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MULTIFEBE: AN OPEN-SOURCE MULTI-DOMAIN INTEGRATED FINITE ELEMENT AND BOUNDARY ELEMENT SOFTWARE

Jacob D.R. Bordón, Ángel G. Vega-Artiles, Guillermo M. Álamo, Luis A. Padrón, Juan J. Aznárez and Orlando Maeso

[1] Instituto Universitario de Sistemas Inteligentes y Aplicaciones Numéricas en Ingeniería
Universidad de Las Palmas de Gran Canaria
Las Palmas de Gran Canaria 35017, Spain
e-mail: {jacobdavid.rodriguezbordon,angelgabriel.vega,
guillermo.alamo,luis.padron,juanjose.aznarez, orlando.maeso}@ulpgc.es
Web page: http://mmc.siani.es

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Abstract. *This paper presents MultiFEBE: an open-source solver for multi-domain integrated Finite Element and Boundary Element models for linear static and time harmonic analysis based on an integration of several BEM-FEM models for soil-shell and soil-beam interaction. The code, developed in the University Institute for Intelligent Systems and Numerical Applications in Engineering (SIANI) of the University of Las Palmas de Gran Canaria [1], allows the static and time-harmonic analysis of many different problem typologies that involve coupling of bounded or unbounded three-dimensional regions with structural elements (beams and shells) [2, 3]. The code allows to model bounded and unbounded potential, viscoelastic and poroelastic three-dimensional regions using the Boundary Element Method; and potential and viscoelastic bounded three-dimensional regions, or structural shell and beam elements, through the Finite Element Method. The code is specially suited to model static and time-harmonic problems that involve coupling between the different types of regions or elements mentioned above, and its more salient features are those related to the modelling of problems in which thin shell structures interact with solid regions, such as in the case of bucket and pile foundations. This paper summarizes very briefly the models and numerical methods implemented in the code, and illustrates its use and capabilities through different application examples.*

1 INTRODUCTION

The scientific community is gradually transitioning to a complete open science, where not only discoveries, theories and studies are available through documents, but also data (data files) and methods (computer codes) are made accesible in standard formats at

public repositories [4]. This open interchange facilitates research and development in science and engineering.
In that sense, this paper aims to present a computational mechanics computer code which is a modern and integrated result of years of developments in our research team. Within computational mechanics, our main research lines are related to the Boundary Element Method (BEM), which was introduced in Spain and in our team by Professor José Domínguez, one of the pioneers of the numerical method [5]. Through the years, specialized and extended computer codes derived from the basic codes published in [6] have been used. For instance, codes for the earthquake analysis of dams including interaction between dam, foundation, reservoir, reservoir sediments [7, 8, 9, 10], where a multi-domain BEM approach (BEM-BEM) allowing the linear dynamic interaction of regions of different nature: fluids (acoustic medium), elastic solids and poroelastic media (Biot's poroelasticity); was developed. It also allowed the dynamic analysis of foundations such as piles and pile groups [11]. Another codes emerged for two-dimensional acoustic analysis of noise barriers [12]. A novel methodology coupling beams modelled with the Finite Element Method (FEM) and the BEM for elastic solids was also developed [2], which allowed a much computationally efficient dynamic analysis of piles and pile groups. Later, this idea of mixed-dimensional coupling (1D FEM structure - 3D BEM medium) was extended for the cases of 1D FEM structure - 2D BEM media (fluid, elastic solid, poroelastic medium) and 2D FEM structure (plates, shells) - 3D BEM media (fluid, elastic solid, poroelastic medium) in a number of papers [13, 14, 3]. Mixed-dimensional coupling between BEM and FEM is particularly interesting for the dynamic study of structures interacting with the surrounding media because it combines the best of both worlds: BEM intrinsic fulfillment of the Sommerfeld radiation condition (for unbounded domains), and FEM ability to model structural elements. Those latter coupling strategies required the derivation and implementation of advanced BEM formulations, called Hypersingular and Dual BEM, for different type of media (fluid, elastic solid, poroelastic medium), which are both analytically and numerically [15] challenging. Given the number of different modelling options available from the developed methodologies, and also the commonalities between them, we decided to organize and integrate them all in a unified and more modern code which is MultiFEBE (available at `https://github.com/mmc-siani-es/MultiFEBE`).

2 MECHANICAL MODELS AND NUMERICAL MODELLING

2.1 Available mechanical models and basic assumptions

MultiFEBE allows to perform mechanical analyses on material bodies whose static or time–harmonic behaviour can be described as one of the following three mathematical models:

- Inviscid fluids that can be represented through the Helmholtz's equation for the propagation of waves in fluids at rest with negligible viscosity,
- Linear–elastic solids that can be represented through the very well known Navier's

equations for elasticity, and

- Poroelastic solids that can be represented through Biot's equations of the poroelasticity [16], when the region is assumed to consist on an elastic solid frame saturated by a compressible viscous fluid.

In all cases, the following classical simplifying hypotheses are assumed: isothermal, isentropic and adiabatic processes, small displacements, isotropy and homogeneity. Loads are considered static or time harmonic. For linear–elastic solids, not only two– and three–dimensional solids are considered, but also some structural elements (bar, beam/arch, plate/shell) are modelled. These simplified models are advantageous because they are more simple to define, computationally more efficient, and their results are more simple to analyze than their three–dimensional counterparts.

2.2 Available numerical models

2.2.1 The Finite Element Method

The Finite Element Method [17, 18] allows the numerical modelling of engineering problems, such as those related to solid and structural mechanics. The governing equations of the problems at hand, described above, are transformed into set to algebraic equations, whose unknowns (active degrees of freedom) describe the relevant field variables (e.g displacements) throughout the domain. This transformation is performed through a weighted residual approach or via the use of the principle of virtual work and through the partition of each domain into a set of subdomains (finite elements) within which the field variables are assummed to vary according to some function (typically polynomials).
On an element level, algebraic equations appear in the form of an equilibrium equation:

$$\left(\mathbf{K}^{(e)} - \omega^2 \mathbf{M}\right) \cdot \mathbf{a}^{(e)} - \mathbf{Q}^{(e)} \cdot \mathbf{f}^{(e)} = \mathbf{q}^{(e)} \tag{1}$$

where $\mathbf{K}^{(e)}$ and $\mathbf{M}^{(e)}$ respectively are the element stiffness and mass matrices, $\mathbf{a}^{(e)}$ is the vector of element degrees of freedom (displacements, rotations, etc.) of each element node, $\mathbf{Q}^{(e)}$ is the matrix which transforms distributed forces $\mathbf{f}^{(e)}$ into equivalent nodal loads, and $\mathbf{q}^{(e)}$ is the element equilibrating load vector. All element equilibrium equations are assembled using compatilibily and equilibrium conditions to form the global equilibrium equation, which together with the boundary conditions allow building a global algebraic system of equations.
The methodology is quite general, versatile, and computationally efficient, which makes the FEM the dominant numerical modelling approach in this field. It has many useful modelling advantages. It can easily handle material heterogeneities and non-linearities (simply by changing material properties on an element level). It also can easily handle structural elements, and their interconection. However, it also has some disadvantages. For example, because the method requires the domain discretization, handling of unbounded domains is difficult and it requires of special devices.

MultiFEBE includes elastic solid finite elements, straight beam finite elements (Euler-Bernoulli and Timoshenko), curved beam finite elements degenerated from the solid (Timoshenko), shell finite elements degenerated from the solid (Reissner-Mindlin), and discrete finite elements (masses, springs, dashpots). For further reading, we refer to Bathe [17] Oñate [19, 20].

2.2.2 The Boundary Element Method

The Boundary Element Method [6, 21] is an alternative numerical method to solve many problems that the FEM can also solve. It is less versatile because it requires at least the fundamental solution of the governing equation, or a Green's function which not only satisfies the governing equation but also some boundary conditions. This requirement constrains the range of applicability, but it also reduces the discretization to the boundary. It also intrinsically fulfills the Sommerfeld radiation condition, so it handles unbounded regions very efficiently. Therefore, it is a very appropriate tool when dealing with wave propagation phenomena through soil, water and air.

The main ingredients of the BEM are the Boundary Integral Equations (BIEs), which once discretized are used to build a solvable linear system of equations. BIEs can be obtained from several starting points, typically from a weighted residual formulation of governing equations or directly from reciprocity relationships [6, 21]. They relate the value of a field variable (displacements, stresses, etc.) at a given point (collocation point $\mathbf{x}^{\mathrm{i}}$) with the value of primary and secondary field variables along the boundary Γ of the domain. If body loads are present, domain integrals also appear in them. For example, for an elastic solid domain, the displacement u_l^{i} and traction t_l^{i} (for a given unit normal $\mathbf{n}$) at a collocation point $\mathbf{x}^{\mathrm{i}} \in \Omega$ can be obtained from:

$$u_l^{\mathrm{i}} + \int_\Gamma t_{lk}^* u_k \, \mathrm{d}\Gamma = \int_\Gamma u_{lk}^* t_k \, \mathrm{d}\Gamma + \int_\Omega u_{lk}^* b_k \, \mathrm{d}\Omega \tag{2a}$$

$$t_l^{\mathrm{i}} + \int_\Gamma s_{lk}^* u_k \, \mathrm{d}\Gamma = \int_\Gamma d_{lk}^* t_k \, \mathrm{d}\Gamma + \int_\Omega d_{lk}^* b_k \, \mathrm{d}\Omega \tag{2b}$$

where u_{lk}^*, t_{lk}^*, d_{lk}^* and s_{lk}^* are the fundamental solution (or Green's function) and derivatives. When $\mathbf{x}^i \to \Gamma$ these BIEs become singular, and an appropriate sense of integration must be considered (Cauchy Principal Value and Hadamard Finite Part). The former BIE is usually called the Singular BIE (SBIE) and the latter is called the Hypersingular BIE (HBIE). With these two BIEs taken individually or in combination (Dual BIE) it is possible to model domains with ordinary boundaries and interfaces (ordinary boundary elements), but also degenerated boundaries such as cracks (crack boundary elements). The discretization of body loads is similarly established by defining what we call body load elements. The discretized form of the BIEs once they are collocated throughout the boundaries can be written as:

$$\mathbf{H} \cdot \mathbf{u} = \mathbf{G} \cdot \mathbf{t} + \mathbf{G}_b \cdot \mathbf{b} \tag{3}$$

where $\mathbf{H}$ and $\mathbf{G}$ are the so-called influence matrices (fully populated), $\mathbf{u}$ and $\mathbf{t}$ gathers respectively the nodal displacements and tractions, $\mathbf{G}_b$ is another fully populated matrix related to the body loads gathered in $\mathbf{b}$. Once boundary conditions are considered, this matrix equation can be written as a linear system of equations which allows obtaining the solution.

MultiFEBE includes plain strain and three-dimensional boundary element modelling of inviscid fluid, elastic solid and Biot's poroelastic medium, including interfaces and crack-like boundaries in all cases. In all cases, the fundamental solution is used (full-space point load solution), except for elastostatics and inviscid fluid where half-space point load Green's function is available. These half-space Green's functions do not require the discretization of the free-surface, which greatly reduces computational time. For further reading, we refer to Brebbia and Domínguez [6, 21], or Bordón [1] for more references and particular details.

2.3 BEM-BEM and BEM-FEM coupling

One of the main distinctive features of the presented software is the integration of many types of coupling between BEM regions (BEM-BEM coupling), and BEM and FEM regions (BEM-FEM coupling).

BEM-BEM couplings arise when two BEM regions of the same, or different, mechanical models (inviscid fluid, elastic solid or poroelastic medium) must interact through an interface. In this case, up to six different types of couplings are possible (see Aznárez et al. [10] for a complete mathematical description of such alternatives).

Regarding BEM-FEM couplings, there are plenty of possibilities within the type of material models and the numerical approaches considered. We have developed those that have more relevance to soil-structure interation phenomena:

- 1D beam structural elements (FEM) embedded in a 3D solid region (BEM), as described, for instance, in Padrón et al. [2] and Álamo et al. [22].
- 1D beam structural elements (FEM) embedded in a 2D inviscid fluid region (BEM) through crack boundary elements as described in Bordón et al. [13].
- 1D beam structural elements (FEM) embedded in a 2D elastic and poroelastic regions (BEM) through crack boundary elements, as described in Bordón et al. [14].
- Shell structural elements (FEM) embedded in 3D elastic, poroelastic and inviscid fluid regions (BEM) through crack boundary elements, as described in Bordón et al. [3].
- Other types of coupling such as the coupling of shell finite elements with external boundary elements, or the coupling of solid finite elements with external boundary elements, which are more widespread in the literature.

In all cases, the coupling is performed on a node-by-node basis, i.e. the so-called engineering direct approach, so finite element mesh and boundary element mesh must be conforming. Although in some cases other more general approaches may be advantageous, this direct approach is cost effective.

3 DESIGN AND FEATURES OF MULTIFEBE

3.1 Design

MultiFEBE is an integrated implementation of all the FEM and the BEM formulations previously mentioned within a single solver. The source code is available at:
`https://github.com/mmc-siani-es/MultiFEBE`.
MultiFEBE is a solver where data is handled via input and output plain text files, i.e. no GUI is included. Notwithstanding, in order to facilitate its usage, it accepts mesh files and it produces post-processing files for Gmsh pre- and post-processor [23]. It also accepts GiD mesh files through a template file [24].
It is written in modern Fortran programming language (Fortran 2003 or later), where source code is written in free source form (unlike old FORTRAN 77 fixed form files). The programming paradigm is mainly procedural, but some object-oriented features are also used (Fortran modules and derived types). The source code is released under `GPLv2` license. The program is linked to `ATLAS` [25] package in order to perform algebra operations within shared-memory multicore systems. At this moment, the program can be compiled with `GNU Fortran 9.4.0` or higher on GNU/Linux environments.
The main project files are organized in folders within `MultiFEBE/` as follows:

`bin/` Executables and auxiliary files.

`doc/` Documentation and tutorials.

`lib/` Source code of libraries (only FBEM at the moment).

- `fbem/` FBEM library.
  - `bin/` Compiled object files.
  - `doc/` Documentation.
  - `include/` Compiled modules.
  - `src/` Source code.
  - `LICENSE` License file.
  - `Makefile` Makefile for compiling the source code.
  - `README.md` Readme file containing information about the library.

`src/` Source code of the application layer.

`LICENSE` License file.

`Makefile` Makefile for compiling the source code.

`README.md` Readme file containing information about the package.

3.2 Available element types

MultiFEBE includes a full set of linear elastic finite elements:

- Solid (or continuum) finite elements:
  - For two-dimensional plane strain problems:
    - 3 or 6 nodes triangular element.
    - 4, 8 or 9 nodes quadrilateral element.
- Structural finite elements:
  - Bar element (2 nodes).
  - Beam elements (doubly symmetric cross-section):
    - 2 or 3 nodes Euler-Bernoulli or Timoshenko straight element.
    - 3 or 4 nodes curved element based on the degeneration from solid (Timoshenko theory).
  - Shell elements (Reissner-Mindlin theory):
    - 3 or 6 nodes triangular and 4, 8 or 6 nodes quadrilateral shell element based on the degeneration from solid (full/selective/reduced integration).
    - 9 nodes quadrilateral MITC element (locking-free).
- Discrete translational and translational-rotational spring/dashpot 2 node elements.
- Discrete mass elements.

It also has a full set of boundary elements for linear elastic (full-space fundamental solution in statics and dynamics, and half-space Green's function in statics), poroelastic (full-space fundamental solution) and inviscid fluid (full-space fundamental solution and half-space Green's function) regions:

- For two-dimensional plane strain problems:
  - 2, 3 or 4 line ordinary or crack boundary elements.
  - 2, 3 or 4 line body load elements.
- For three-dimensional problems:
  - 3 or 6 triangular and 4, 8 or 9 quadrilateral ordinary or crack boundary elements.
  - 2, 3 or 4 line body load elements (only in elastic solid BE regions).
  - 3 or 6 triangular and 4, 8 or 9 quadrilateral body load elements.

4 APPLICATION EXAMPLES

This section presents three application examples to illustrate some of the capabilities of the software. First, a static analysis on a two–dimensional problem with a simple reference analytical solution, is presented. Then, a similar problem, but defined in a three–dimensional cube and subjected to time–harmonic loads, is shown. Finally, a more complex problem involving the computation of the static response of a floating pile is presented. In all cases, the results are compared against reference solutions, to test the validity of the software. More details on each one of the cases can be found among the documentation of the code.

4.1 Static analysis of a 2D plain strain elastic square

Fig. 1 shows the geometry for a static analysis of a plain strain elastic square (left side of the figure) together with a representation of the boundary element mesh employed for its solution (right side of the figure). It is worth noting that, due to the different boundary conditions set on each boundary, nodes must be duplicated in the corners.
Required material and geometric properties are the Young's modulus E, the Poisson's ratio ν, the square side length L and the static load P. Self-weight is not considered.
The analytical solution to this problem in terms of displacement and stress fields can be written as:

$$\begin{aligned} u_1 &= \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} P x_1 \\ u_2 &= 0 \\ \sigma_{11} &= P \\ \sigma_{22} &= \frac{\nu}{1-\nu} P \\ \sigma_{12} &= 0 \end{aligned} \tag{4}$$

where tractions are defined as $t_i = \sigma_{ij} n_j$ (n_j are the components of the outward unit normal components). The displacement field is linear along x_1-direction and constant (null) along x_2-direction. Also, the only non-null stresses (σ_{11} and σ_{22}) are constant.
The problem is solved for $L = 1$ m, $E = 1$ N/m^2, $\nu = 0.25$ and $P = 1$ N/m^2 and the analytical and numerical solutions are shown in Table 1. It can be seen that the numerical solution is in perfect agreement with the analytical solution.

4.2 Time harmonic analysis of a 3D strain elastic cube

Fig. 2 shows the geometry for a time harmonic analysis of a 3D elastic cube. Required material and geometric properties are the Young's modulus E, the Poisson's ratio ν, the density ρ, shear modulus μ, damping coefficient ξ, the cube side length L and the time harmonic load P. Load P can be regarded as a P-wave travelling along the x_1 direction. Self-weight is not considered.

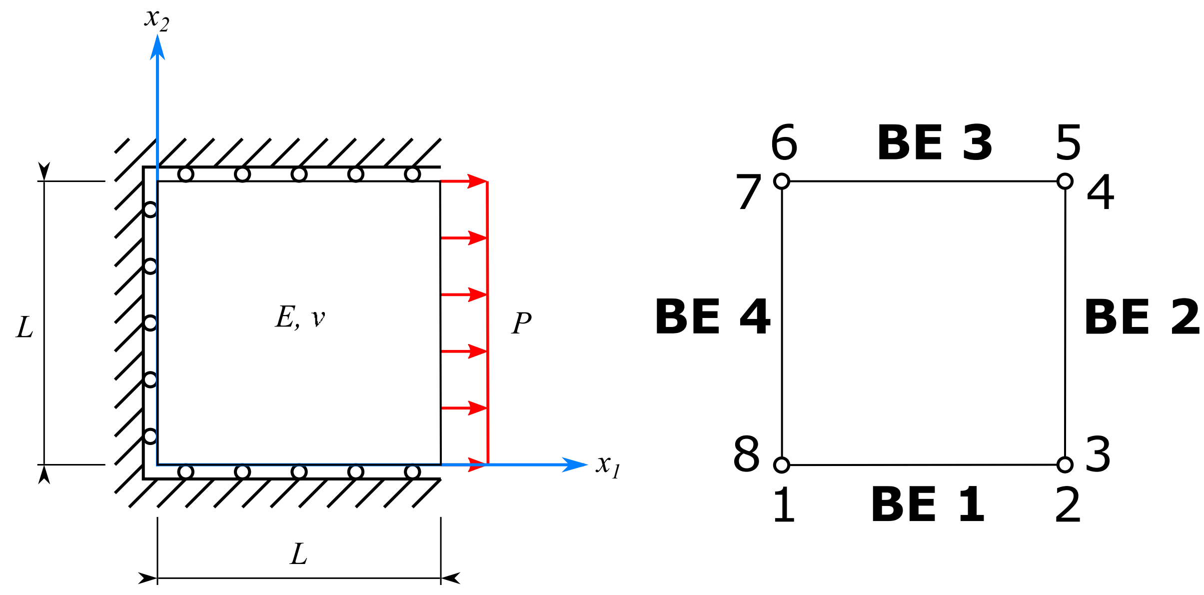


Figure 1: 2D plain strain elastic square.

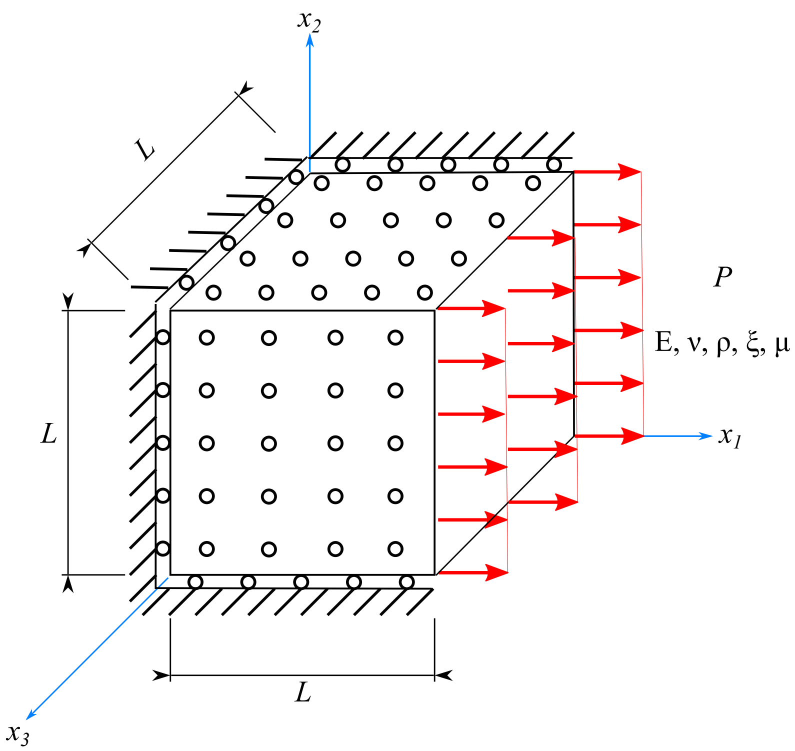


Figure 2: 3D harmonic elastic cube.

Node	Variable	Analytical solution	Numerical solution
3	u_1	$0.8\overline{3}$	0.833333
3	u_2	0	$0.3116271 \cdot 10^{-6}$
3	t_1	1	1.000000
3	t_2	0	0.000000
4	u_1	$0.8\overline{3}$	0.833333
4	u_2	0	$-0.3116271 \cdot 10^{-6}$
4	t_1	1	1.000000
4	t_2	0	0.000000

Table 1: Results of the 2D static elastic square.

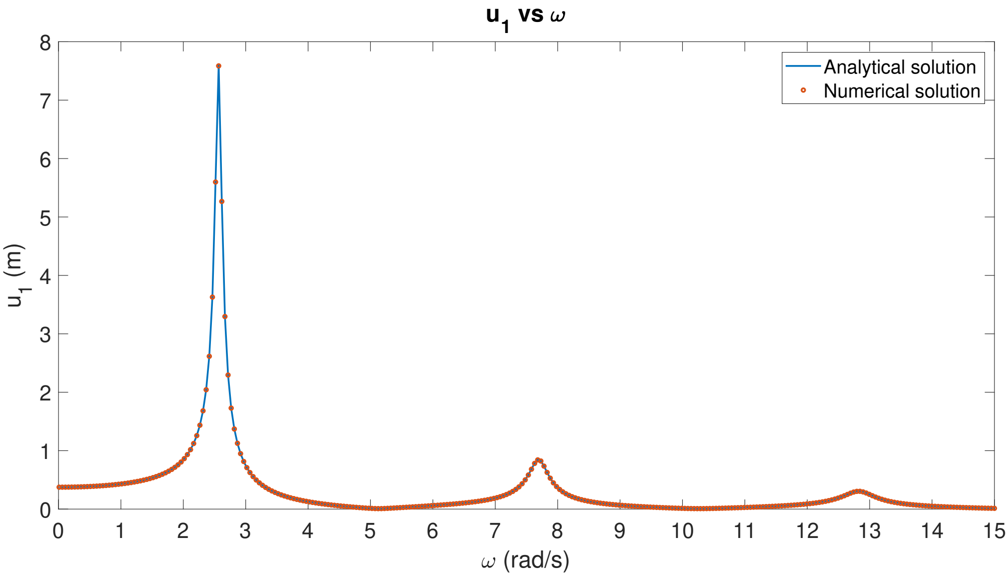


Figure 3: Results of the 3D harmonic elastic cube.

The analytical solution to this problem can be obtained by means of the following equations:

$$\begin{aligned} & k = \omega / c \\ & c_p = \mathrm{Re(c_p)} \cdot \sqrt{1 + 2\mathrm{i}\xi} \\ & \mathrm{Re(c_p)} = \sqrt{(\lambda + 2\mu)/\rho} \\ & u(x_1, \omega) = -\frac{\mathrm{e}^{-\mathrm{i}kx_1} - \mathrm{e}^{\mathrm{i}kx_1}}{(\lambda + 2\mu) \cdot \mathrm{i}k \cdot (\mathrm{e}^{-\mathrm{i}kL} + \mathrm{e}^{\mathrm{i}kL})} \end{aligned} \tag{5}$$

where natural frequencies can be calculated as:

$$\omega_n = \frac{Re(c_p)\pi}{2L} n; \quad n = 1, 3, 5... \tag{6}$$

The problem is solved for $L = 1$ m, $E = 1$ N/m^2, $\nu = 0.25$, $\mu = 1$, $\rho = 1$ kg/m^3, $\xi = 0.02$ and $P = 1$ N/m^2. First, second and third natural frequencies are:

$$\begin{aligned} \omega_1 &= 2.6 \text{ rad/s} \\ \omega_3 &= 7.7 \text{ rad/s} \\ \omega_5 &= 12.8 \text{ rad/s} \end{aligned} \tag{7}$$

The comparison between the analytical and numerical solutions is shown on Fig. 3. It can be seen that the numerical solution agrees perfectly with the analytical solution.

4.3 Floating pile

In this application example, the horizontal pressure distribution along different floating piles due to a static load is obtained. The obtained results can be directly compared against the literature [26], which serves as a validation case for this coupled model.

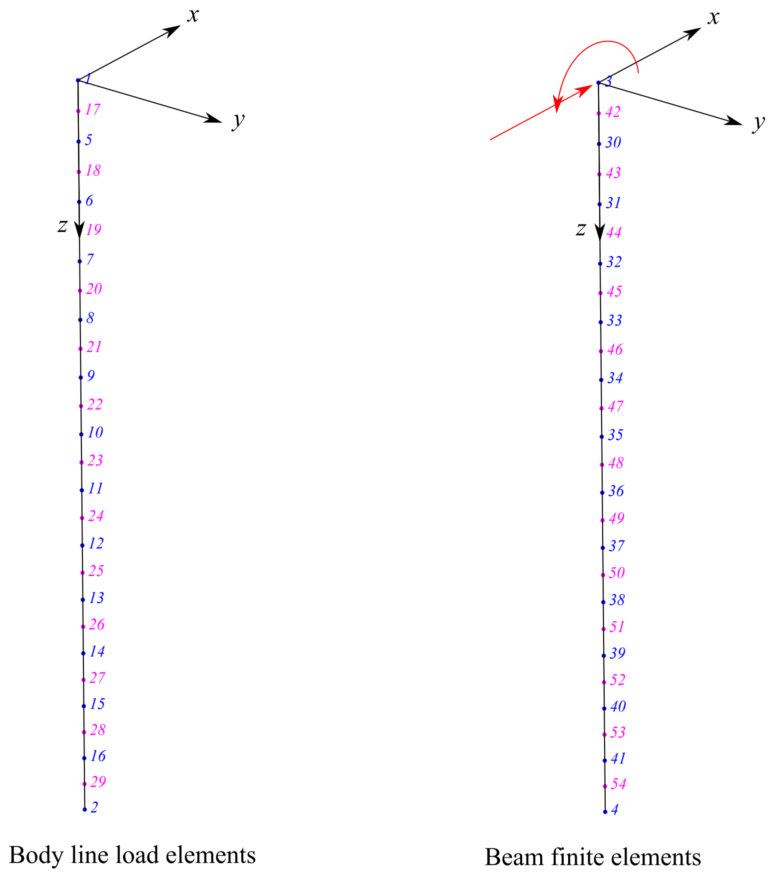
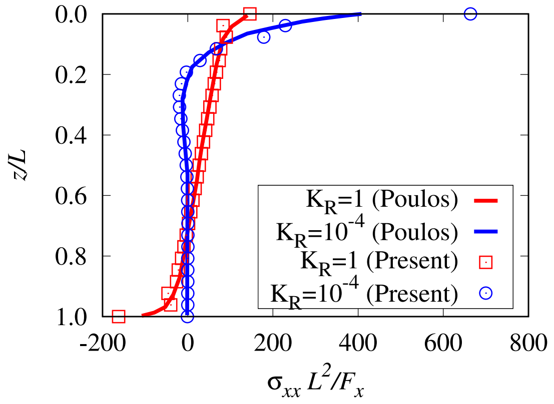
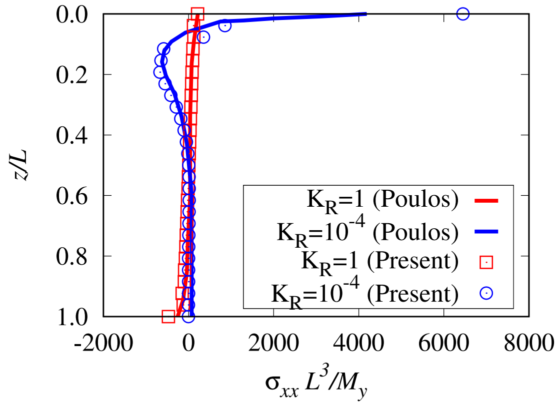


Figure 4: Mesh for the floating pile example. Boundary element mesh consists only on body line load elements, which geometrically coincides to beam finite element mesh.

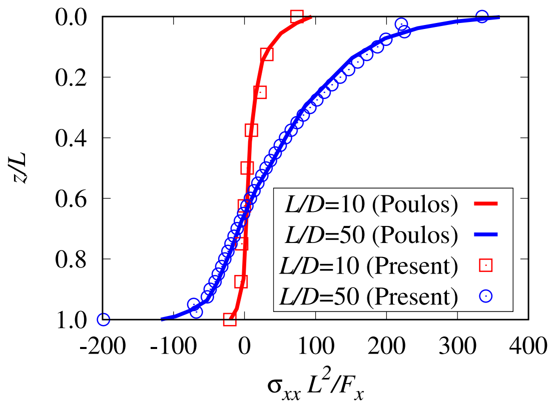
A floating pile is a foundation with a slender cylindrical shape which is buried into the soil. The structure is mainly one-dimensional, thus it is modelled as a finite element



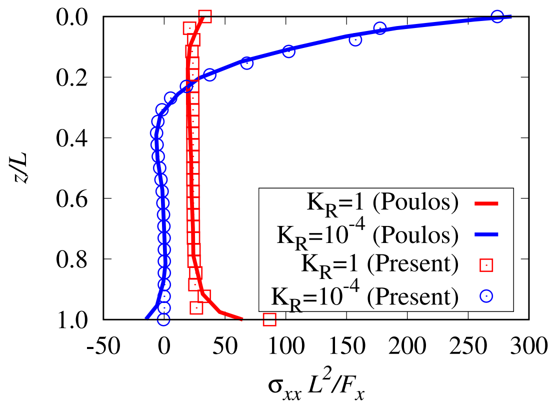
(a) Free-head and horizontal load F_x applied. Pile with $L/D = 25$, and soil $\nu_s = 0.5$.



(b) Free-head and moment load M_y applied. Pile with $L/D = 25$, and soil $\nu_s = 0.5$.



(c) Free-head and horizontal load F_x applied. Pile flexibility $K_R = 0.01$, and soil $\nu_s = 0.5$.



(d) Fixed-head and horizontal load F_x applied. Pile with $L/D = 25$, and soil $\nu_s = 0.5$.

Figure 5: Pressure distribution along floating piles for several cases. Reference results are taken from [26].

region consisting on beam finite elements. The soil is a half-space (unbounded domain), and thus it is modelled as a boundary element region. For this boundary element region, the half-space Green's function (Mindlin solution) is used, so no discretization is needed for the free-surface. The soil-pile interaction is introduced by including in the boundary element region a body line load whose discretization must be conforming to the beam finite elements. Fig. 4 shows the devised mesh for the problem.

Fig. 5 shows the dimensionless pressure along a floating pile for several cases. The results from [26] are shown together with the obtained results from the present model. The cases differ in pile slenderness L/D (L is the pile length and D is the pile diameter), pile head contraint (free-head or fixed-head where rotation is constrained), loading type (horizontal force F_x or moment M_y), and pile flexibility factor $K_R = E_\text{p}I_\text{p}/(E_\text{s}L^4)$. It is observed a very good agreement except around the pile tip and head, where small discrepancies appear presumably due to the simplification introduced by a one-dimensional modelling of the pile-soil interaction.

5 CONCLUSIONS

The boundary elements – finite elements code MultiFEBE, developed at the University Institute SIANI of the Universidad de Las Palmas de Gran Canaria, has been briefly described in this paper. The major original features of this code are related to its capabilities in coupling regions and structural elements modeled through finite and boundary elements, in order to be able to use the advantages of both methodologies, and be able to model complicated problems involving, for instance, cracks in elastic and poroelastic regions, or the direct interaction between structural elements such as beams and shells, embedded in bounded or unbounded fluid, elastic or poroelastic domains.
This program has been published as open–source code at https://github.com/mmc-siani-es/MultiFEBE so that the computational mechanics community can use, modify and enhanced the code. Documentation and tutorials are also provided together with the source code.

6 ACKNOWLEDGMENTS

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