which shows the first inequality. Next,

$$\begin{aligned}
F_{2n+1}F_{2n+2}L_{2n+3} &= \frac{L_{6n+6}+2L_{2n+1}}{5} \\
&= \frac{L_{6n+6}-2(-1)^{3n+3}}{5}+\frac{2L_{2n+1}}{5}+\frac{2(-1)^{3n+3}}{5} \\
&< \frac{L_{6n+6}-2(-1)^{3n+3}}{5}+\frac{2(L_{3n+4}+L_{3n+2})}{5}+1 \\
&= F_{3n+3}^2+2F_{3n+3}+1 \\
&= (F_{3n+3}+1)^2,
\end{aligned}$$

which proves the second inequality.

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, High School Summer Research Group at the Citadel (Ethan Curb, Peyton Matheson, Aiden Milligan, Cameron Moening, Virginia Rhett Smith and Ell Torek), Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.

Three Infinite Series of Reciprocals of Products of Fibonacci Numbers

B-1302 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 60.1, February 2022)

Evaluate

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1})^2}$

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+3}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+3})^2}$

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{2(-1)^n}{(F_nF_{n+3})^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1})^2}$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

(i) Note that both series are absolutely convergent, therefore their sum is equal to

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}}+\frac{(-1)^n}{(F_nF_{n+1})^2}\right)=\sum_{n=1}^{\infty}\frac{F_nF_{n+1}+(-1)^n}{(F_nF_{n+1})^2}.$$

Because of d'Ocagne's identity, we have

$$F_nF_{n+1}+(-1)^n=F_{n+2}F_{n-1}=(F_{n+1}+F_n)(F_{n+1}-F_n)=F_{n+1}^2-F_n^2.$$

The last series telescopes, and its sum is

$$\sum_{n=1}^{\infty}\frac{F_nF_{n+1}+(-1)^n}{(F_nF_{n+1})^2}=\sum_{n=1}^{\infty}\left(\frac{1}{F_n^2}-\frac{1}{F_{n+1}^2}\right)=1.$$

(ii) As in (i), the sum is equal to

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+3}}+\frac{(-1)^n}{(F_nF_{n+3})^2}\right)=\sum_{n=1}^{\infty}\frac{F_nF_{n+3}+(-1)^n}{(F_nF_{n+3})^2}.$$

Again, d'Ocagne's identity implies that

$$F_nF_{n+3}+(-1)^n=F_{n+2}F_{n+1}=\frac{(F_{n+2}+F_{n+1})^2-(F_{n+2}-F_{n+1})^2}{4}=\frac{F_{n+3}^2-F_n^2}{4}.$$

The last series telescopes, and its sum is

$$\sum_{n=1}^{\infty}\frac{F_nF_{n+3}+(-1)^n}{(F_nF_{n+3})^2}=\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{F_n^2}-\frac{1}{F_{n+3}^2}\right)=\frac{1}{4}\left(1+1+\frac{1}{4}\right)=\frac{9}{16}.$$

(iii) Let the sums in the three problems be A, B, and C, respectively. Taking into account the absolute convergence of the involved series, we find

$$C=2B-A+\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}}-\frac{2}{F_nF_{n+3}}\right).$$

Because $F_{n+3}-2F_{n+1}=F_n$,

$$\sum_{n=1}^{\infty}\left(\frac{1}{F_nF_{n+1}}-\frac{2}{F_nF_{n+3}}\right)=\sum_{n=1}^{\infty}\frac{1}{F_{n+1}F_{n+3}}=\sum_{n=1}^{\infty}\left(\frac{1}{F_{n+1}F_{n+2}}-\frac{1}{F_{n+2}F_{n+3}}\right)=\frac{1}{2}.$$

Therefore,

$$C=2\cdot\frac{9}{16}-1+\frac{1}{2}=\frac{5}{8}.$$

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Charles Cook, Dmitry Fleischman, Robert Frontczak, Albert Stadler, Andrés Ventas, and the proposer.

When Do We Reach the Same Floor?

B-1303 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 60.1, February 2022)

Prove that $\left\lfloor x+\dfrac{1}{x}\right\rfloor=\left\lfloor x^2+\dfrac{1}{x^2}\right\rfloor$ if and only if $\dfrac{1}{\alpha}<x<\alpha$.

Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.

We have

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2. \tag{1}$$

(i) Assume $\frac{1}{\alpha}<x<\alpha$. Then,

$$0>\frac{1}{x}(x-\alpha)\left(x-\frac{1}{\alpha}\right)=x+\frac{1}{x}-\alpha-\frac{1}{\alpha}=x+\frac{1}{x}-\sqrt{5}. \tag{2}$$

By (2) and the AM-GM inequality, we have

$$2\le x+\frac{1}{x}<\sqrt{5}.$$