F_k - 1} \right)^2 \geq 1.$$

Also solved by Thomas Achammer, Michel Bataille, I. V. Fedak, Wei-Kai Lai, Edwin Daniel Patiño Osorio (undergraduate), Ángel Plaza, Albert Stadler, and the proposer.

It's Jensen Again!

B-1323 Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.
(Vol. 61.1, February 2023)

Let n be a positive integer. Show that ${F_n}^{F_n} + {L_n}^{L_n} \geq 2{F_{n+1}}^{F_{n+1}}$.

Solution by Brian D. Beasley, Simpsonville, SC.

For $x \geq 1$, let $f(x) = x^x$. Then $f'(x) = x^x(\ln x + 1)$, so

$$f''(x) = x^{x-1} + x^x(\ln x + 1)^2.$$

Because $f''(x) > 0$ for $x \geq 1$, the function $f(x)$ is convex on $[1, \infty)$. Hence, for any λ in $(0, 1)$ and any $a \leq b$ in $[1, \infty)$, Jensen's inequality asserts that

$$\lambda f(a) + (1 - \lambda) f(b) \geq f(\lambda a + (1 - \lambda) b).$$

Taking $\lambda = 1/2$, $a = F_n$, and $b = L_n$ yields

$$F_n^{F_n} + L_n^{L_n} \geq 2 \left(\frac{F_n + L_n}{2} \right)^{\frac{F_n + L_n}{2}}.$$

Because $F_n + L_n = 2F_{n+1}$, the proof is complete.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, I. V. Fedak, Robert Frontczak, Richard Gonzalez Hernandez and Edwin Daniel Patiño Osorio (both undergraduates) (jointly), Won Kyun Jeong, Jacob Juillerat, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Henry Ricardo, Albert Stadler, David Terr, Daniel Văcaru, Andrés Ventas, and the proposer.

The Limit of the Fractional Part

B-1324 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 61.1, February 2023)

Let $\{x\}$ denote the fractional part of the real number x. Evaluate $\lim_{n\to\infty} \left\{\alpha F_{2n}^2\right\}$ and $\lim_{n\to\infty} \left\{\alpha F_{2n-1}^2\right\}$.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For the first limit, we will first show that $\left\{\alpha F_{2n}^2\right\} = \alpha F_{2n}^2 - F_{2n}F_{2n+1} + 1$. Notice that this is equivalent to showing that

$$F_{2n}F_{2n+1} - 1 < \alpha F_{2n}^2 < F_{2n}F_{2n+1}.$$

After some algebra, we can rewrite it as

$$\alpha^{4n+1} + \beta^{4n+1} - \alpha - \beta - 5 < \alpha^{4n+1} - \beta^{4n-1} - 2\alpha < \alpha^{4n+1} + \beta^{4n+1} - \alpha - \beta,$$

which simplifies to

$$(\alpha - \beta)(1 - \beta^{4n}) - 5 < 0 < (\alpha - \beta)(1 - \beta^{4n}).$$

Because this is obviously true, our claim is established. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \left\{\alpha F_{2n}^2\right\} &= \lim_{n\to\infty} \left(\alpha F_{2n}^2 - F_{2n}F_{2n+1} + 1\right) \\
&= \lim_{n\to\infty} \frac{-\beta^{4n-1} - 2\alpha - \beta^{4n+1} + \alpha + \beta + 5}{5} \\
&= 1 - \frac{\alpha - \beta}{5} = 1 - \frac{\sqrt{5}}{5}.
\end{aligned}$$

For the second limit, we first want to show that $\left\{\alpha F_{2n-1}^2\right\} = \alpha F_{2n-1}^2 - F_{2n-1}F_{2n}$, which is equivalent to

$$F_{2n-1}F_{2n} < \alpha F_{2n-1}^2 < F_{2n-1}F_{2n} + 1.$$

After some algebra, we can rewrite it as

$$\alpha^{4n-1} + \beta^{4n-1} + \alpha + \beta < \alpha^{4n-1} - \beta^{4n-1} + 2\alpha < \alpha^{4n-1} + \beta^{4n-1} + \alpha + \beta + 5,$$

and simplify it to

$$2\beta^{4n-1} < \alpha - \beta < 2\beta^{4n-1} + 5.$$

Because this is true, the claim is established. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \left\{\alpha F_{2n}^2\right\} &= \lim_{n\to\infty} \left(\alpha F_{2n-1}^2 - F_{2n-1}F_{2n}\right) \\
&= \lim_{n\to\infty} \frac{-\beta^{4n-1} + 2\alpha - \beta^{4n+1} - \alpha - \beta}{5} \\
&= \frac{\alpha - \beta}{5} = \frac{\sqrt{5}}{5}.
\end{aligned}$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Richard Gonzalez Hernandez (undergraduate), Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.

Sum of Eight Consecutive Tribonacci Numbers

B-1325 Proposed by Hans J. H. Tuenter, Toronto, Canada. (Vol. 61.1, February 2023)

Let $\{\mu_n\}$ be a sequence that follows the recurrence relation $\mu_{n+3} = \mu_{n+2} + \mu_{n+1} + \mu_n$, with arbitrary initial values μ_0, μ_1, and μ_2. Prove that, for such a generalized Tribonacci sequence, the sum of eight consecutive numbers always equals four times the seventh of these numbers.