Surface mesh smoothing and its application to match curves

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Abstract

In this work we focus our attention on two aspects related to the node movement in surface meshes: smoothing of triangular meshes defined on surfaces and the adaption of these meshes to match given curves or contours.

The quality improvement of the mesh is obtained by an iterative process in which each node of the mesh is moved to a new position that minimizes a certain objective function. The objective function is derived from some algebraic quality measure [1, 2] of the *local submesh*, that is, the set of triangles connected to the adjustable or *free node*.

When we deal with meshes defined on surfaces we have to impose some restrictions to the movement of the free node. Firstly, is clear that such node must be sited on the surface after optimizing. But, this is not the only constraint. If we allow the free node to move on the surface without imposing any other restriction, only guided by the improvement of the quality, the optimization procedure can construct a high-quality local mesh, but with this node in an *unacceptable* position. To avoid this problem the optimization is done in the *parametric mesh*, where the presence of barriers in the objective function maintains the free node inside the feasible region. In this way, the original problem on the surface is transformed into a two-dimensional one on the *parametric space*. In our case, the parametric space is a plane, chosen in terms of the local mesh, in such a way that this mesh can be optimally projected performing a *valid* mesh, that is, without inverted elements.

We use the flexibility that provides this techniques to adapt a given surface mesh to a curve defined on it. The idea consists on displacing the nodes close to the curve to positions sited on the curve. The process is repeated until the it is correctly approximated (interpolated) by a set of linked edges of the mesh.

The determination of which nodes can be projected on the curve is accomplished by analyzing if there is a position on the curve on which the free node can be projected without inverting any triangle of its local submesh. The optimal position of the free node on the curve is determined attending to the quality of the local submesh.

Sometimes we lack an analytic expression of the curve to be interpolated and, instead, it is given by a set of aligned points with a density high enough. This is the case, for example, of data supplied by digitalized maps describing coastal shores or river banks.

All these questions will be conveniently supported by examples.

Introduction

For 2-D or 3-D meshes the quality improvement [1] can be obtained by an iterative process in which each node of the mesh is moved to a new position that minimizes an objective function [2]. This function is derived from a quality measure of the local mesh. We have chosen, as a starting point in section 2, a 2-D objective function that presents a barrier in the boundary of the *feasible region* (set of points where the free node could be placed to get a *valid* local mesh, that is, without *inverted elements*). This barrier has an important role because it avoids the optimization algorithm to create a tangled mesh when it starts with a valid one. Nevertheless, objective functions constructed by algebraic quality measures are only directly applicable to inner nodes of 2-D or 3-D meshes, but not to its boundary nodes. To overcome this problem, the local mesh, $M(p)$, sited on a surface Σ, is orthogonally projected on a plane P (the existence and search of this plane will be discuss in section 3) in such a way that it performs a valid local mesh $N(q)$. Therefore, it can be said that $M(p)$ is *geometrically conforming* with respect to P. Here p is the free node on Σ and q is its projection on P. The optimization of $M(p)$ is got by the appropriated optimization of $N(q)$. To do this we try to get *ideal* triangles in $N(q)$ that become equilateral in $M(p)$. In general, when the local mesh $M(p)$ is on a surface, each triangle is placed on a different plane and it is not possible to define a feasible region on Σ. Nevertheless, this region is perfectly defined in $N(q)$ as it is analyzed in section 2.1.

To construct the objective function in $N(q)$, it is first necessary to define the objective function in $M(p)$ and, afterward, to establish the connection between them. A crucial aspect for this construction is to keep the barrier of the 2-D objective function. This is done with a suitable approximation in the process that transforms the original problem on Σ into an entirely two-dimensional one on P. We develop this approximation in section 2.2.

The optimization of $N(q)$ becomes a two-dimensional iterative process. The optimal solutions of each two-dimensional problem form a sequence $\{\mathbf{x}^k\}$ of points belonging to P. We have checked in many numerical test that $\{\mathbf{x}^k\}$ is always a convergent sequence. It is important to underline that this iterative process only takes into account the position of the free node in a discrete set of points, the points on Σ corresponding to $\{\mathbf{x}^k\}$ and, therefore, it is not necessary that the surface is smooth. Indeed, the surface determined by the piecewise linear interpolation of the initial mesh is used as a reference to define the geometry of the domain.

If the node movement only responds to an improvement of the quality of the mesh, it can happen that the optimized mesh loses details of the original surface. To avoid this problem, every time the free node p is moved on Σ, the optimization process only allows a small distance between the centroid of the triangles of $M(p)$ and the underlaying surface (the true surface, if it is known, or the piece-wise linear interpolation, if it is not).

There are several alternatives to the previous method. For example, Garimella et al. [3] develop a method to optimize meshes in which the nodes of the optimized mesh are kept close to the original positions by imposing the Jacobians of the current and original meshes to be also close. Frey et al. [4] get a control of the gap between the mesh and the surface by modifying the element-size (subdividing the longest edges and collapsing the shortest ones) in terms of an approximation of the smallest principal curvatures radius associated to the nodes. Rassineux et al. [5] also use the smallest principal curvatures radius to estimate the element-size compatible with a prescribed gap error. They construct a geometrical model by using the Hermite diffuse interpolation in which local operations like edge swapping, node removing, edge splitting, etc. are made to adapt the mesh size and shape. More accurate approaches, that have into account the directional behavior of the surface, have been considered by Frey in [6].

Application of our proposed optimization technique is shown in section 4.

1 Construction of the Objective Function

As it is shown in [2], [7], and [8] we can derive optimization functions from *algebraic quality measures* of the elements belonging to a local mesh. Let us consider a triangular mesh defined in $\mathbb{R}^2$ and let t be an triangle in the physical space whose vertices are given by $\mathbf{x}_k = (x_k, y_k)^T \in \mathbb{R}^2$, $k = 0, 1, 2$. First, we are going to introduce an algebraic quality measure for t. Let t_R be the reference triangle with vertices $\mathbf{u}_0 = (0,0)^T$, $\mathbf{u}_1 = (1,0)^T$, and $\mathbf{u}_2 = (0,1)^T$. If we choose $\mathbf{x}_0$ as the translation vector, the affine map that takes t_R to t is $\mathbf{x} = A\mathbf{u} + \mathbf{x}_0$, where A is the Jacobian matrix of the affine map referenced to node $\mathbf{x}_0$, given by $A = (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0)$. We will denote this type of affine maps as $t_R \xrightarrow{A} t$. Let now t_I be an *ideal* triangle (not necessarily equilateral) whose vertices are $\mathbf{w}_k \in \mathbb{R}^2$, $(k = 0, 1, 2)$ and let $W_I = (\mathbf{w}_1 - \mathbf{w}_0, \mathbf{w}_2 - \mathbf{w}_0)$ be the Jacobian matrix, referenced to node $\mathbf{w}_0$, of the affine map $t_R \xrightarrow{W_I} t_I$; then, we define $S = AW_I^{-1}$ as the weighted Jacobian matrix of the affine map $t_I \xrightarrow{S} t$. In the particular case that t_I was the equilateral triangle t_E, the Jacobian matrix $W_I = W_E$ will be defined by $\mathbf{w}_0 = (0,0)^T$, $\mathbf{w}_1 = (1,0)^T$ and $\mathbf{w}_2 = (1/2, \sqrt{3}/2)^T$.

We can use matrix norms, determinant or trace of S to construct algebraic quality measures of t. For example, the Frobenius norm of S, defined by $|S| = \sqrt{tr\,(S^T S)}$, is specially indicated because it is easily computable. Thus, it is shown in [1] that $q_\eta = \frac{2\sigma}{|S|^2}$ is an algebraic quality measure of t , where $\sigma = \det(S)$. We use this quality measure to construct an objective function. Let $\mathbf{x} = (x, y)^T$ be the position vector of the free node, and let S_m be the weighted Jacobian matrix of the m-th triangle of a valid local mesh of M triangles. The objective function associated to m-th triangle is $\eta_m = \frac{|S_m|^2}{2\sigma_m}$, and the corresponding objective function for the local mesh is the n-norm of $(\eta_1, \eta_2, \ldots, \eta_M)$,

$$\left|K_\eta\right|_n(\mathbf{x}) = \left[\sum_{m=1}^{M} \eta_m^n(\mathbf{x})\right]^{\frac{1}{n}} \tag{1}$$

This objective function presents a barrier in the boundary of the feasible region that avoids the optimization algorithm to create a tangled mesh when it starts with a valid one.

Previous considerations and definitions are only directly applicable for 2-D (or 3-D) meshes, but some of them must be properly adapted when the meshes are located on an arbitrary surface. For example, the concept of valid mesh is not clear in this situation because neither the concept of inverted element is. We will deal with these questions in next subsections.

1.1 Similarity Transformation for Surface- and Parametric Meshes

Suppose that for each local mesh $M(p)$ placed on the surface Σ, that is, with all its nodes on Σ, it is possible to find a plane P such that the orthogonal projection of $M(p)$ on P is a valid mesh $N(q)$. Moreover, suppose that we define the axes in such a way that the x, y-plane coincide with P. If, in the feasible region of $N(q)$, it is possible to define the surface Σ by the parametrization $\mathbf{s}(x, y) = (x, y, f(x, y))$, where f is a continuous function, then, we can optimize $M(p)$ by an appropriate optimization of $N(q)$. We will refer to $N(q)$ as the *parametric mesh*. The basic idea consists on finding the position $\bar{q}$ in the feasible region of $N(q)$ that makes $M(p)$ be an optimum local mesh. To do this, we search *ideal* elements in $N(q)$ that become equilateral in $M(p)$. Let $\tau \in M(p)$ be a triangular element on Σ whose vertices are given by $\mathbf{y}_k = (x_k, y_k, z_k)^T$, $(k = 0, 1, 2)$ and t_R be the reference triangle in P (see Figure 1). If we choose $\mathbf{y}_0$ as the translation vector, the affine map $t_R \overset{A_\pi}{\to} \tau$ is $\mathbf{y} = A_\pi \mathbf{u} + \mathbf{y}_0$, where A_π is its Jacobian matrix, given by

$$A_\pi = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \\ z_1 - z_0 & z_2 - z_0 \end{pmatrix} \tag{2}$$

Now, consider that $t \in N(q)$ is the orthogonal projection of τ on P. Then, the vertices of t are $\mathbf{x}_k = \Pi \mathbf{y}_k = (x_k, y_k)^T$, $(k = 0, 1, 2)$, where $\Pi = (\mathbf{e}_1, \mathbf{e}_2)^T$ is 2×3 matrix of the affine map $\tau \overset{\Pi}{\to} t$, being $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the canonical basis in $\mathbb{R}^3$ (the associated projector from $\mathbb{R}^3$ to P, considered as a subspace of $\mathbb{R}^3$, is $\Pi^T \Pi$). Taking $\mathbf{x}_0$ as translation vector, the affine map $t_R \overset{A_P}{\to} t$ is $\mathbf{x} = A_P \mathbf{u} + \mathbf{x}_0$, where $A_P = \Pi A_\pi$ is its Jacobian matrix

$$A_P = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} \tag{3}$$

Therefore, the 3×2 matrix of the affine map $t \overset{T}{\to} \tau$ is

$$T = A_\pi A_P^{-1} \tag{4}$$

Let V_π be the subspace spanned by the column vectors of A_π and let π be the plane defined by V_π and the point $\mathbf{y}_0$. Our goal is to find the *ideal* triangle $t_I \subset P$, moving q on P, such that t_I is mapped by T into an equilateral one, $\tau_E \subset \pi$. In general, the strict fulfillment of this requirement is only possible if $N(q)$ is formed by a unique triangle.

Due to $\text{rank}(A_\pi) = \text{rank}(A_P) = 2$, it exists a unique factorization $A_\pi = QR$, where Q is an orthogonal matrix ($Q^T Q = I$) and R is an upper triangular one with $[R]_{ii} > 0$ $(i = 1, 2)$. The columns of the 3×2 matrix Q define an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ that spans V_π, so we can see Q as the matrix of the affine map $t_R \overset{Q}{\to} \tau_R$ and R as the 2×2 Jacobian matrix of the affine map $\tau_R \overset{R}{\to} \tau$ (see Figure 1). As $t_R \overset{W_E}{\to} t_E$ and Q is an orthogonal matrix that keeps the angles and norms of the vectors, then $t_E \overset{Q}{\to} \tau_E$ and, therefore

$$QW_E = A_\pi R^{-1} W_E \tag{5}$$

is the 3×2 Jacobian matrix of affine map $t_R \overset{QW_E}{\to} \tau_E$. On the other hand, we define on the plane π

$$S = RW_E^{-1} \tag{6}$$

as the 2×2 weighted Jacobian matrix of the affine map that transforms the equilateral triangle into the physical one, that is, $\tau_E \overset{S}{\to} \tau$.

We have chosen as ideal triangle in π the equilateral one $(\tau_I = \tau_E)$, then, the Jacobian matrix W_I of the affine map $t_R \overset{W_I}{\to} t_I$ is calculated by imposing the condition $TW_I = QW_E$, because $t_R \overset{TW_I}{\to} \tau_I$ and $t_R \overset{QW_E}{\to} \tau_E$. Taking into account (5), it yields

$$TW_I = A_\pi R^{-1} W_E \tag{7}$$

and, from (4), we obtain

$$W_I = A_P R^{-1} W_E \tag{8}$$

so we define on P the *ideal-weighted* Jacobian matrix of the affine map $t_I \overset{S_I}{\to} t$ as $S_I = A_P W_I^{-1}$. From (8) it results

$$S_I = A_P W_E^{-1} R A_P^{-1} \tag{9}$$

and, from (6)

$$\begin{aligned} S_I &= A_P W_E^{-1} S W_E A_P^{-1} \\ &= A_P W_E^{-1} S \left(A_P W_E^{-1}\right)^{-1} = S_E S S_E^{-1} \end{aligned} \tag{10}$$

where $S_E = A_P W_E^{-1}$ is the *equilateral-weighted* Jacobian matrix of the affine map $t_E \overset{S_E}{\to} t$. Finally, from (10), we obtain the next similarity transformation.

$$S = S_E^{-1} S_I S_E \tag{11}$$

Therefore, it can be said that the matrices S and S_I are *similar.*

1.2 Optimization on the Parametric Space

It might be used S, as it is defined in (6), to construct the objective function and, then, solve the optimization problem. Nevertheless, this procedure has important disadvantages. First, the optimization of $M(p)$, working on the true surface, would require the imposition of the constraint $p \in \Sigma$.

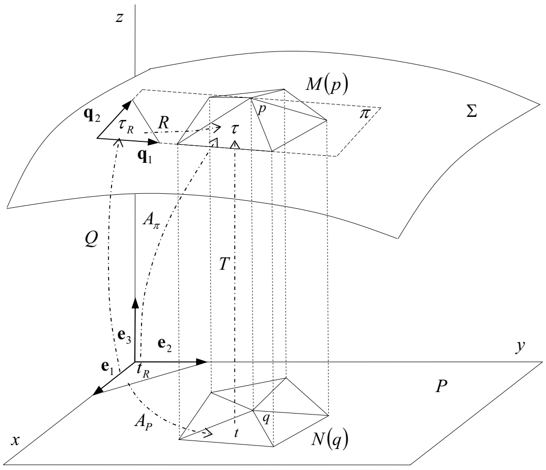

Figure 1: Local surface mesh $M(p)$ and its associated parametric mesh $N(q)$

It would complicate the resolution of the problem because, in many cases, Σ is not defined by a smooth function. Moreover, when the local mesh $M(p)$ is on a curved surface, each triangle is sited on a different plane and the objective function, constructed from S, lacks barriers. It is impossible to define a feasible region in the same way as it was done at the beginning of this section. Indeed, all the positions of the free node, except those that make $\det(S) = 0$ for any triangle, produce correct triangulations of $M(p)$. However, for many purposes as, for example, to construct a 3-D mesh from the surface triangulation, there are unacceptable positions of the free node.

To overcome these difficulties we propose to carry out the optimization of $M(p)$ in an indirect way, working on $N(q)$. With this approach the movement of the free node will be restricted to the feasible region of $N(q)$, which avoids to construct unacceptable surface triangulations. It all will be carried out using an approximate version of the similarity transformation given in (11).

Let us consider that $\mathbf{x} = (x, y)^T$ is the position vector of the free node q, sited on the plane P. If we suppose that Σ is parametrized by $\mathbf{s}(x, y) = (x, y, f(x, y))$, then, the position of the free node p on the surface is given by $\mathbf{y} = (x, y, f(x, y))^T = (\mathbf{x}, f(\mathbf{x}))^T$.

Note that $S_E = A_P W_E^{-1}$ only depends on $\mathbf{x}$ because W_E is constant and A_P is a function of $\mathbf{x}$. Besides, $S_I = A_P W_I^{-1}$ depends on $\mathbf{y}$, due to $W_I = A_P R^{-1} W_E$, and R is a function of $\mathbf{y}$. Thus, we have $S_E(\mathbf{x})$ and $S_I(\mathbf{y})$. We shall optimize the local mesh $M(p)$ by an iterative procedure maintaining constant $W_I(\mathbf{y})$ in each step. To do this, at the first step, we fix $W_I(\mathbf{y})$ to its initial value, $W_I^0 = W_I(\mathbf{y}^0)$, where $\mathbf{y}^0$ is given by the initial position of p. So, if we define $S_I^0(\mathbf{x}) = A_P(\mathbf{x})(W_I^0)^{-1}$, we approximate the similarity transformation (11) as

$$S^0(\mathbf{x}) = S_E^{-1}(\mathbf{x}) S_I^0(\mathbf{x}) S_E(\mathbf{x}) \tag{12}$$

Now, the construction of the objective function is carried out in a standard way, but using S^0 instead of S. So, we obtain the objective function for a given triangle $\tau \subset \pi$

$$\eta^0(\mathbf{x}) = \frac{|S^0(\mathbf{x})|^2}{2\sigma^0(\mathbf{x})} \tag{13}$$

where $\sigma^0(\mathbf{x}) = \det(S^0(\mathbf{x}))$.

With this approach the optimization of the local mesh $M(p)$ is transformed into a two-dimensional problem without constraints, defined on $N(q)$, and, therefore, it can be solved with low computational cost. Furthermore, if we write W_I^0 as $A_P^0 (R^0)^{-1} W_E$, where $A_P^0 = A_P(\mathbf{x}^0)$ and $R^0 = R(\mathbf{y}^0)$, it is straightforward to show that S^0 can be simplified as

$$S^0(\mathbf{x}) = R^0 \left(A_P^0\right)^{-1} S_E(\mathbf{x}) \tag{14}$$

and our objective function for the local mesh is

$$\left|K_\eta^0\right|_n(\mathbf{x}) = \left[\sum_{m=1}^{M} \left(\eta_m^0\right)^n(\mathbf{x})\right]^{\frac{1}{n}} \tag{15}$$

Let now analyze the behavior of the objective function when the free node crosses the boundary of the feasible region. If we denote $\alpha_P = \det(A_P)$, $\alpha_P^0 = \det\left(A_P^0\right)$, $\rho^0 = \det\left(R^0\right)$, $\omega_E = \det(W_E)$ and taking into account (14), we can write $\sigma^0 = \rho^0 \left(\alpha_P^0\right)^{-1} \alpha_P \omega_E^{-1}$. Note that ρ^0, α_P^0, and ω_E are constants, so η^0 has a singularity when $\alpha_P = 0$, that is, when q is placed on the boundary of the feasible region of $N(q)$. This singularity determines a barrier in the objective function that prevents the optimization algorithm to take the free node outside this region. This barrier does not appear if we use the exact weighted Jacobian matrix S, given in (6), due to $\det(R) = R_{11} R_{22} > 0$.

Suppose that $\mathbf{x}^1 = \bar{\mathbf{x}}^0$ is the minimizing point of (15). As this objective function has been constructed by keeping $\mathbf{y}$ in its initial position, $\mathbf{y}^0$, then $\mathbf{x}^1$ is only the first approximation to our problem. This result is improved updating the objective function at $\mathbf{y}^1 = (\mathbf{x}^1, f(\mathbf{x}^1))^T$ and, then, computing the new minimizing position, $\mathbf{x}^2 = \bar{\mathbf{x}}^1$. This local optimization process is repeated, obtaining a sequence $\{\mathbf{x}^k\}$ of optimal points, until a convergence criteria is verified. We have experimentally verified in numerous tests, involving continuous functions to define the surface Σ, that this algorithm converges.

Let us consider P as an optimal projection plane (this aspect will be discussed in next section). In order to prevent a loss of the details of the original geometry, our optimization algorithm evaluates the difference of heights ($[\Delta z]$) between the centroid of the triangles of $M(p)$ and the reference surface, every time a new position $\mathbf{x}^k$ is calculated. If this distance exceeds a threshold, $\Delta(p)$, the movement of the node is aborted and the previous position is stored. This threshold $\Delta(p)$ is established attending to the size of the elements of $M(p)$. In concrete, the algorithm evaluates the average distance between the free node and the nodes connected to it, and takes $\Delta(p)$ as percentage of this distance. Other possibility is to fix $\Delta(p)$ as a constant for all local meshes. In the particular case in which we have an explicit representation of the surface by a function $f(x, y)$, $\Delta(p)$ can be established as a percentage of the maximum difference of heights between the original surface and the initial mesh.

2 Search of the Optimal Projection Plane

The former procedure needs a plane in which the local mesh, $M(p)$, is projected conforming a valid mesh, $N(q)$. If this plane exists it is not unique, because a small rotation of the coordinate system produces another valid projection plane, that is, another plane in which $N(q)$ is valid. We have observed that the number of iterations required by our procedure depends on the chosen plane. In general, this number is less if the plane is well *faced* to $M(p)$. We have to find the rotation of reference system x, y, z such that the new x', y'-plane, P', is optimal with respect to a suitable criterion.

We will denote $N(q')$ as the projection of $M(p)$ onto P' and t' the projection of the physical triangle $\tau \in M(p)$ onto P'. Let $A'_P = (\mathbf{x}'_1 - \mathbf{x}'_0, \mathbf{x}'_2 - \mathbf{x}'_0)$ be the matrix associated to the affine map that takes the reference element defined on P' to t', then, the area of t' is given by $\frac{1}{2}|\alpha'_P|$ where $\alpha'_P = \det(A'_P)$.

Our goal is to find a coordinate system rotation such that $\sum_{m=1}^{M} \alpha'_{P_m}$ is maximum satisfying the constraints $\alpha'_{P_m} = \det\left(A'_{P_m}\right) > 0$ for all the triangles of $N(q')$, that is, $m = 1, ..., M$. In [9] a method to determine a projection plane is considered but without the enforcement of these constraints.

According to Euler's rotation theorem, any rotation may be described using three angles. The so-called *x-convention* is the most common definition. In this convention, the rotation is given by Euler angles (ϕ, θ, ψ), where the first rotation is by an angle $\phi \in [0, 2\pi]$ about the z-axis, the second is by an angle $\theta \in [0, \pi]$ about the x-axis, and the third is by an angle $\psi \in [0, 2\pi]$ about the z-axis (again).

Let $\Phi(\phi, \theta, \psi)$ be the Euler's rotation matrix such that $\mathbf{y}' = \Phi\mathbf{y}$, then, the Jacobian matrix $A_\pi = (\mathbf{y}_1 - \mathbf{y}_0, \mathbf{y}_2 - \mathbf{y}_0)$ associated to the triangle τ of $M(p)$, defined in (2), can be spanned on the rotated coordinate system as $A'_\pi = (\mathbf{y}'_1 - \mathbf{y}'_0, \mathbf{y}'_2 - \mathbf{y}'_0) = \Phi A_\pi$. Thus, the Jacobian matrix A'_P is written as $A'_P = \Pi A'_\pi = \Pi\Phi A_\pi$. With these considerations it is easy to proof that the value of α'_P is

$$\begin{aligned} \alpha'_P &= \det(\Pi\Phi A_\pi) = m_1 \sin(\phi)\sin(\theta) \\ &+ m_2 \sin(\theta)\cos(\phi) + m_3\cos(\theta) \end{aligned} \qquad (16)$$

where m_i is the minor obtained by deleting the i-th row of A_π. Note that equation (16) only depends on ϕ and θ angles, as was to be expected.

Although the above maximization problem can be solved taken into account the constraints, we propose an unconstrained approach.

Let us consider, as a first attempt, the objective function $\sum_{m=1}^{M} (\alpha'_{P_m})^{-1}(\phi, \theta)$. The minimization of this function tends to maximize the values of α'_{P_m} and, due to the barrier that appears when $\alpha'_{P_m} = 0$ for some triangle of $N(q')$, the values of α'_{P_m} are maintained positive if the minimization algorithm starts at an interior point, that is, a point (ϕ_0, θ_0) belonging to the set Ψ of angles (ϕ, θ) such that $\alpha'_{P_m}(\phi, \theta) > 0$ for ($m = 1, ..., M$). On the other hand, if any $\alpha'_{P_m} < 0$ the barrier prevents to reach the required minimum. In next paragraph we propose a method to find an interior point (ϕ_0, θ_0) of Ψ to be used as a starting point in the minimization algorithm.

Let $G = [\mathbf{g}_m]$ be the $3 \times M$ matrix formed by the vectors, $\mathbf{g}_m$, normal to the triangles of $M(p)$. A solution of the inequality system (if it exists) $G^T\mathbf{g} > \mathbf{0}$ provides a direction, defined by vector $\mathbf{g}$, such that all the triangles of $M(p)$ can be projected on a plane, normal to the unitary vector $\mathbf{n} = \frac{\mathbf{g}}{\|\mathbf{g}\|}$, so that $\alpha'_{P_m} > 0$ for ($m = 1, ..., M$). Then, it only remains to find the angles ϕ_0 and θ_0 in which the coordinate system needs to be rotated to get the z' axis to point in the direction of $\mathbf{n}$. More precisely, the angles ϕ_0 and θ_0 are the solution of the equation $\Phi^T(\phi_0, \theta_0, 0)\,\mathbf{e}_3 = \mathbf{n}$, where $\mathbf{e}_3 = (0, 0, 1)^T$. If the inequality system has not solution, then, there is not any valid projection plane for this local mesh, against the premise done in section 2.1. In this case, the local optimization procedure maintains the free node p at its initial position.

We have observed that the previous objective function has computational difficulties as the optimization algorithms use discrete steps to search the optimal point. A step leading outside the region Ψ may indicate a decrease in the value of the objective function and take to a false solution. To overcome this problem we propose a modification of the objective function in such a way that it will be regular all over $\mathbb{R}^3$ and its barrier will be "smoothed". The modification consists of substituting α'_{P_m} by $h(\alpha_{P_m})$, where $h(\alpha)$ is the positive and increasing function introduced in [10] and given by

$$h(\alpha) = \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4\delta^2}) \tag{17}$$

being the parameter $\delta = h(0)$. The behavior of $h(\alpha)$ in function of δ parameter is such that, $\lim_{\delta\to 0} h(\alpha) = \alpha, \forall\alpha \geq 0$ and $\lim_{\delta\to 0} h(\alpha) = 0, \forall\alpha \leq 0$. The characteristics of h function and its application in the context of mesh untangling and smoothing have been studied in [10] and [11]. Thus, the proposed objective function for searching the projection plane is

$$\Omega(\phi,\theta) = \sum_{m=1}^{M} \frac{1}{h(\alpha'_{P_m}(\phi,\theta))} \tag{18}$$

A crucial property is that the angles that minimize the original and modified objective functions are nearly identical when δ is *small*. Details about the determination of δ value for 3-D triangulations can be found in [11].

3 Matching curves defined on surfaces

Node movement provides a surface mesh the ability to match an arbitrary curve. Suppose that the surface mesh, M, is projectable on a unique plane P forming a parametric mesh, N. If C is a curve defined on P, our objective is to move some nodes of N, projecting them on C, to get an interpolation of C by edges of N. Note that, associated to this interpolation, there is a mapped interpolation on M. To achieve this objective we have to decide which nodes of N can be projected on C without inverting any triangle of its local submesh. More accurately, we say that a free node q is projectable on C if it exists any point of C, say q', such that the resulting local submesh $N(q)$ has not any inverted triangle after carrying q to the position of q'. In general, if q is projectable, its possible placement on C is not unique and, therefore, we have to determine the "best" position to relocate q. To decide which is the best position of this node we could think on minimizing the objective function $\left|K_\eta^0\right|_n(\mathbf{x})$ [14] subject to the constrained $\mathbf{x} \in C$. Nevertheless, this function only works properly when $N(q)$ is not tangled. To overcome this problem we propose to modify this objective function following the criteria developed in [11]. This modification consists of substituting σ^0 in (13) by the positive and increasing function $h(\sigma^0)$, so that the barrier associated with the singularities of $\left|K_\eta^0\right|_n(\mathbf{x})$ will be eliminated and the new function will be smooth all over $\mathbb{R}^2$. If $\left|K_\eta'^0\right|_n(\mathbf{x})$ is the modified objective function, the problem of finding the optimal position to project the free node on C is

$$\text{minimize } \left|K_\eta'^0\right|_n(\mathbf{x}), \text{ subject to } \mathbf{x} \in C \tag{19}$$

The objective function $\left|K_\eta'^0\right|_n$ strongly penalizes the negative values of σ^0, so that, the minimization process of (19) leads to the construction of a local submesh $N(q)$ without inverted triangles, provided it is possible. Then, if $\bar{\mathbf{x}}$ is the minimizing position of (19) and $\sigma^0(\bar{\mathbf{x}}) > 0$ for all triangle of $N(q)$, we conclude that q is projectable on C and $\bar{\mathbf{x}}$ is its optimal position.

The projection of a free node on C can give rise to a local mesh with very poor quality. This effect is partly palliated after smoothing the remainder nodes, following the procedure described in section 2.2. Moreover, we have observed that the final mesh has better quality if the constraint $\sigma^0(\bar{\mathbf{x}}) > 0$ is substituted by the most restrictive one $\sigma^0(\bar{\mathbf{x}}) > \epsilon$ for all triangle of $N(q)$, were $\epsilon > 0$ is a decreasing parameter that tends to zero as the number of global iterations increases.

The nodes are inserted in the curve without specific criterion, just according to the increasing order of its numeration. This produces situations in which some sections of the curve C can not be interpolated by edges of N without removing some nodes previously projected on C. The figure 2(a) shows a scheme of this problem and figures 2(b) and (c) explain the way to solve it by a convenient displacement of the two extreme nodes.

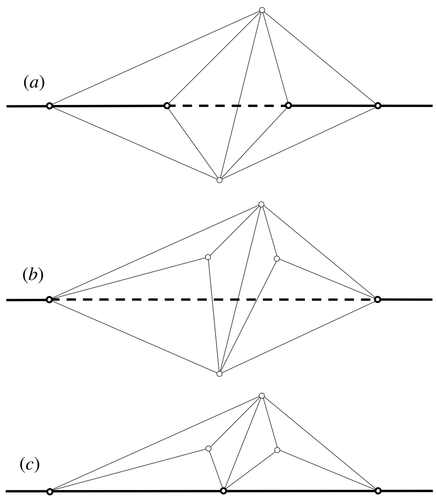

Figure 2: The line (in bold) is non-recoverable if the two extreme nodes are not moved (a). The extreme nodes are removed from the line (b) until another one takes its place (c)

In some applications we lack an analytic expression of the curve to be interpolated. Only a set of aligned points

$\{q_c\}$ that approximately describes a contour is available. This is the case, for example, of data supplied by digitalized maps describing coastal shores or river banks. To approach this situation we solve a discrete version of (19). Given local submesh $N(q)$, we analyze if q is projectable on any point of $\{q_c\}$, that is, we check if the condition $\sigma^0(\mathbf{x}) > \epsilon$ for all triangle of $N(q)$ is satisfied when $\mathbf{x}$ cover $\{q_c\}$. Among the positions $\mathbf{x}$ that satisfy previous condition we choose the optimal point, $\bar{\mathbf{x}}$, as the one that minimizes $\left|K_\eta'^0\right|_n$. We must underline that this problem is correctly defined only if the density of points of $\{q_c\}$ is high enough. Typically, the distance between contiguous points of $\{q_c\}$ must be much shorter than the distances between adjacent nodes of N.

Usually, most of nodes of N are very far from any point of $\{q_c\}$ and, therefore, they are not projectable, so it is advantageous to have a previous knowledge of which nodes are candidates to be projected. A possibility is to associate to each node of both N and $\{q_c\}$ the square of a regular grid in which it is included. Let us suppose that the size of these squares is $d_{max} \times d_{max}$, being d_{max} the maximum edge present at the mesh. We can take a quick decision about if the node q is candidate to be projectable on $\{q_c\}$ only by inspecting the region, S_q, formed by the square that contains q and the surrounding squares. Firstly, we find the subset $\{q_c'\}$ of points belonging to $\{q_c\}$ and included in S_q. If $\{q_c'\} \neq \emptyset$, we analyze if q is projectable on $\{q_c'\}$ as it was explained above. Note that the distance between q and any point of $\{q_c\}$ not in S_q is greater than d_{max} and, consequently, outside the feasible region of $N(q)$ (the feasible region of $N(q)$ is included in the circle of radius d_{max} and center q).

4 Application

4.1 Application to scanned objects

In this subsection the proposed technique is applied to smooth a mesh obtained from *http://www.cyberware.com/*. The object is a screwdriver (see Figure 3) with 27150 triangles and 13577 nodes.

The projection plane for this surface triangulation have been chosen in terms of the local mesh to be analyzed. We have used the objective function (1) with $n = 2$.

The average quality for this application is increased from 0.822 to 0.920 in four iterations, see Figure 4. The worst 500 triangles increases its average quality from 0.486 to 0.704. It is important to remark that the original geometry is almost preserved in the optimization process. The quality curves are shown in Figure 5. This curve is obtained by sorting the elements in increasing order of its quality, $q(e)$.

We have fixed $\Delta(p)$ to 10% of average distance between the free node and the nodes connected to it. The number of not moved nodes by the algorithm with this election of $\Delta(p)$ have been 85 in the first iteration, 167 in the second, 187 in the third, and 193 in the fourth one. We remark that the quality curves from the first to the fourth iteration are very close. In particular, the algorithm only needs one iteration to reach an average quality 0.907.

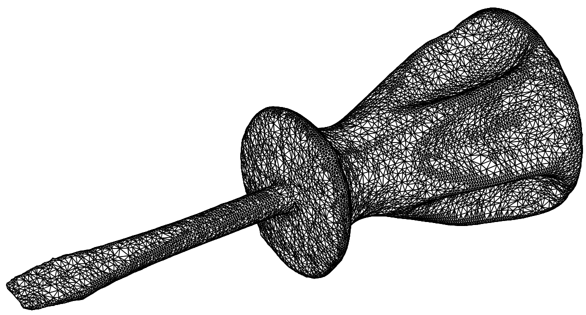

Figure 3: Original mesh of a screwdriver from *http://www.cyberware.com/*.

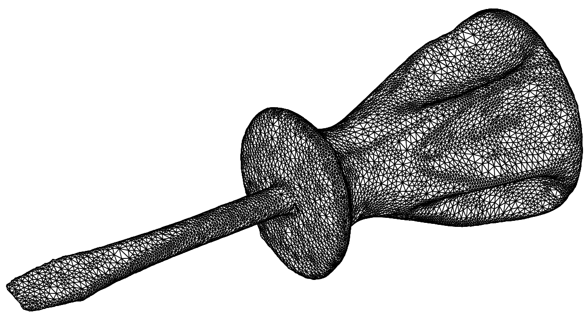

Figure 4: Optimized mesh of the screwdriver after four iterations.

4.2 Mesh adaption to prescribed contours in orographic surfaces

In many cases of environmental modelling, there are some contour lines which determinate certain characteristics of the studied region. For example, in wind simulation [13] the well definition of contour lines of very steep slopes may be very important for obtaining accurate results, since a change in the direction of edges of the mesh can strongly affect the computed wind. Thus, an accurate mesh must be adapted to follows these contours lines. Figure 10 shows the adaption of a nested mesh, related to a region of the north west of Gran Canaria Island, to the shore line (plotted by points in

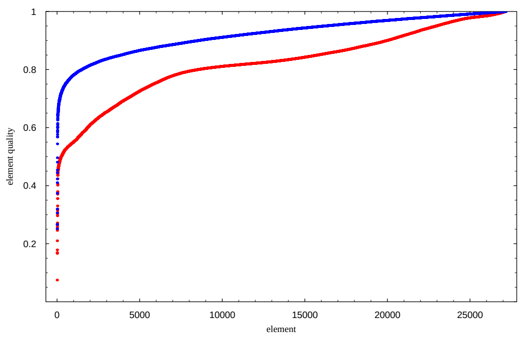

Figure 5: Quality curves for the initial (red line) and optimized (blue line) meshes for the screwdriver.

red). A detail of the initial and adapted meshes are shown in figures 6) and 7, respectively.

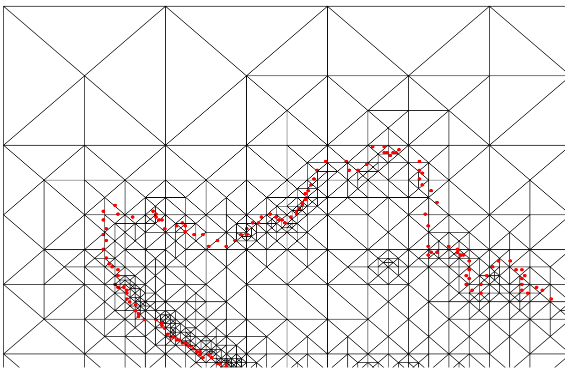

Figure 6: A detail of the coast in the north west of Gran Canaria Island corresponding to the not adapted mesh.

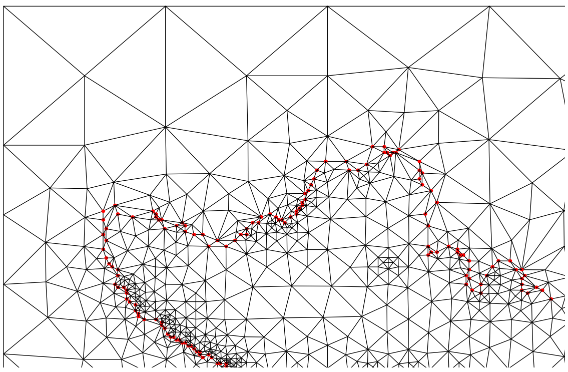

Figure 7: A detail of the coast in the north west of Gran Canaria Island corresponding to the adapted mesh.

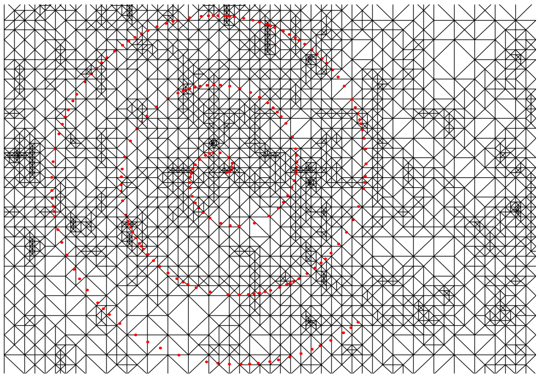

Figure 8: Region defined in the surrounding of Arucas Mountain (Gran Canaria Island). Contour is defined as an spiral line around the mountain to which the initial mesh must be adapted.

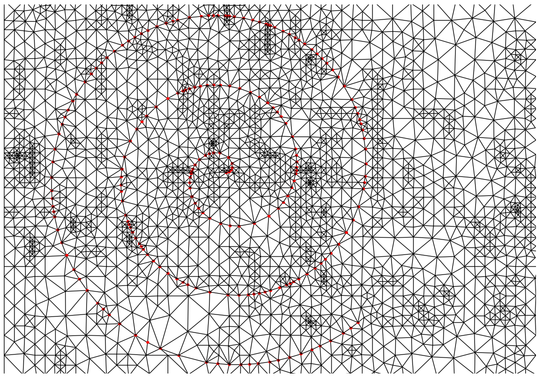

Figure 9: Region defined in the surrounding of Arucas Mountain. Contour plots and adapted mesh.

The second example corresponds to a mesh of another region of Gran Canaria Island in the surrounding of Arucas Mountain (figure 8) that is adapted to a spiral around the mountain (an imaginary road), see figure 9. In this view, we can clearly see how the edges of the mesh end up being placed on the curve. Figure 11 shows a 3-D view detail of the adapted mesh.

5 Conclusions and Future Research

We have developed an algebraic method to optimize triangulations defined on surfaces. Its main characteristic is that the original problem is transformed into a fully two-dimensional sequence of approximate problems on the parametric space.

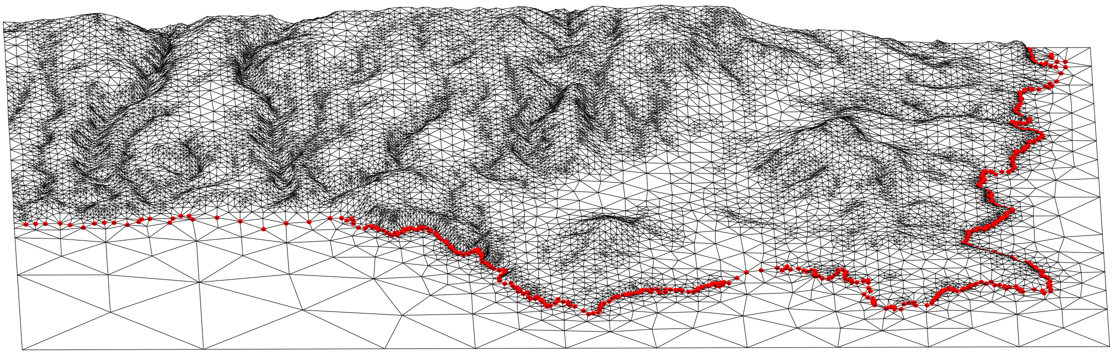

Figure 10: Region defined in the north west of Gran Canaria Island. Contour plots and adapted mesh.

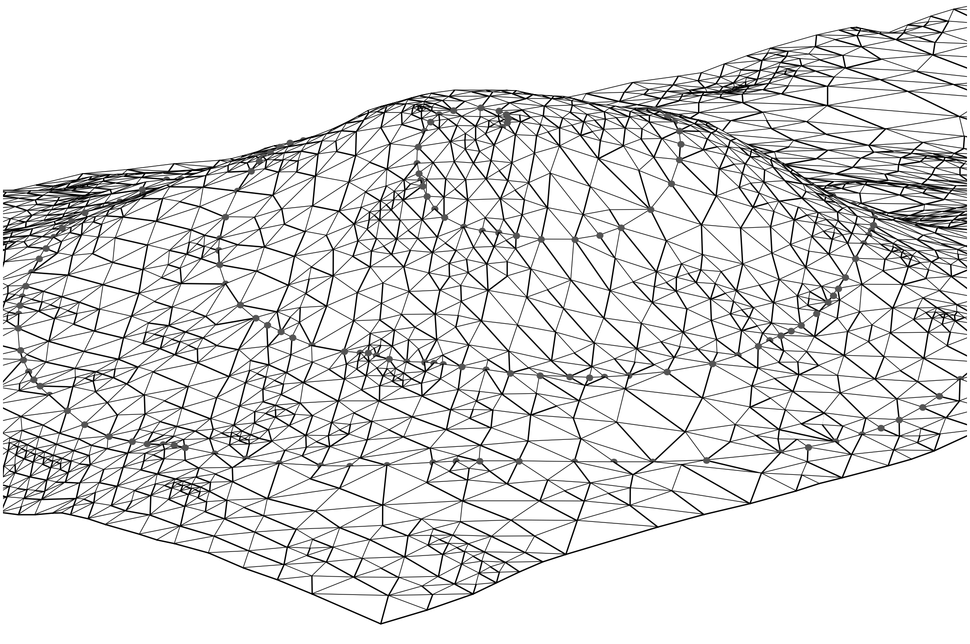

Figure 11: A detail of the adapted mesh to the spiral in Arucas Mountain.

This characteristic allows the optimization algorithm to deals with surfaces that only need to be continuous. Moreover, the barrier exhibited by the objective function in the parametric space prevents the algorithm to construct unacceptable meshes.

We have also introduced a procedure to find an optimal projection plane (our parametric space) based on the minimization of a suitable objective function. We have observed that correct choice of this plane plays a relevant role.

We have shown how the technique of surface mesh smoothing can be used to match an arbitrary curve. This last application requires the mesh can be projected in a a unique projection plane. We propose the generalization of these ideas to avoid the this restriction. Also, we think that the this procedure could be extrapolated for matching surfaces defined in 3-D meshes.

The optimization process includes a control on the gap between the optimized mesh and the reference surface that avoids to lose details of the original geometry. In this work we have used a piecewise linear interpolation to define the reference surface when the true surface is not known, but it would be also possible to use a more regular interpolation, for example, the proposed in [5]. Likewise, it would be possible to introduce a more sophisticated criterion for the gap control, by using a local refinement/derefinement techniques, that takes into account the curvature of the surface [4], [5], [6].

In the present work we have only considered a sole objective function obtained from an isotropic and area independent algebraic quality metric. Nevertheless, the framework that establishes the *algebraic quality measures* [1] provides us the possibility to construct anisotropic and area sensitive objective functions by using a suitable metric.

In future works we will use the present smoothing technique for improving the mesh quality of the boundary of 3-D domain triangulations defined over complex terrains [12]. A simultaneous smoothing and untangling procedure [11] could be applied to inner nodes of the domain after. Authors have developed this tetrahedral mesh generator for wind field simulation in realistic problems [13].

Acknowledgments

This work has been supported by the Spanish Government and FEDER, grant contracts: REN2001-0925-C03-02/CLI and CGL2004-06171-C03-02/CLI.

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