Thus, $\alpha_k^2 + \beta_k^2 = k^2 + 2$, and

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{G_{k,mn-4j} + G_{k,mn+4j}}{G_{k,mn}} = (k^2+2)^{2n} - \binom{2n}{n}.$$

Editor's Note: Greubel obtained the following generalization for any integer $q \geq 0$:

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{F_{mn-2pj} + F_{mn+2pj}}{F_{mn}} = \begin{cases} L_{2q}^{2n} - \binom{2n}{n} & \text{if } p = 2q, \\ 5^n F_{2q+1}^{2n} - \binom{2n}{n} & \text{if } p = 2q+1; \end{cases}$$

and Tuenter showed that, for any integers a, b and n, where $n \geq 0$,

$$\sum_{j=0}^{n} \binom{2n}{n-j} \frac{F_{b-4aj} + F_{b+4aj}}{F_b} = L_{2a}^{2n} + \binom{2n}{n}.$$

Also solved by I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Haydn Gwyn (undergraduate), Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, David Terr, Hans J. H. Tuenter, and the proposer.

A Logarithmic Inequality

B-1284 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Gran Canaria, Spain.
(Vol. 59.1, February 2021)

Let $(x_n)_{n\geq 0}$ be the sequence recurrently defined by $x_{n+1} = x_n + x_{n-1}$ for $n \geq 1$, with initial conditions $x_0 \geq 0$ and $x_1 \geq 1$. For $n \geq 2$, prove that

$$\ln\left(\frac{1}{n-1}\left(\frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}}\right)\right) \geq \frac{2}{n-1}\left(\frac{x_0}{x_3} + \frac{x_1}{x_4} + \cdots + \frac{x_{n-2}}{x_{n+1}}\right).$$

Solution 1 by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let $f(t) = \ln t - \frac{2(t-1)}{t+1}$. Then, $f(1) = 0$ and $f'(t) = \frac{1}{t} - \frac{4}{(t+1)^2} = \frac{(t-1)^2}{t(t+1)^2} \geq 0$ for all $t \geq 1$. Therefore, $\ln t \geq \frac{2(t-1)}{t+1}$ for all $t \geq 1$. Using $x_0 \geq 0$ and $x_1 > 0$, we gather that $x_{k+1}/x_k \geq 1$ for all positive integers k. Thus, for $n \geq 2$,

$$\begin{aligned}
\ln\left(\frac{1}{n-1}\left(\frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}}\right)\right) &\geq \ln\left(\frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \cdot \cdots \cdot \frac{x_n}{x_{n-1}}\right)^{\frac{1}{n-1}} \\
&= \frac{1}{n-1}\left(\ln\frac{x_2}{x_1} + \ln\frac{x_3}{x_2} + \cdots + \ln\frac{x_n}{x_{n-1}}\right) \\
&\geq \frac{2}{n-1}\left(\frac{\frac{x_2}{x_1} - 1}{\frac{x_2}{x_1} + 1} + \frac{\frac{x_3}{x_2} - 1}{\frac{x_3}{x_2} + 1} + \cdots + \frac{\frac{x_n}{x_{n-1}} - 1}{\frac{x_n}{x_{n-1}} + 1}\right) \\
&= \frac{2}{n-1}\left(\frac{x_0}{x_3} + \frac{x_1}{x_4} + \cdots + \frac{x_{n-2}}{x_{n+1}}\right).
\end{aligned}$$

Solution 2 by Hideyuki Ohtsuka, Saitama, Japan.

For $s > t > 0$, the log mean inequality asserts that

$$\frac{s-t}{\ln s - \ln t} \le \frac{s+t}{2},$$

that is,

$$\frac{2(s-t)}{s+t} \le \ln s - \ln t.$$

This inequality also holds for $s = t > 0$. Putting $s = x_{k+1}$ and $t = x_k$ for $k \ge 1$ in the above inequality, we find

$$\frac{2x_{k-1}}{x_{k+2}} \le \ln x_{k+1} - \ln x_k.$$

Therefore,

$$\sum_{k=1}^{n-1} \frac{2x_{k-1}}{x_{k+2}} \le \sum_{k=1}^{n-1} \big(\ln x_{k+1} - \ln x_k \big) = \ln x_n - \ln x_1 = \ln \frac{x_n}{x_1}.$$

Using the AM-GM inequality, we have

$$(n-1)\ \ln \left(\frac{1}{n-1} \sum_{k=1}^{n-1} \frac{x_{k+1}}{x_k} \right) \ge (n-1) \ln \left(\prod_{k=1}^{n-1} \frac{x_{k+1}}{x_k} \right)^{\frac{1}{n-1}} = \ln \frac{x_n}{x_1}.$$

Combining the last two results, we obtain the desired inequality.

Also solved by Michel Bataille, Dmitry Fleischman, Haydn Gwyn (undergraduate), Albert Stadler, and the proposer.

It Is the Binomial Theorem!

B-1285 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 59.1, February 2021)

Let $i = \sqrt{-1}$. For any integer $n \ge 0$, prove that

(i) $\displaystyle\sum_{k=-n}^{n} \binom{2n}{n+k} \left(e^{\frac{2k\pi i}{5}} + e^{\frac{4k\pi i}{5}} \right) = L_{2n};$

(ii) $\displaystyle\sum_{k=-n}^{n} \binom{2n}{n+k} \left(e^{\frac{k\pi i}{5}} + (-1)^n e^{\frac{3k\pi i}{5}} \right) = \big(\sqrt{5}\,\big)^n L_n.$

Solution by Michel Bataille, Rouen, France.

We first derive a lemma: for $\theta \in \mathbb{R}$, we have

$$\sum_{k=-n}^{n} \binom{2n}{n+k} e^{ik\theta} = e^{-in\theta} \sum_{j=0}^{2n} \binom{2n}{j} e^{ij\theta}$$
$$= e^{-in\theta}(1+e^{i\theta})^{2n} = \left(e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}} \right)^{2n} = \left(2\cos\frac{\theta}{2} \right)^{2n}$$