(A) Because $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}=\alpha$, it is enough to prove that $F_n^3 < (4/\alpha^2)^{2^n}$, which follows because $4/\alpha^2 > 2$, and $2^{2^n} > F_n^3$.

(B) Analogously, it is enough to prove that $L_n^3 < (9/\alpha^2)^{2^n}$, which follows because $9/\alpha^2 > 3$, and $3^{2^n} > L_n^3$.

Reference

[1] A. Hershfeld, *On infinite radicals*, Amer. Math. Monthly, **42** (1935), 419–429.

Also solved by Michel Bataille, Dmitry Fleischman, Albert Stadler, Andrés Ventas, and the proposer.

Lagrange Interpolation Formula

B-1294 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Gran Canaria, Spain, and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.
(Vol. 59.3, August 2021)

Let $A(x)$ and $B(x)$ be polynomials of degree n such that $A(i) = F_i$ and $B(i) = L_i$, respectively, for every i with $0 \le i \le n$. Find the values of $A(n+1)$ and $B(n+1)$.

Solution by Ernest James (undergraduate student), the Citadel, Charleston, SC.

(A) Using the Lagrange interpolation formula, we find

$$A(x) = \sum_{k=0}^{n} F_k \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{x-j}{k-j}.$$

Now we see that

$$A(n+1) = \sum_{k=0}^{n} F_k \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{n+1-j}{k-j} = \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+1)}{n-k+1}\binom{n}{k} F_k.$$

By combining Binet's formula and the binomial theorem, we have

$$\sum_{k=0}^{n} \binom{n}{k} F_k (-x)^{n-k} = \frac{1}{\sqrt{5}}\left[(\alpha-x)^n - (\beta-x)^n\right].$$

Next, multiply both sides by $n+1$ and integrate over $0 \le x \le 1$:

$$\int_0^1 \sum_{k=0}^{n} (n+1)\binom{n}{k} F_k (-x)^{n-k}\, dx = \int_0^1 \frac{n+1}{\sqrt{5}}\left[(\alpha-x)^n - (\beta-x)^n\right] dx.$$

Doing so, we obtain

$$A(n+1) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+1)}{n-k+1}\binom{n}{k} F_k = \left[1-(-1)^n\right] F_{n+1}.$$

(B) We can see, similarly to what we did above, that $B(n+1) = \left[1+(-1)^n\right] L_{n+1}$.

Editor's Note: Frontczak remarked that a general result can be found in [1].

Reference

[1] A. M. Alt, *Numerical sequences and polynomials*, Arhimede Mathematical Journal, **2** (2019), 114–120.

Also solved by Ulrich Abel, Michael R. Bacon and Charles K. Cook (jointly), Michel Bataille, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.

A Sum with Reciprocals of the Central Binomial Coefficients

B-1295 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 59.3, August 2021)

Given an even integer r, prove that

$$\sum_{n=0}^{\infty} \frac{L_{rn}}{2n+1}\left(\frac{4}{L_r}\right)^n \binom{2n}{n}^{-1} = \frac{L_r \pi}{2}.$$

Solution by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

From [2, Theorem 2.4], we know that

$$\sum_{n=0}^{\infty} \frac{4^n x^n}{(2n+1)\binom{2n}{n}} = \frac{1}{x}\sqrt{\frac{x}{1-x}} \arctan\left(\sqrt{\frac{x}{1-x}}\right), \qquad |x| < 1.$$

Setting $x = \alpha^r/L_r$, we immediately obtain (recall that r is even)

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\alpha^{rn}}{(2n+1)\binom{2n}{n}}\left(\frac{4}{L_r}\right)^n &= \frac{L_r}{\alpha^r}\sqrt{\frac{\alpha^r}{L_r-\alpha^r}} \arctan\left(\sqrt{\frac{\alpha^r}{L_r-\alpha^r}}\right) \\ &= \frac{L_r}{\alpha^r}\sqrt{\frac{\alpha^r}{\beta^r}} \arctan\left(\sqrt{\frac{\alpha^r}{\beta^r}}\right) \\ &= L_r \arctan\left(\alpha^r\right).\end{aligned}$$

Similarly, with $x = \beta^r/L_r$, we obtain

$$\sum_{n=0}^{\infty} \frac{\beta^{rn}}{(2n+1)\binom{2n}{n}}\left(\frac{4}{L_r}\right)^n = L_r \arctan\left(\beta^r\right).$$

Adding the two identities yields

$$\sum_{n=0}^{\infty} \frac{L_{rn}}{(2n+1)\binom{2n}{n}}\left(\frac{4}{L_r}\right)^n = L_r\big[\arctan(\alpha^r) + \arctan(\beta^r)\big] = \frac{L_r\pi}{2},$$

where, in the last step, we have used the identity $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x)\,\frac{\pi}{2}$.

Editor's Note: Some solvers used the power series [1]

$$\sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x\,\arcsin x}{\sqrt{1-x^2}}$$

to derive their solutions.