which may be proved by induction. For $m = 1$, the identity holds. If it holds for some integer $m \geq 1$, then it is enough to prove that

$$\frac{(F_{2rm}F_{2rm+2r})^2}{F_{4r}} + F_{2r(m+1)}^2 F_{4r(m+1)} = \frac{(F_{2r(m+1)}F_{2r(m+1)+2r})^2}{F_{4r}},$$

or

$$\frac{F_{2rm}^2}{F_{4r}} + F_{4r(m+1)} = \frac{F_{2rm+4r}^2}{F_{4r}}.$$

The last identity follows from $F_{2rm+4r}^2 - F_{2rm}^2 = F_{4r}F_{4rm+4r}$, which is, again, a consequence of Catalan's identity.

Solution 2 by the Proposer.

As a special case of Catalan's identity, we have

$$F_{a+b}^2 - F_{a-b}^2 = F_{2a}F_{2b}.$$

Letting $n = 2m$, we have

$$\begin{aligned}
L_{2r}\sum_{k=1}^{n}(-1)^k(F_{rk}F_{r(k+1)})^2 &= L_{2r}\sum_{k=1}^{m}\left[-(F_{r(2k-1)}F_{2rk})^2 + (F_{2rk}F_{r(2k+1)})^2\right] \\
&= \sum_{k=1}^{m} L_{2r}F_{2rk}^2\left(F_{2rk+r}^2 - F_{2rk-r}^2\right) = \sum_{k=1}^{m} L_{2r}F_{2rk}^2 \cdot F_{4rk}F_{2r} \\
&= \sum_{k=1}^{m} F_{2rk}^2 \cdot F_{4rk}F_{4r} = \sum_{k=1}^{m} F_{2rk}^2\left(F_{2rk+2r}^2 - F_{2rk-2r}^2\right) \\
&= \sum_{k=1}^{m}\left(F_{2r(k+1)}^2F_{2rk}^2 - F_{2rk}^2F_{2r(k-1)}^2\right) = F_{2r(m+1)}^2F_{2rm}^2 - F_{2r}^2F_0^2 \\
&= (F_{rn}F_{rn+2r})^2.
\end{aligned}$$

Therefore, we obtain the desired identity.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Kristen Hartz (undergraduate), Won Kyun Jeong, Muzahim Mamedov, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, and Andrés Ventas.

Fibonacci and Lucas Subscripts

B-1312 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 60.3, August 2022)

For any positive integer n, find closed form expressions for the sums

$$\sum_{k=1}^{n} L_{F_k}L_{F_{k+1}}F_{F_k}F_{F_{k+1}} \qquad \text{and} \qquad \sum_{k=1}^{n} L_{L_k}L_{L_{k+1}}F_{L_k}F_{L_{k+1}}.$$

Solution 1 by Jason L. Smith, Richland Community College, Decatur, IL.

The summand in each sum has the form of $F_{G_k}L_{G_k}F_{G_{k+1}}L_{G_{k+1}}$ for a generalized Fibonacci sequence $\{G_k\}_{k=1}^{\infty}$. Using the Fibonacci double-angle identity $F_mL_m = F_{2m}$, this immediately becomes $F_{2G_k}F_{2G_{k+1}}$. Using the product formula $F_sF_t = \frac{1}{5}\left[L_{s+t} - (-1)^t L_{s-t}\right]$, the summand can be written as

$$\frac{1}{5}\left(L_{2G_{k+1}+2G_k} - L_{2G_{k+1}-2G_k}\right) = \frac{1}{5}\left(L_{2G_{k+2}} - L_{2G_{k-1}}\right).$$

We now see that each sum telescopes:

$$\begin{aligned}\sum_{k=1}^{n} F_{G_k}L_{G_k}F_{G_{k+1}}L_{G_{k+1}} &= \frac{1}{5}\sum_{k=1}^{n}\left(L_{2G_{k+2}} - L_{2G_{k-1}}\right)\\ &= \frac{1}{5}\left(L_{2G_{n+2}} + L_{2G_{n+1}} + L_{2G_n} - L_{2G_2} - L_{2G_1} - L_{2G_0}\right).\end{aligned}$$

The first sum uses $G_k = F_k$, in which case we also observe that $L_{2F_2} + L_{2F_1} + L_{2F_0} = L_2 + L_2 + L_0 = 8$. Therefore, the first sum becomes

$$\sum_{k=1}^{n} F_{F_k}L_{F_k}F_{F_{k+1}}L_{F_{k+1}} = \frac{1}{5}\left(L_{2F_{n+2}} + L_{2F_{n+1}} + L_{2F_n} - 8\right).$$

The second sum uses $G_k = L_k$, where we observe that $L_{2L_2} + L_{2L_1} + L_{2L_0} = L_6 + L_2 + L_4 = 28$. Thus, the second sum becomes

$$\sum_{k=1}^{n} F_{L_k}L_{L_k}F_{L_{k+1}}L_{L_{k+1}} = \frac{1}{5}\left(L_{2L_{n+2}} + L_{2L_{n+1}} + L_{2L_n} - 28\right).$$

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

Using the product formulas (which are easy to derive from Binet's formulas)

$$F_aL_b = F_{a+b} + (-1)^b F_{a-b}, \qquad \text{and} \qquad L_aF_b = F_{a+b} - (-1)^b F_{a-b},$$

we find

$$\begin{aligned}\sum_{k=1}^{n} L_{F_k}L_{F_{k+1}}F_{F_k}F_{F_{k+1}} &= \sum_{k=1}^{n}\left(F_{F_{k+1}}L_{F_k}\right)\left(L_{F_{k+1}}F_{F_k}\right)\\ &= \sum_{k=1}^{n}\left[F_{F_{k+2}} + (-1)^{F_k}F_{F_{k-1}}\right]\left[F_{F_{k+2}} - (-1)^{F_k}F_{F_{k-1}}\right] = \sum_{k=1}^{n}\left(F_{F_{k+2}}^2 - F_{F_{k-1}}^2\right)\\ &= F_{F_{n+2}}^2 + F_{F_{n+1}}^2 + F_{F_n}^2 - F_{F_0}^2 - F_{F_1}^2 - F_{F_2}^2 = F_{F_{n+2}}^2 + F_{F_{n+1}}^2 + F_{F_n}^2 - 2.\end{aligned}$$

Similarly,

$$\sum_{k=1}^{n} L_{L_k}L_{L_{k+1}}F_{L_k}F_{L_{k+1}} = \sum_{k=1}^{n}\left(F_{L_{k+2}}^2 - F_{L_{k-1}}^2\right) = F_{L_{n+2}}^2 + F_{L_{n+1}}^2 + F_{L_n}^2 - 6.$$

Editor's Notes: Greubel obtained $\frac{1}{5}\left(L_{F_n}^2 + L_{F_{n+1}}^2 + L_{F_{n+2}}^2 - 6\right)$, and $\frac{1}{5}\left(L_{L_n}^2 + L_{L_{n+1}}^2 + L_{L_{n+2}}^2 - 11\right)$.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Kenny B. Davenport, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Albert Stadler, Seán M. Stewart, David Terr, and the proposer.